# Post algebras and pseudo-Post algebras *

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Post algebras were first considered by Rosenbloom [6] and have been studied in recent years by Epstein [3], Traczyk [8], [9] and Dwinger [2]. From the work of Epstein and Traczyk it follows that a distributive lattice with $0, 1$ is a Post algebra if and only if it is the coproduct of a Boolean algebra and a finite chain. Similarly a distributive lattice with $0, 1$ will be called a pseudo-Post algebra if it is the coproduct of a pseudo-Boolean algebra and a finite chain. The first part of the paper deals with the theory of Post algebras and pseudo-Post algebras.

At the end of the paper we consider an application to logic. The notion of validity in classical and intuitionistic logic may be defined semantically by the methods of Tarski and Kripke (1) respectively. If we replace the two truth-values occurring in these definitions by a system of $n$ truth-values, we obtain what may be referred to as classical $n$-valued logic and intuitionistic $n$-valued logic respectively. The representation theory of Post algebras and pseudo-Post algebras can be used to establish the completeness of suitable axiomatizations of these logics. We consider classical and intuitionistic $n$-valued propositional calculi from this point of view in Section 6.

Consider the category of distributive lattices with $0, 1$ and $0, 1$-preserving homomorphisms; the objects and morphisms of this category will be referred to simply as lattices and homomorphisms. A lattice is called non-degenerate if it contains the two-element lattice $2$ as a sublattice. A pseudo-Boolean algebra is a lattice in which the element $a \supset b = \max\{x \in L\colon x \wedge a \leqslant b\}$ exists for every pair of elements $a, b$; we shall write $-a = \max\{x \in L\colon x \wedge a = 0\}$. A Boolean algebra is a lattice in which for every element $a$ there exists an element $a'$ such that

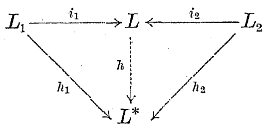
$$a \wedge a' = 0 , \quad a \vee a' = 1 ;$$

* The results of this paper (with the exception of Theorems 5 and 10) were communicated at Professor Mostowski's seminar in the Mathematical Institute of the Polish Academy of Sciences, Warsaw, in October and November 1966.

(1) Cf. Kripke [4].

clearly every Boolean algebra is a pseudo-Boolean algebra with $a \supset b = a' \vee b$ and $-a = a'$. The meet $a \wedge b$ of the elements $a, b$ will often be denoted by $ab$.

**1. Coproducts.** A lattice $L$ is said to be the coproduct of lattices $L^1$ and $L_2$ if there exist homomorphisms $i_1: L_1 \to L$ and $i_2: L_2 \to L$ such that whenever $h_1: L_1 \to L^*$ and $h_2: L_2 \to L^*$ are homomorphisms into a lattice $L^*$ there exists a unique homomorphism $h: L \to L^*$ such that $h_1 = h \circ i_1$ and $h_2 = h \circ i_2$.




The existence and basic properties of the coproduct may be derived from results in general algebra.

THEOREM 1. *Any two lattices $L_1$ and $L_2$ have a coproduct $L$ which is unique up to isomorphism; the coproduct $L$ is generated by the union of the images of the mappings $i_1: L_1 \to L$ and $i_2: L_2 \to L$ and if $L_1$ and $L_2$ are non-degenerate then these mappings are injective.*

Proof. Every equational class of algebras has coproducts. Indeed, using a construction of Sikorski [7], it can be shown that a class of algebras has coproducts if it has free algebras and is closed under the formation of homomorphic images. The uniqueness of the coproduct $L$ can be proved in any class of algebras. So, too, can the fact that $i_1(L_1) \cup i_2(L_2)$ generates $L$. To prove that $i_1$ is injective, let $h_1: L_1 \to L_1$ be the identity and let $h_2: L_2 \to L_1$ be any homomorphism (²). We know that there exists a homomorphism $h: L \to L_1$ such that $h \circ i_1 = h_1$; since $h_1$ is injective, so too is $i_1$. A similar argument shows that $i_2$ is injective, and this completes the proof.

If we agree to restrict our attention to non-degenerate lattices, then in view of Theorem 1 we may always suppose that the lattices $L_1$ and $L_2$ are sublattices of the coproduct $L$.

An explicit construction for the coproduct may be given as follows (cf. Sikorski [7]): if $L_1$ and $L_2$ are identified with suitable lattices of sets then their coproduct $L$ may be identified with the lattice of sets generated by the sets of the form $A_1 \times A_2$ where $A_1 \in L_1$ and $A_2 \in L_2$. It is also of interest to note the following algebraic characterization of the coproduct which was communicated to the author by W. Holsztyński: a lattice $L$ is the coproduct of its sublattices $L_1$ and $L_2$ if and only if it is generated by their union and for any elements $a_1, b_1 \in L_1$ and $a_2, b_2 \in L_2$ such that $a_1 \wedge a_2 \leqslant b_1 \vee b_2$ we have $a_1 \leqslant b_1$ or $a_2 \leqslant b_2$.

(²) There exist homomorphisms $h_2: L_2 \to L_1$ in virtue of the prime ideal theorem.



**2. The lattices $[D]_n$.** Let $D$ be an arbitrary lattice and let $E$ be a finite chain with $n \geqslant 2$ elements. The coproduct of the lattices $D$ and $E$ will be denoted by $[D]_n$.

We suppose that $E$ consists of the elements

(1) $$0 = e_0 < e_1 < \ldots < e_{n-1} = 1 .$$

If a lattice $L$ contains $D$ and $E$ as sublattices, then by a monotone representation (w. r. t. $D$ and $E$) of an element $x \in L$ we mean a representation of the form

(2) $$x = \bigvee_{i=1}^{n-1} a_i e_i \quad (a_1 \geqslant \ldots \geqslant a_{n-1}) ,$$

where $a_1, \ldots, a_{n-1} \in D$. If $x$ and $y$ have monotone representations $x = \bigvee_{i=1}^{n-1} a_i e_i$ and $y = \bigvee_{i=1}^{n-1} b_i e_i$, then it follows by means of the distributive law and (1) that $x \vee y$ and $x \wedge y$ have monotone representations

(3) $$x \vee y = \bigvee_{i=1}^{n-1} (a_i \vee b_i) e_i , \qquad x \wedge y = \bigvee_{i=1}^{n-1} (a_i \wedge b_i) e_i .$$

We now determine the structure of the lattice $[D]_n$.

THEOREM 2. *If a lattice $L$ contains $D$ and $E$ as sublattices then $L$ is isomorphic to $[D]_n$ if and only if every element of $L$ has a unique monotone representation w.r.t. $D$ and $E$.*

Proof. If $S$ is the set of all elements of $[D]_n$ which have monotone representations, then we see from (3) that $S$ is a sublattice of $[D]_n$. Since $S$ includes $D$ and $E$ it follows from Theorem 1 that $S$ coincides with $[D]_n$; hence every element of $[D]_n$ has a monotone representation. By the defining property of $[D]_n$ there exists for each $i (i = 1, \ldots, n-1)$ a unique homomorphism $D_i: [D]_n \to D$ such that $D_i$ reduces to the identity on $D$ and

$$D_i(e_k) = \begin{cases} 1 & i \leqslant k \\ 0 & i > k \end{cases} \quad (k = 0, 1, \ldots, n-1);$$

operating on (2) with $D_i$ we obtain

$$D_i(x) = a_i \quad (i = 1, \ldots, n-1) ,$$

which implies that the monotone representation is unique.

From (3) we see that any two lattices in which every element has a unique monotone representation are isomorphic. Hence if every element of $L$ has a unique monotone representation then it follows that $L$ is isomorphic to $[D]_n$. This completes the proof of the theorem.

As an immediate consequence of Theorem 2 and the relations (3) we obtain the following construction for $[D]_n$:

THEOREM 3. *The lattice $[D]_n$ is isomorphic to the lattice of all formal expressions* (2) *combined according to* (3). *Equivalently, $[D]_n$ is isomorphic to the sublattice of $D^{n-1}$ consisting of all elements $(a_1, \ldots, a_{n-1})$ such that $a_1 \geqslant \ldots \geqslant a_{n-1}$.*

Since every distributive lattice may be represented as a lattice of open sets, it follows from Theorem 2 that every $[D]_n$ may be represented as a lattice of lower semicontinuous $n$-valued functions.

**3. Post algebras and pseudo-Post algebras.** If $D$ is a Boolean algebra then $[D]_n$ is called the Post algebra of order $n$ over $D$. Similarly if $D$ is a pseudo-Boolean algebra then $[D]_n$ is called the pseudo-Post algebra of order $n$ over $D$. Clearly every Post algebra is a pseudo-Post algebra, but not conversely.

The above definition of Post algebras is equivalent to those appearing in the literature, as can be seen by comparing Theorem 2 with results of Traczyk [8]. We note that Chang and Horn [1] define a generalized Post algebra as the lattice of all continuous functions from a Boolean space to a discretely topologized chain; if we consider only chains with $0, 1$ this can be shown to be equivalent to saying that a lattice is a generalized Post algebra iff it is the coproduct of a Boolean algebra and an arbitrary chain.

The operations $D_1, \ldots, D_{n-1}$ and the constants $e_0, e_1, \ldots, e_{n-1}$ are defined in any lattice $[D]_n$. We show now that if $[D]_n$ is a Post algebra or pseudo-Post algebra then the pseudo-Boolean operations $\supset$ and $-$ are also defined. The fact that $-$ is defined in any Post algebra was noted by Epstein [3].

THEOREM 4. *Each pseudo-Post algebra $[D]_n$ is a pseudo-Boolean algebra containing $D$ and $E$ as pseudo-Boolean subalgebras. The following identities hold* $(i = 1, \ldots, n-1)$:

$$D_i(x \supset y) = \bigwedge_{j=1}^{i} \big(D_j(x) \supset D_j(y)\big)\,, \tag{4}$$

$$D_i(-x) = -D_1(x)\,. \tag{5}$$

Proof. Suppose $x, y \in [D]_n$ have monotone representations $x = \bigvee_{i=1}^{n-1} a_i e_i$ and $y = \bigvee_{i=1}^{n-1} b_i e_i$. If we let

$$c_i = \bigwedge_{j=1}^{i} (a_j \supset b_j) \qquad (i = 1, \ldots, n-1)\,,$$



then there exists an element $z$ with the monotone representation $z = \bigvee_{i=1}^{n-1} c_i e_i$. We shall prove that $z = \max\{u\colon\ u \wedge x \leqslant y\}$; i.e., for any $u \in [D]_n$,

$$u \leqslant z \Longleftrightarrow u \wedge x \leqslant y\,.$$

If $u \leqslant z$ then for each $i = 1, \ldots, n-1$ we have

$$D_i(u) \leqslant D_i(z) = \bigwedge_{j=1}^{i} \big(D_j(x) \supset D_j(y)\big) \leqslant D_i(x) \supset D_i(y)\,;$$

from this it follows that

$$D_i(u \wedge x) = D_i(u) \wedge D_i(x) \leqslant D_i(y) \qquad (i = 1, \ldots, n-1)\,,$$

so that $u \wedge x \leqslant y$. Conversely if $u \wedge x \leqslant y$, then for each $j$

$$D_j(u) \wedge D_j(x) \leqslant D_j(y)\,;$$

it follows that for each $j \leqslant i$

$$D_i(u) \leqslant D_j(u) \leqslant D_j(x) \supset D_j(y)\,,$$

and so we have

$$D_i(u) \leqslant D_i(z) \qquad (i = 1, \ldots, n-1)\,,$$

from which we conclude that $u \leqslant z$.

We have proved that $[D]_n$ is pseudo-Boolean. If $x, y \in D$ then $z \in D$ and similarly if $x, y \in E$ then $z \in E$; hence $D$ and $E$ are pseudo-Boolean subalgebras and it is consistent to write $z = x \supset y$. Equation (4) holds by definition, and we obtain (5) on setting $y = 0$ in (4). This completes the proof.

For many purposes Boolean algebras and pseudo-Boolean algebras are best considered as algebras $B = (B, \wedge, \vee, \supset, -)$. Similarly we may consider Post algebras and pseudo-Post algebras as algebras

$$L = (L, \wedge, \vee, \supset, -, D_1, \ldots, D_{n-1}, e_0, e_1, \ldots, e_{n-1})\,.$$

The one-element algebra will also be regarded as a Post algebra, although the underlying lattice is degenerate.

We may ask to what extent the additional operations are uniquely determined. It is clear that if $D$ and $E$ are given then all operations are uniquely determined. A more precise answer is given by the following theorem, part of which was obtained by Dwinger [2], whose argument may be extended to give an alternative proof.

THEOREM 5. *If $L$ is a Post algebra then the order $n$ and the operations of $L$ are uniquely determined. If $L$ is a pseudo-Post algebra over a given pseudo-Boolean algebra $D$ then the order $n$ and the operations of $L$ are uniquely determined.*

Proof. We note first that the operations $\supset$ and $-$ are uniquely determined in any pseudo-Boolean algebra. If $L$ is a Post algebra over the Boolean algebra $D$, then $D$ coincides with the set of elements of the form $-x$; indeed by (5) $-x = -D_1(x) \in D$, while conversely if $z \in D$ then $z$ is of the form $-x$ where $x = -z$. Thus $D$ is uniquely determined, and it suffices to prove the second part of the theorem. To do this we have only to show that the elements of $E$ are uniquely determined. However this is a consequence of the following assertions: (i) $e_0 = 0$; (ii) if $e_{i-1}$ exists and is less than 1, then $e_i$ is the least $x \in L$ which satisfies the condition

$$\text{for all } a \in D\,, \quad \text{if } x \leqslant a \vee e_{i-1} \quad \text{then} \quad a = 1\,. \tag{6}$$

The proof of (ii) is as follows. If $e_i \leqslant a \vee e_{i-1}$ where $a \in D$, then operating on both sides of the inequality with $D_i$ we obtain $a = 1$, so that $x = e_i$ satisfies (6); on the other hand if $x$ satisfies (6) then setting $a = D_i(x)$ and using the fact that $x = \bigvee_{j=1}^{n-1} D_j(x) e_j \leqslant e_{i-1} \vee D_i(x)$ we obtain $D_i(x) = 1$ or $x \geqslant e_i$, so that $x = e_i$, is the least $x$ which satisfies (6). This completes the proof of the theorem.

We observe that the order $n$ and the operations of a pseudo-Post algebra are not uniquely determined in general. For example, the coproduct of chains $D$ and $E$ of lengths $m$ and $n$ respectively may be considered either as the pseudo-Post algebra of order $n$ over $D$ or the pseudo-Post algebra of order $m$ over $E$.

**4. Equational characterization.** We now give simple equational characterizations for the class of Post algebras and the class of pseudo-Post algebras. The class of Post algebras was characterized by means of equations in Traczyk [9]. We note that the class of pseudo-Boolean algebras $(L, \wedge, \vee, \supset, -)$ is equationally definable (cf., for example, Rasiowa and Sikorski [5] p. 124). If $L$ is a pseudo-Boolean algebra we denote by $0, 1$ the uniquely determined zero and unit.

THEOREM 6. *In order that $L = (L, \wedge, \vee, \supset, -, D_1, \ldots, D_{n-1}, e_0, e_1, \ldots \ldots, e_{n-1})$ should be a pseudo-Post algebra of order $n$ it is necessary and sufficient that $L$ be a pseudo-Boolean algebra and that the following identities hold* $(i = 1, \ldots, n-1)$:

$$D_i(x \wedge y) = D_i(x) \wedge D_i(y)\,, \tag{7}$$

$$D_i(x \vee y) = D_i(x) \vee D_i(y)\,, \tag{8}$$

$$D_i\big(D_k(x)\big) = D_k(x) \qquad (k = 1, \ldots, n-1)\,, \tag{9}$$

$$D_i(e_k) = \begin{cases} 1, & i \leqslant k \\ 0, & i > k \end{cases} \qquad (k = 0, 1, \ldots, n-1)\,, \tag{10}$$

$$x = \bigvee_{i=1}^{n-1} D_i(x) e_i\,. \tag{11}$$



*The necessary and sufficient condition that $L$ be a Post algebra is obtained if we add the identity*

$$D_1(x) \vee -D_1(x) = 1\,. \tag{12}$$

Proof. In each case the necessity of the condition is clear; let us prove the sufficiency. Suppose $L$ is non-degenerate. Setting $x = e_k$ in (11) and applying (10) we see that $e_k = \bigvee_{i=1}^{n-1} D_i(e_k) e_i \geqslant e_{k-1}$ $(k = 1, \ldots, n-1)$; also substituting $x = e_0$ resp. $x = 1$ in (11) we obtain $e_0 = 0$ and $e_{n-1} = 1$. Since $0 \neq 1$ it follows from (10) that the $e_i$ are all distinct; thus we have

$$0 = e_0 < e_1 < \ldots < e_{n-1} = 1\,,$$

and so the chain $E$ is a sublattice of $L$.

By (9) the mappings $D_i (i = 1, \ldots, n-1)$ have a common image $D$. Since by (7), (8), (10) these mappings are lattice homomorphisms, $D$ is a sublattice of $L$. We show that $D$ is closed under $\supset$. If $x, y \in D$ then $D_i(x) = x$ and $D_i(y) = y$; applying $D_i$ to the inequality $(x \supset y) \wedge x \leqslant y$ we obtain $D_i(x \supset y) \wedge x \leqslant y$, i.e.

$$D_i(x \supset y) \leqslant x \supset y \qquad (i = 1, \ldots, n-1)\,;$$

applying $D_k$ to this inequality we obtain $D_i(x \supset y) \leqslant D_k(x \supset y)$, and so $D_1(x \supset y) = \ldots = D_{n-1}(x \supset y)$; hence by (11) we obtain $x \supset y = D_i(x \supset y) \in D$. Thus $D$ is a pseudo-Boolean sublattice of $L$, or a Boolean sublattice if (12) holds.

Operating on (11) with $D_i$ we have $D_i(x) \geqslant \bigvee_{k=1}^{n-1} D_k(x) D_i(e_k) \geqslant D_{i+1}(x)$, so that $D_1(x) \geqslant \ldots \geqslant D_{n-1}(x)$. Hence (11) gives a monotone representation for each element of $L$. If $x = \bigvee_{i=1}^{n-1} a_i e_i$ is a monotone representation for $x$, then applying $D_i$ we obtain $D_i(x) = a_i$ $(i = 1, \ldots, n-1)$, and so the monotone representation is unique. By Theorem 2 it follows that $L$ is the coproduct $[D]_n$ of $D$ and $E$. The operations in $[D]_n$ were defined by means of certain properties which are also enjoyed by the corresponding operations in $L$. Since the operations in $[D]_n$ are uniquely determined by $D$ and $E$, it follows that the operations in $L$ coincide with their counterparts in $[D]_n$. Hence $L = (L, \wedge, \vee, \supset, -, D_1, \ldots, D_{n-1}, e_0, e_1, \ldots, e_{n-1})$ is a pseudo-Post algebra (resp. Post algebra), and the proof is complete.

We note that it is possible to reduce the number of primitive operations for pseudo-Post algebras in view of the identities

$$D_i(x) = D_{n-1}(e_i \supset x) \qquad (i = 1, \ldots, n-1)\,.$$

However this would make the equational characterization more complicated.

**5. Representation theory.** The simplest pseudo-Post algebra is the Post algebra $E = [2]_n$. Beginning with this algebra we can construct new pseudo-Post algebras as follows. Let $\Lambda$ be any partially ordered set. Consider the set $E^{(\Lambda)}$ of all families $\{a_\lambda\}_{\lambda\in\Lambda}$ in $E^\Lambda$ which satisfy the following condition for all $\lambda, \mu \in \Lambda$:

$$a_\lambda \leqslant a_\mu \quad \text{whenever} \quad \lambda \leqslant \mu . \tag{13}$$

We define the operations in $E^{(\Lambda)}$ by setting

$$\{a_\lambda\} \wedge \{b_\lambda\} = \{a_\lambda \wedge b_\lambda\},$$
$$\{a_\lambda\} \vee \{b_\lambda\} = \{a_\lambda \vee b_\lambda\},$$
$$\{a_\lambda\} \supset \{b_\lambda\} = \{\inf_{\mu\geqslant\lambda} (a_\mu \supset b_\mu)\},$$
$$-\{a_\lambda\} = \{\inf_{\mu\geqslant\lambda} (-a_\mu)\},$$
$$D_i(\{a_\lambda\}) = \{D_i(a_\lambda)\} \qquad (i = 1, \ldots, n-1),$$
$$e_i = \{e_i\} \qquad (i = 0, 1, \ldots, n-1).$$

By Theorem 6 we see that $E^{(\Lambda)}$ is indeed a pseudo-Post algebra: it is only necessary to verify that $E^{(\Lambda)}$ is a pseudo-Boolean algebra and that the identities (7)–(11) hold. Clearly $E^{(\Lambda)} = [2^{(\Lambda)}]_n$ where $2^{(\Lambda)}$ is the pseudo-Boolean algebra of all elements of $2^\Lambda$ which satisfy (13). If $\Lambda$ is discretely ordered (i.e. $\lambda \leqslant \mu$ only if $\lambda = \mu$) then $E^{(\Lambda)}$ is simply the Post algebra $E^\Lambda$ of all $E$-valued functions on $\Lambda$, with the operations defined point-wise. The following representation theorem was first proved in the case of Post algebras by Wade [10].

THEOREM 7. *Every pseudo-Post algebra can be embedded in a pseudo-Post algebra of the form $E^{(\Lambda)}$. Every Post algebra can be embedded in a Post algebra of the $E^\Lambda$.*

Proof. It is easy to see that if $D$ and $D'$ are pseudo-Boolean algebras, then any pseudo-Boolean homomorphism $\varphi$: $D \to D'$ can be extended in a unique manner to a pseudo-Post homomorphism $h$: $[D]_n \to [D']_n$, given by

$$h(x) = \bigvee_{i=1}^{n-1} \varphi(D_i(x))\, e_i;$$

further it is clear that $h$ is one-one iff $\varphi$ is one-one. Thus to prove the first part of the Theorem it suffices to show that any pseudo-Boolean algebra $D$ can be embedded in a pseudo-Boolean algebra of the form $2^{(\Lambda)}$. Let $\Lambda$ be the set of all prime filters $\lambda$ of $D$, partially ordered by inclusion. If $\varphi_\lambda$ is the characteristic function of the set $\lambda$, define

$$\varphi(x) = \{\varphi_\lambda(x)\}_{\lambda\in\Lambda}.$$

Obviously $\varphi$ is a lattice homomorphism $D \to 2^{(\Lambda)}$; let us show that it preserves $\supset$. This amounts to showing that $x \supset y \in \lambda$ iff for all $\mu \geqslant \lambda$, $x \in \mu$ implies $y \in \mu$. In one direction this is trivial; for the other we note that if $x \supset y \notin \lambda$ then the filter generated by $\lambda \cup \{x\}$ is disjoint from the ideal generated by $y$ and so may be extended to a prime filter $\mu$ not containing $y$. Thus $\varphi$: $D \to 2^{(\Lambda)}$ is a pseudo-Boolean homomorphism. For any two distinct elements $x, y \in D$ there exists a prime filter $\lambda$ which contains exactly one of $x$ and $y$; hence $\varphi$ is one-one. Thus the first part of the theorem is proved. If $D$ is Boolean then $\Lambda$ is discretely ordered, and so $E^{(\Lambda)} = E^\Lambda$; this proves the second part of the theorem.

**6. Many-valued propositional calculi.** An $n$-valued propositional calculus is set up in the following way. We choose a set $V$ of propositional variables and a family $\{\omega_t\}_{t\in T}$ of connectives. In addition we take $E = \{e_0, e_1, \ldots, e_{n-1}\}$ as the set of truth-values and for each connective $\omega_t$ we choose a corresponding truth-function, which we also denote by $\omega_t$. The set of formulas will be denoted by $S$, and particular formulas will be denoted by $\alpha, \beta, \ldots$

We now define classical and intuitionistic validity for formulas of the $n$-valued propositional calculus. This will be done in such a way as to minimize the differences between the two definitions. It will be clear that the definition of classical validity which we give is equivalent to the usual one, and that the definition of intuitionistic validity is equivalent to that of Kripke [4] when $n = 2$.

If $\Lambda$ is any set, then a family $\{h_\lambda\}_{\lambda\in\Lambda}$ of mappings $h_\lambda$: $S \to E$ is said to be a classical valuation if for each $\omega_t$ we have for all $\lambda \in \Lambda$

$$h_\lambda(\omega_t(\alpha, \beta, \ldots)) = \omega_t(h_\lambda(\alpha), h_\lambda(\beta), \ldots).$$

A formula $\alpha \in S$ is classically valid if for each classical valuation $\{h_\lambda\}_{\lambda\in\Lambda}$ we have $h_\lambda(\alpha) = 1$ for all $\lambda \in \Lambda$.

If $\Lambda$ is any partially ordered set, then a monotone (3) family $\{h_\lambda\}_{\lambda\in\Lambda}$ of mappings $h_\lambda$: $S \to E$ is said to be an intuitionistic valuation if for each $\omega_t$ we have for all $\lambda \in \Lambda$

$$h_\lambda(\omega_t(\alpha, \beta, \ldots)) = \inf_{\mu\geqslant\lambda} \omega_t(h_\mu(\alpha), h_\mu(\beta), \ldots).$$

(3) The family $\{h_\lambda\}_{\lambda\in\Lambda}$ is said to be monotone if for each $\alpha \in S$ we have $h_\lambda(\alpha) \leqslant h_\mu(\alpha)$ whenever $\lambda \leqslant \mu$.

A formula $\alpha \in S$ is intuitionistically valid if for each intuitionistic valuation $\{h_\lambda\}_{\lambda \in \Lambda}$ we have $h(\alpha) = 1$ for all $\lambda \in \Lambda$.

We shall limit our attention in what follows to the case where the primitive truth functions are the operations of the Post algebra $E$:

(14) $$\wedge, \vee, \supset, -, D_1, \ldots, D_{n-1}, e_0, e_1, \ldots, e_{n-1}.$$

There is no essential loss of generality in this, since it can be shown that any formula is classically and intuitionistically equivalent to some formula containing only the connectives (14).

If we regard $S$ as an algebra equipped with the operations (14), then the definitions of classical and intuitionistic validity may be formulated algebraically. It is easy to see that the equation

$$h(\alpha) = \{h_\lambda(\alpha)\}_{\lambda \in \Lambda}$$

determines a one-one correspondence between classical (intuitionistic) valuations and homomorphisms $h\colon S \to E^{\Lambda}$ ($h\colon S \to E^{(\Lambda)}$). Hence a formula $\alpha \in S$ is classically (intuitionistically) valid iff for each (partially ordered) set $\Lambda$ we have $h(\alpha) = 1$ for every homomorphism $h\colon S \to E^{\Lambda}$ ($h\colon S \to E^{(\Lambda)}$). An application of Theorem 7 yields the following result:

THEOREM 8. *For a formula $\alpha$ to be classically (intuitionistically) valid it is necessary and sufficient that for each Post algebra (pseudo-Post algebra) $L$ we have $h(\alpha) = 1$ for every homomorphism $h\colon S \to L$.*

We shall now characterize classical and intuitionistic validity by means of axioms. Let (A1)–(A10) be a system of axioms for Heyting's propositional calculus; we add the following schemes [4] ($i = 1, \ldots, n-1$):

(A11) $D_i(\alpha \wedge \beta) \equiv D_i(\alpha) \wedge D_i(\beta)$,

(A12) $D_i(\alpha \vee \beta) \equiv D_i(\alpha) \vee D_i(\beta)$,

(A13) $D_i(\alpha \supset \beta) \equiv \bigwedge_{j=1}^{i} \big(D_j(\alpha) \supset D_j(\beta)\big)$,

(A14) $D_i(-\alpha) \equiv -D_1(\alpha)$,

(A15) $D_i\big(D_j(\alpha)\big) \equiv D_j(\alpha)$ $(j = 1, \ldots, n-1)$,

(A16) $D_i(e_j)$ $(j = i, \ldots, n-1)$,

(A17) $-D_i(e_j)$ $(j = 0, \ldots, i-1)$,

(A18) $\alpha \equiv \bigvee_{i=1}^{n-1} \big(D_i(\alpha) \wedge e_i\big)$,

(A19) $D_{j+1}(\alpha) \supset D_j(\alpha)$ $(0 < j < n-1)$,

(A20) $D_1(\alpha) \vee -D_1(\alpha)$.

We take (A1)–(A19) as axiom schemes for the intuitionistic propositional calculus and (A1)–(A20) for the classical propositional calculus.

[4] The formula $(\alpha \equiv \beta)$ is an abbreviation for the formula $(\alpha \supset \beta) \wedge (\beta \supset \alpha)$.



In either case the rule of inference is *modus ponens*. We now prove the completeness of these axiomatizations.

THEOREM 9. *A formula is intuitionistically valid iff it is derivable from* (A1)–(A19); *a formula is classically valid iff it is derivable from* (A1)–(A20).

Proof. It is easy to see from Theorem 6 that any formula derivable from (A1)–(A19) is intuitionistically valid. Consider the relation $\alpha \sim \beta$ which holds when $(\alpha \equiv \beta)$ is derivable from (A1)–(A19). It can be shown that this relation is a congruence on $S$ and that the quotient algebra $S/\sim$ is a pseudo-Post algebra. By Theorem 8 the natural homomorphism $S \to S/\sim$ carries any intuitionistically valid formula into the unit element of $S/\sim$; hence every intuitionistically valid formula is derivable from (A1)–(A19). This proves the first part of the Theorem; the second part is proved similarly.

We now describe a transformation $\alpha \to \alpha^*$ of the formulas of the $n$-valued calculus into formulas of the two-valued calculus such that $\alpha$ is classically (intuitionistically) valid iff $\alpha^*$ is classically (intuitionistically) valid in the ordinary two-valued sense.

We arrange the propositional variables of the $n$-valued calculus in a sequence $p, q, \ldots$ and similarly we arrange the propositional variables of the two-valued calculus in a sequence

(15) $$p_1, \ldots, p_{n-1}, q_1, \ldots, q_{n-1}, \ldots$$

The set $S_0$ of formulas of the two-valued propositional calculus is the least set which contains each member of the sequence (15) and which contains $\alpha \wedge \beta$, $\alpha \vee \beta$, $\alpha \supset \beta$ and $-\alpha$ whenever it contains $\alpha$ and $\beta$.

The mappings $\delta_i\colon S \to S_0$ $(i = 1, \ldots, n-1)$ are defined inductively as follows:

$$\delta_i(p) = p_1 \wedge \ldots \wedge p_i,$$
$$\delta_i(q) = q_1 \wedge \ldots \wedge q_i,$$
$$\vdots$$
$$\delta_i(\alpha \wedge \beta) = \delta_i(\alpha) \wedge \delta_i(\beta),$$
$$\delta_i(\alpha \vee \beta) = \delta_i(\alpha) \vee \delta_i(\beta),$$
$$\delta_i(\alpha \supset \beta) = \bigwedge_{j=1}^{i} \big(\delta_j(\alpha) \supset \delta_j(\beta)\big),$$
$$\delta_i(-\alpha) = -\delta_1(\alpha),$$
$$\delta_i(D_j\alpha) = \delta_j(\alpha) \quad (j = 1, \ldots, n-1),$$
$$\delta_i(e_j) = \begin{cases} 1 & i \leqslant j \\ 0 & i > j \end{cases} \quad (j = 0, 1, \ldots, n-1).$$

Here 1 and 0 may be taken to be the formulas $(p_1 \supset p_1)$ and $(p_1 \wedge -p_1)$ respectively.

Theorem 10. *A formula $\alpha$ of the $n$-valued propositional calculus is classically (intuitionistically) valid iff the corresponding formula $\delta_{n-1}(\alpha)$ of the two-valued propositional calculus is classically (intuitionistically) valid.*

Proof. It is easy to check that for any Boolean or pseudo-Boolean algebra $B$ there is a one-one correspondence between homomorphisms $h\colon S_0 \to B$ and homomorphisms $f\colon S \to [B]_n$ determined by the relation

$$D_i(f(\alpha)) = h(\delta_i(\alpha)) \qquad (i = 1, \ldots, n-1)\,.$$

Hence the result follows from Theorem 8.

Theorem 10 gives a solution of the decision problem for the classical and intuitionistic $n$-valued propositional calculus. It is also easy to deduce the following analogues of some well-known properties of Heyting's propositional calculus.

Corollary. *If $\alpha$ and $\beta$ are formulas of the $n$-valued propositional calculus then $\alpha \vee \beta$ is intuitionistically valid iff either $\alpha$ or $\beta$ is intuitionistically valid. The formula $-\alpha$ is classically valid iff it is intuitionistically valid.*

Those formulas which contain only the connectives $\wedge, \vee, \supset, -$ are common to all $n$-valued propositional calculi; we shall denote by $J_n$ resp. $K_n$ the set of all such formulas which are intuitionistically resp. classically valid in the $n$-valued propositional calculus. From the fact that every pseudo-Boolean algebra is embedded in a pseudo-Post algebra while conversely every pseudo-Post algebra is itself pseudo-Boolean, it follows that each $J_n$ coincides with the set of formulas provable in Heyting's propositional calculus. In contrast the sets $K_n$ form a strictly decreasing chain $K_2 \supset K_3 \supset \ldots$

We note that one can set up classical and intuitionistic $n$-valued predicate calculi which may also be studied by algebraic methods. However it will be clear from the above that this would merely involve a straightforward generalization of the corresponding two-valued theory.

## References


[1] C. C. Chang and A. Horn, *Prime ideal characterization of generalized Pos algebras*, in *Lattice theory*, Providence (1960), pp. 43–48.

[2] P. Dwinger, *Notes on Post algebras* I, II, Indag. Math. 28 (1966), pp. 462–478.

[3] G. Epstein, *The Lattice theory of Post algebras*, Trans. Amer. Math. Soc. 95 (1960), pp. 300–317.

[4] S. Kripke, *Semantical analysis of intuitionistic logic* I, in *Formal systems and recursive functions*, Amsterdam (1965), pp. 92–130.





[5] H. Rasiowa and R. Sikorski, *The mathematics of metamathematics*, Warszawa (1963).

[6] P. C. Rosenbloom, *Post algebras* I, *Postulates and general theory*, Amer. J. Math. 64 (1942), pp. 167–188.

[7] R. Sikorski, *Products of abstract algebras*, Fund. Math. 39 (1952), pp. 211–228.

[8] T. Traczyk, *Axioms and some properties of Post algebras*, Colloq. Math. 10 (1963), pp. 193–209.

[9] T. Traczyk, *An equational definition of a class of Post algebras*, Bull. Acad. Pol. Sci. 12 (1964), pp. 147–149.

[10] L. I. Wade, *Post algebras and Rings*, Duke Math. J. 12 (1945), pp. 389–395.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