Compactness and chromatic number

by

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This paper is a continuation of [5], in which we gave an example of an atomic-compact relational structure which is not a retract of a compact topological relational structure. That example was a graph of infinite chromatic number; in this paper we show that infinite chromatic number is necessary for such an example. That is, we generalize the notion of chromatic number and show (Theorem 3.2) that an atomic-compact relational structure is a retract of a compact topological relational structure if and only if none of its chromatic numbers is an infinite cardinal.

§ 0 contains the preliminaries. In § 1 we develop the notion of chromatic number, and in § 2 we relate this notion to the notion of pure extension. The main result is in § 3. § 4 contains a characterization of retracts of compact topological relational structures in terms of ultrafilters.

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0. Preliminaries. We let $\mathfrak{A} = \langle A, R_i \rangle_{i \in \mathbb{N}}$ denote a relational structure (with no operations and no constants). If $A$ is a (compact) topological space (not necessarily Hausdorff), and each $n$-ary $R_i$ is closed in $A^n$, then we say that $\mathfrak{A}$ is a (compact) topological relational structure. Consult [7] for the following notions: formula with constants in $\mathfrak{A}$, satisfiability of a set of formulas with constants in $\mathfrak{A}$, purity, homomorphism, atomic compactness, and retract. We will not distinguish between the designation of a predicate symbol and the relation to which it refers. We take $x$ and $y$ to be variables, and $a$ to be any fixed element of $A$. We take "$x = y$" as representing equality in the formal language, whereas "$x \neq y$" will mean that $x$ and $y$ are the same variable.

If $R$ is an $n$-ary relation on the set $A$, we let $\overline{R}$ denote the closure of $R$ in $(\beta A)^n$, where $\beta A$ denotes the Stone-Čech compactification of $A$. If $\mathfrak{A}$ is as above, then $\beta \mathfrak{A}$ denotes the relational structure $(\beta A, R_i)$ of $\beta A$. $\beta A$ will be taken as equal to the set of ultrafilters on $A$, which will be

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denoted $\lambda$, $\mu$, $\nu$ (possibly subscripted). For information on the topology of $\beta A$, consult e.g. [2].

By cardinal, finite or infinite, denoted $\kappa$, we mean an initial ordinal (which we take to be the set of smaller ordinals). We let $\aleph$ denote a finite cardinal. Also we adopt a symbol $\infty$ with the convention that $\kappa < \infty$ for every $\kappa$.

1. Chromatic numbers. Let us be given $\kappa$, a positive integer, and $\Sigma$ an equivalence relation on the set $\{1, 2, \ldots, \kappa\}$.

Definition 1.1. $\Sigma(\kappa, \kappa)$ is the relational structure $\langle \kappa, S, \rangle$, where $S$ is the $\kappa$-ary relation defined as follows:

(*) $\langle x_1, \ldots, x_\kappa \rangle \in S$ iff $i j = x_i = x_j$.

Definition 1.2. Let $R$ be an $\kappa$-ary relation on the set $\kappa$. $\chi_\kappa(R)$, the $\kappa$-chromatic number of $R$, is the least $\kappa$ such that there is a homomorphism $F : \langle \kappa, R \rangle \to \Sigma(\kappa, \kappa)$. If no such homomorphism exists, we put $\chi_\kappa(R) = \infty$.

To see that this definition generalizes the usual definition of chromatic number, we will show that $\kappa$ is a symmetric antireflexive binary relation on $\kappa$. Now if we take $\epsilon$ such that $1 \in \epsilon$, then $\Sigma(\kappa, \kappa)$ is just the complete graph on $\kappa$ vertices. Thus the homomorphism $F$ in Definition 1.2 is a coloring of the graph $\langle \kappa, R \rangle$ in $\kappa$ colors. It is easy to check that our definition agrees with that of Erdős and Hajnal [1, Definition 2.8], in the case of uniform set-system.

Lemma 1.3. Suppose there is a homomorphism $F : \langle \kappa, R \rangle \to \langle \kappa, S \rangle$. Then $\chi_\kappa(R) \leq \chi_\kappa(S)$.

Lemma 1.4. Suppose $\mathcal{W} = \langle \kappa, T \rangle$ is a compact topological relational structure. Then $\chi_\kappa(T)$ is finite or $\infty$.

Proof. Suppose that $\chi_\kappa(T)$ is an infinite cardinal. We let $\theta$ be the set of equivalence relations on $\kappa$ such that every $\sigma$-equivalence class is finite. $\theta$ becomes a directed set if we take $\theta_1 \leq \theta_2$ to mean that $\sigma_2 \prec \sigma_1$. Since $\chi_\kappa(T)$ is finite, it follows, by (*), that for each $\sigma \in \theta$, we may find $\langle \sigma_1, \ldots, \sigma_\kappa \rangle \in T$ such that $\sigma_2 \prec \sigma_1$ and $\langle \sigma_1, \ldots, \sigma_\kappa \rangle \in T$. Take $i$ to be one member of one $\sigma$-equivalence class, and some subset of $\theta_0$ converging to $\sigma_i$. Continuing this process, we clearly can find $\langle \sigma_1, \ldots, \sigma_\kappa \rangle \in T = T_i$ such that $\kappa \sigma_1 = \kappa \sigma_2 = \kappa \sigma_3 = \kappa \sigma_4 = \cdots$. Thus $\chi_\kappa(T) = \infty$. Q.E.D.

Corollary 1.5. Suppose $\mathcal{W} = \langle \kappa, R \rangle$ is a retract of a compact topological relational structure. Then $\chi_\kappa(R)$ is finite or $\infty$.

Corollary 1.6. If $\chi_\kappa(R) = \kappa < \kappa$, then $\chi_\kappa(R) = \kappa$. If $\chi_\kappa(R) > \kappa$, then $\chi_\kappa(R) = \infty$.

Proof. Suppose first that $\chi_\kappa(R) = \kappa$. Then there is a homomorphism $F : \langle \kappa, R \rangle \to \Sigma(\kappa, \kappa)$. Since $\Sigma(\kappa, \kappa)$ is finite, hence compact, we may extend $F$ to a homomorphism $\beta F : \langle \beta A, R \rangle \to \Sigma(\kappa, \kappa)$. Thus $\chi_\kappa(R) = \kappa$. The reverse inequality holds by Lemma 1.3. If $\chi_\kappa(R) > \kappa$, then $\chi_\kappa(R) = \kappa$, by Lemma 1.3. Thus by Lemma 1.4, $\chi_\kappa(R) = \infty$. Q.E.D.

2. Chromatic number and purity. In order to state and prove our main theorem, we need the following notion of derived relations in a relational structure.

Definition 2.1. Let $\mathcal{U} = \langle A, R \rangle$ be a relational structure. We let $\mathcal{D}(\mathcal{U})$ (the derived relational structure) be the relational structure with carrier $A$ and containing all relations derived from relations of $\mathcal{U}$ by permutation of variables, Cartesian products and substitution of constants of $A$. That is, the set $W$ of relations of $\mathcal{D}(\mathcal{U})$ is the smallest set of relations, containing the relations of $\mathcal{U}$, and subject to the following:

(D1) Let $R$ be an $n$-ary relation $\langle W \rangle$, and let $\pi$ be a permutation of $\{1, \ldots, n\}$. Then $W$ contains the relation $\mathcal{D}(\mathcal{U})$ defined by:

$\langle \pi(1), \ldots, \pi(n) \rangle \in \mathcal{D}(\mathcal{U})$ iff $\langle x_{\pi(1)}, \ldots, x_{\pi(n)} \rangle \in R$.

(D2) Let $R$ be an $m$-ary relation $\langle W \rangle$, and let $S$ be an $n$-ary relation $\langle W \rangle$. Then $W$ contains the $(n+m)$-ary relation $T$ defined by:

$\langle \pi(1), \ldots, \pi(n), \eta(1), \ldots, \eta(m) \rangle \in T$ iff $\langle \pi(1), \ldots, \pi(n), \eta(1), \ldots, \eta(m) \rangle \in R$.

(D3) Let $R$ be an $(n+1)$-ary relation $\langle W \rangle$, and let $a \in A$. Then $W$ contains the $n$-ary relation $S$ defined by:

$\langle a, \pi(1), \ldots, \pi(n) \rangle \in S$ iff $\langle a, x_{\pi(1)}, \ldots, x_{\pi(n)} \rangle \in R$.

Lemma 2.2. If $\mathcal{W}$ is a compact topological relational structure, then $\mathcal{D}(\mathcal{W})$ is a compact topological relational structure under the same topology.

Lemma 2.3. Let $R$ and $S$ be as in (D1). Then

$\langle \mu(1), \ldots, \mu(n) \rangle \in \mathcal{D}(\mathcal{U})$ iff $\langle \mu_{\pi(1)}, \ldots, \mu_{\pi(n)} \rangle \in \mathcal{D}(\mathcal{U})$.

Lemma 2.4. Let $R, S$ and $T$ be as in (D2). Then

$\langle \pi(1), \ldots, \pi(n), \mu(1), \ldots, \mu(m) \rangle \in T$ iff $\langle \pi(1), \ldots, \pi(n), \mu(1), \ldots, \mu(m) \rangle \in R$.

Lemma 2.5. Let $R$ and $S$ be as in (D3). Then

$\langle \pi(1), \ldots, \pi(n), \mu(1), \ldots, \mu(m) \rangle \in T$ iff $\langle a, \pi(1), \ldots, \pi(n), \mu(1), \ldots, \mu(m) \rangle \in S$.

Proof. Let $U$ and $V$ be compact Hausdorff spaces, with $p : U \times V \to V$ the first co-ordinate projection, and $e$ an isolated point of $V$. One has to prove that $p[\beta A \cap (U \times \{e\})] = p[\beta A \cap (U \times \{e\})]$. It is easy to check.
the inclusion $\subseteq$. The reverse inclusion follows, since any net converging to an element of $U \times (w)$ must eventually be in $U \times (w)$, because $U \times (w)$ is open. Q.E.D.

**Lemma 2.6.** An embedding $F: \mathfrak{U} \rightarrow \mathfrak{M}$ is a pure embedding if there is a homomorphism $G: \mathfrak{B} \rightarrow \mathfrak{G}$ such that $G \circ F$ is an elementary embedding.

*Proof.* Essentially the same as [11], Lemma 2.2.

**Lemma 2.7.** If $\mathfrak{U}$ is pure in $\beta \mathfrak{M}$, then $\mathfrak{U}$ is pure in $\beta \mathfrak{B}$. 

*Proof.* By Lemma 2.6, there is a homomorphism $G: \beta \mathfrak{M} \rightarrow \mathfrak{G}$ such that $G \circ J$ is an elementary embedding, where $J$ is the natural inclusion of $\mathfrak{U}$ into $\beta \mathfrak{M}$. But by Lemmas 2.3-3, the relations of $\beta \mathfrak{M}$ are defined by the same positive formulæ as are their closures in $\beta \mathfrak{A}$. Thus we may apply the Theorem of Marczewski [3] to see that $G$ is a homomorphism for these derived relations. Thus $\mathfrak{U}$ is pure in $\beta \mathfrak{B}$. Q.E.D.

**Lemma 2.8.** Suppose $\mathfrak{U}$ is pure in $\beta \mathfrak{A}$, and let $R$ be an $n$-ary relation of $\mathfrak{A}$. Then $x_y(R)$ is finite or $\infty$.

*Proof.* If $x_y(R)$ is an infinite cardinal, then by Cor. 1.6, $x_y(R) = \infty$. Thus there exists $(\mu_1, \ldots, \mu_n) \in R$, such that $\mu_j$ implies $\mu_i = \mu_i$. Since $\mathfrak{A}$ is pure in $\beta \mathfrak{A}$ by Lemma 2.7, we have $(\mu_1, \ldots, \mu_n) \in R$, such that $\mu_j$ implies $\mu_i = \mu_i$. Thus $x_y(R) = \infty$, which is a contradiction. Q.E.D.

**Lemma 2.9.** Suppose $\mathfrak{U}$ is a retract of a compact topological relational structure, and $R$ is an $n$-ary relation of $\mathfrak{U}$. Then $x_y(R)$ is finite or $\infty$.

*Proof.* Similar to the above proof, using Cor. 1.5, Lemmas 2.2-5 and 2.7, and [3].

In the rest of this paper, when we say "every chromatic number of a structure $\mathfrak{U}$, we mean all $x_y(E)$, for every relation $E$ of $\mathfrak{U}$, and for every appropriate $y$.

**Theorem 2.10.** $\mathfrak{U}$ is pure in $\beta \mathfrak{A}$ iff every chromatic number of $\mathfrak{A}$ is finite or $\infty$.

*Proof.* The necessity of the condition was shown in Lemma 2.8. To see sufficiency, let us assume that every chromatic number of $\mathfrak{A}$ is finite or $\infty$. We will let $\Sigma$ be a finite set of atomic formulæ of $\mathfrak{A}$ with constants in $\mathfrak{U}$, which is satisfiable in $\beta \mathfrak{A}$. We replace any occurrence of a constant $a$ in a relational formula by a new variable $x_a$ and adjoin the formula "%a = a%". Thus we may assume that no new constant appears in a relational formula in $\Sigma$. We prove that such a $\Sigma$ is satisfiable in $\mathfrak{A}$. The proof is by induction on the number of formulæ in $\Sigma$.

Case I. $\Sigma$ contains no relational formulæ, but only equalities. In this case there is essentially nothing to prove.

Case II. $\Sigma$ contains "%a = a%" (where $a$ and $b$ need not be distinct). Form $\Sigma'$ from $\Sigma$ by discarding "%a = a%", and replacing every occurrence of $b$ by $a$. By induction, $\Sigma'$ is satisfiable in $\beta \mathfrak{A}$, and thus so is $\Sigma$.

**Case III.** $\Sigma$ contains at least two relational formulæ, "%a(x_1, ..., x_0)%" and "%b(y_1, ..., y_m)%". Let $T$ be supplied by (D2). Form $\Sigma'$ from $\Sigma$ by discarding these two formulæ and adjoinning the formulæ "%a(x_1, ..., x_0, y_1, ..., y_m)%", "%b(y_1, ..., y_m)%", since $\Sigma$ is satisfiable in $\beta \mathfrak{A}$, then by Lemma 2.4, $\Sigma'$ is satisfiable in $\beta \mathfrak{A}$ by induction. $\Sigma'$ is satisfiable in $\beta \mathfrak{A}$ by induction. Thus by (D2), $\Sigma$ is satisfiable in $\beta \mathfrak{A}$.

**Case IV.** $\Sigma$ contains "%a = a%" and exactly one relational formula. We form $\Sigma'$ from $\Sigma$ in the following way. First we discard "%a = a%". Then by (D1) we replace the given relational formula with a formula "%a(x_1, ..., x_2, ..., y_m)%", where no $y_i$ is $a$. Then we replace this formula with the formula "%a(y_1, ..., y_m)%", given by one application of (D3) for each appearance of $x$. By Lemmas 2.3 and 2.4, $\Sigma'$ is satisfiable in $\beta \mathfrak{A}$, and hence, by induction, $\Sigma'$ is satisfiable in $\mathfrak{A}$. Thus by (D1) and (D3), $\Sigma$ is satisfiable in $\mathfrak{A}$.

**Case V.** $\Sigma$ consists of exactly one relational formula, "%a(x_1, ..., x_0)%". We define the equivalence relation $\equiv$ by taking $x_j$ to mean $x_i = x_i$. Since $\Sigma$ is satisfiable in $\beta \mathfrak{A}$, $x_y(R)$ is finite. Thus by Cor. 1.6, $x_y(R)$ is finite. Thus, by hypothesis, $x_y(R) = \infty$. This means that $\Sigma$ is satisfiable in $\mathfrak{A}$.

Thus we have shown that $\mathfrak{A}$ is pure in $\beta \mathfrak{A}$. Since $\mathfrak{A}$ is a retract of $\mathfrak{A}$, it is clear that $\mathfrak{U}$ is pure in $\beta \mathfrak{A}$. Q.E.D.

3. The main result. The following easy theorem is a modification of a proposition of Mycielski [4, p. 4]. The main change is that we no longer require the Hausdorff axiom. This modified theorem is true even for algebras if we require that the graph of each $n$-ary operation be closed in $A^{n\times}$. 

**Theorem 3.1.** Every retract of a compact topological relational structure is atomic-compact.

In [5] it was shown that the converse to Theorem 3.1 is false. But we can state the following theorem relating compactness and atomic compactness.

**Theorem 3.2.** The following are equivalent:

(i) $\mathfrak{U}$ is a retract of $\beta \mathfrak{A}$.

(ii) $\mathfrak{U}$ is a retract of a compact Hausdorff topological relational structure.

(iii) $\mathfrak{U}$ is a retract of a compact topological relational structure.

(iv) $\mathfrak{U}$ is atomic-compact, and every chromatic number of $\mathfrak{A}$ is finite or $\infty$.

*Proof.* Obviously (i) = (ii) = (iii). By Lemma 2.9 and Theorem 3.1, (iii) = (iv). Finally, (iv) = (i) by Theorem 2.10 and [7, Theorem 2.3]. Q.E.D.
The two conditions of (iv) above are independent. [3] gives an example of an atomic-compact structure with at least one chromatic number an infinite cardinal. On the other hand, Warfield has shown [6] that every Abelian group is pure in its Bohr compactification. Thus if $\mathbb{K}$ is the Abelian group of integers under addition, every chromatic number of $\mathbb{K}$ is finite or $\infty$, but $\mathbb{K}$ is not atomic-compact [4, p. 2].

Suppose $\mathbb{K}$ is a relational structure which is really an algebra, that is, each relation of $\mathbb{K}$ is the graph of an operation. If $\mathbb{K}$ is atomic-compact, and every chromatic number of $\mathbb{K}$ is finite or $\infty$, then $\mathbb{K}$ is a retract of a compact topological relational structure. If each operation is unary, then $\mathbb{K}$ is actually a retract of a compact topological algebra, since $\mathbb{K}$ is an algebra. Otherwise, it is an open problem whether $\mathbb{K}$ is a retract of a compact topological algebra.

**Corollary 3.3**. Suppose $\mathbb{K}$ is atomic-compact and every finite subset of $\mathbb{K}$ is a retract of a compact topological relational structure. Then $\mathbb{K}$ is a retract of a compact topological relational structure.

4. **Ultrarapid characterization of retracts of compact topological relational structures.** The following theorem is due to Weglorz [7, Theorem 2.3].

**Theorem 4.1**. A relational structure $\mathbb{K}$ is atomic-compact iff $\mathbb{K}$ is a retract of every ultrapower of $\mathbb{K}$.

Assuming that $\mathbb{K}$ is actually a retract of $\beta \mathbb{K}$, we will give a novel proof of the necessity of the condition. If $\mathbb{K}/\mu$ is any ultrapower of $\mathbb{K}$, then an element of $\mathbb{K}/\mu$ is a class of functions $f: I \rightarrow A$, any two of which agree almost everywhere. Given such a function $f: I \rightarrow A$, we denote by $f(\mu)$ the ultrapower on $A$, $(K \subseteq A: f^{-1}(K) \in \mu$. If $\mathbb{K}$ is a retract of $\beta \mathbb{K}$ under the ultrapower mapping $\mathcal{F}$, then $\mathcal{F}(f(\mu))$ is an element of $A$. It is simple to check that $f(\mu)$ does not depend on the choice of $\mathcal{F}$ in the class representing an element of $\mathbb{K}/\mu$. It is also easy to check that in this way we have constructed a retraction of the ultrapower $\mathbb{K}/\mu$ onto $\mathbb{K}$. This construction is an improvement on Theorem 4.1.1 in the following way. It is obvious that if $f(\mu) = g(\mu)$, then the class of $f$ and the class of $g$ are mapped to the same element of $\mathbb{K}$ under this retraction. Thus we have the following definition and theorem.

**Definition 4.2**. A consistent retraction of an ultrapower $\mathbb{K}/\mu$ onto $\mathbb{K}$ is a function $\mathcal{F}$ from $\mathbb{K}/\mu$ onto $\mathbb{K}$, such that $f(\mu) = \mathcal{F}(\mu)$ for each $\mu$ in the class of $f(\mu)$. This is called a consistent retraction of every ultrapower of $\mathbb{K}$.

**Theorem 4.3**. A relational structure $\mathbb{K}$ is a retract of a compact topological relational structure iff $\mathbb{K}$ is a consistent retraction of every ultrapower of $\mathbb{K}$.

Proof. We have already found consistent retracts of ultrapowers of relational topological relational structures. To prove the converse, we will need the following definition and lemma.

**Definition 4.4**. An ultrafilter $\mathcal{F}$ on the set $I$ will be called universal for the set $A$, if given any ultrafilter $\mu$ on $A$, there is a map $f: I \rightarrow A$ such that $f(\mu) = \mu$.

**Lemma 4.5**. Let $A$ be any set. Then there is a set $\mathcal{F}$ and an ultrafilter $\mu$ on $\mathcal{F}$ such that $\mathcal{F}$ is universal for $A$.

Proof. Take $\mathcal{F}$ to be the set $\{\alpha \subseteq A: \mu(\alpha) = 1\}$, then let $\mu$ be an ultrafilter on $\mathcal{F}$ extending the filter of subsets of the form $\mathcal{F} \cap \{x: x \in \alpha\}$, where $\mu \in \mathcal{F} \cap \{x: x \in \alpha\}$. This is the $\mu$-th co-ordinate projection of $I$ onto $A$, and $\mu \in \mathcal{F}$.

Returning to the proof of Theorem 4.3, let $\mathcal{F}$ be an ultrafilter on some set $I$ which is universal for the set $\bigcup_{\alpha < \beta} A_{\alpha}$, and let $\mathcal{F}$ be a consistent retraction of $\mathbb{K}/\mu$ onto $\mathbb{K}$. It is clear that we have a consistently defined mapping, $C$, from $\mathbb{K}$ to $\mathbb{K}$. We need only show that $\mathcal{F}$ is a homomorphism.

Suppose $\mathcal{F}$ is an $n$-ary relation of $\mathbb{K}$ and we are given $\mu_1, \ldots, \mu_n \in \mathcal{F}$.

Let $\mathcal{F}$ be the filter on $\mathcal{F}$ generated by $\mathcal{F}$ and all products $K_1 \times \ldots \times K_n$, where each $K_i \in \mu_i$. Since $\mu_1, \ldots, \mu_n \in \mathcal{F}$, $\mathcal{F}$ is a proper filter, and thus we may extend $\mathcal{F}$ to an ultrafilter $\mathcal{F}$ on $A$. Since $\mathcal{F}$ is universal, there is a map $f: I \rightarrow A$ such that $f(\mathcal{F}) = \mathcal{F}$. Since $\mathcal{F} \in \mathcal{F}$, it follows that almost everywhere, $f_1(i), \ldots, f_n(i) \in \mu_i$, where $f_i(i)$ is the $i$th component of $f(i)$, and where $i \in I$. Thus since $\mathcal{F}$ is a homomorphism, it follows that $\mathcal{F}(f_1(\mu)), \ldots, \mathcal{F}(f_n(\mu)) \in \mathcal{F}$, where each $\mu_i$ is the class in $\mathbb{K}/\mu$ with representative $f_i$. But since each $f_i(\mu) = \mu_i$, it follows that $\mathcal{F}(\mu_1), \ldots, \mathcal{F}(\mu_n) \in \mathcal{F}$, and thus that $\mathcal{F}$ is a homomorphism. Q.E.D.

**References**


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