3. $(\delta', \delta)$ satisfies condition (ii). Suppose $|p - q| < f_{\delta'}(a)$. Choose $a' = a$ and $a''$ so that $0 < 2f_{\delta'}(a'') < f_{\delta'}(a) - |p - q|$

4. Finally, $F(a, b) = 0$ for $0 < a, b < 1$. Suppose $a, b, c$ given. Choose $x$ so that $f_{\delta'}(2x) < f_{\delta'}(x) + f_{\delta'}(x)$. Then $p, q, r$ such that $|p - q| < f_{\delta'}(x)$, $|q - r| < f_{\delta'}(x)$, and $|p - r| > f_{\delta'}(2x)$.

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Sequents in many valued logic II *

by

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The notions of validity in classical and intuitionistic logic may be defined semantically by the methods of Tarski [5] and Kripke [2] respectively, if we replace the two truth-values occurring in these definitions by a system of $M$ truth-values, we obtain what may be referred to as classical $M$-valued logic and intuitionistic $M$-valued logic respectively. Gentzen [1] gives sequent calculi $LK$ and $LJ$ for classical and intuitionistic logic. The present work is concerned with the many valued analogues of these calculi. We shall limit our attention here to propositional logic; some remarks about predicate logic will be made at the end of the paper. We show that for each choice of $M$-valued truth-functions there exist corresponding sequent calculi $LKM$ and $LM$ for classical $M$-valued logic and intuitionistic $M$-valued logic respectively.

The relation between these calculi is similar to that between $LK$ and $LJ$.

We note that the calculus $LKM$ differs from the sequent calculus constructed in [3] (§1) in that the notion of sequent is more restricted.

We take $M = \{0, 1, ..., M-1\} (M \geq 2)$ as the set of truth-values and consider a fixed system of $M$-valued truth-functions $f_{n}, M^{n} \rightarrow M$ ($k = 1, ..., w$). We also choose a set $\mathfrak{A}$ of $k$ atomic statements and connectives $P_{k}$ of degree $r_{k}$ ($k = 1, ..., w$), thus determining the set $\mathfrak{A}$ of statements. We denote statements by the letters $a, b, c, ...,$ and finite sets of statements by $\Gamma, \Delta, ...$

A sequent is an expression of the form

$$(1) f_{n}[\Gamma_{1}] ... [\Gamma_{M-1}] [\Delta_{1}] ... [\Delta_{M-1}]$$

where for each $a \in \mathfrak{A}$ the set $\{w : a \in \Gamma_{w}\}$ is the complement of an interval of $M$. Thus if $a \in \Gamma_{w}$ then either $a \in \Gamma_{w}$ for all $w' < w$ or $a \in \Gamma_{w'}$ for all $w' > w$. Sequents will be denoted by the letters $\Pi, \Sigma, ...$. We observe that the notion of sequent as here defined coincides with that used in [3] only in the case $M = 2$.

* This paper is a sequel to [3]. We note that p. 32 line 18 of [3] should read: $a \rightarrow \mathfrak{A} \cup (\forall a \rightarrow \mathfrak{A}) \rightarrow \mathfrak{A}$.
Proof. Any subset $S$ of $M'$ can be expressed as the union of at most $\lfloor (M' + 1)/2 \rfloor$ Cartesian products of intervals of $M$. Indeed, if $M$ is even then $M'$ is the sum of $\frac{1}{2} (M' - 1)$ two-element products of intervals, and if $M$ is odd then $M'$ is the sum of $\frac{1}{2} (M' - 1)$ two-element products of intervals together with a single one-element set; in either case $M'$ is the sum of $\lfloor (M' + 1)/2 \rfloor$ one- or two-element products of intervals; we obtain the desired representation of $S$ by forming the intersections of $S$ with each of these one- or two-elements products of intervals, since each such intersection is obviously a product of intervals. The first part of the lemma now follows by an application of this remark to the sets

$$S = \{(x_1, \ldots, x_n) \in M': f_d(x_1, \ldots, x_n) > m\}$$

and

$$S = \{(x_1, \ldots, x_n) \in M': f_d(x_1, \ldots, x_n) < m\}$$

respectively. The set $S = \{(m_1, \ldots, m_n); m_1 + \ldots + m_n = 0 \mod 2\}$ has $\lfloor (M' + 1)/2 \rfloor$ elements but includes no Cartesian product of intervals with more than one element; from this we deduce that the bound $\lfloor (M' + 1)/2 \rfloor$ is best possible.

Let $f_d$ be a truth-function of degree $r = r_k$ and let $m$ be a truth-value. For any $x_1, \ldots, x_n \in \Gamma$ the sentence

$$\Gamma_k \models \\Gamma \models \text{true}$$

will be denoted by $\mathcal{H}_k(\eta_1, \ldots, \eta_n) \models \eta_i \models \Gamma$ or by $\mathcal{H}_k(\eta_1, \ldots, \eta_n) \models \eta_i$.

We now describe the rules of the weakest calculi $L_kM$ and $L_kM$. Both calculi have the following "weakening" rule:

$$\Gamma \models \Sigma \models \Gamma_k$$

In addition both calculi have the following introduction rules for each $P_k$ and $m$:

$$\begin{align*}
(P_k, m)^- &\vdash \mathcal{H}_k(\eta_1, \ldots, \eta_n) \models \eta_i \models \Gamma \\
P_k &\vdash \mathcal{H}_k(\eta_1, \ldots, \eta_n) \models \eta_i \models \Gamma \\
(P_k, m)^* &\vdash \mathcal{H}_k(\eta_1, \ldots, \eta_n) \models \eta_i \models \Gamma
\end{align*}$$

The only difference between the two calculi lies in the fact that in $L_kM$ the rule $(P_k, m)^*$ may be applied unrestrictedly whereas in $L_kM$ we require that

$$\mathcal{H}(0) \models \mathcal{H}(1) \models \ldots \models \mathcal{H}(M-1)$$

A sequent $\Pi$ is said to be fundamental if there exist a statement $\sigma$ such that $\sigma$ occurs in every place of $\Pi$. In either calculus the provable
order used in this proof is the reflexivity; this property however is essential.

**Theorem 2.** A sequent is provable in $\mathbf{LJ}_M$ if and only if it is intuitionistically valid.

**Proof.** In view of what has been proved already, it suffices to show that every valid sequent is provable.

If $\Omega$ is an unprovable sequent then there exists an unprovable sequent $\Omega'$ such that

$$\Omega \subseteq \Omega'$$

and such that for each connective $F_k$ and each truth-value $m$,

$$\text{if } [F_k a_1, ..., a_n] \subseteq \Omega' \text{ then } H_i(a_1, ..., a_n) \subseteq \Omega'$$

This may be seen as follows. If $[F_k a_1, ..., a_n] \subseteq \Omega$ then, because $\Omega$ is unprovable, there exists $i \in I^-$ such that $\Omega H_i(a_1, ..., a_n) = \Omega'$ is unprovable; now apply the same argument to $\Omega'$ with respect to a different sequent $[F_k a_1, ..., a_n] \subseteq \Omega'$; continuing in this way we obtain a sequent $\Omega, \Omega', \Omega'', ..., $ which must terminate after a finite number of steps in a sequent $\Omega''$ with the desired properties (7) and (8).

If $\Omega$ is an unprovable sequent and $\Sigma = [F_k a_1, ..., a_n] \subseteq \Omega$ then there exists an unprovable sequent $\Delta^\Sigma$ such that

$$\text{if } [a] \subseteq \Omega \text{ then } [a]^\Sigma \subseteq \Delta^\Sigma (a \in \mathbb{S}, i \in IM),$$

and such that

$$\text{if } [a] \subseteq \Omega \text{ then } [a]^\Sigma \subseteq \Delta^\Sigma$$

This is achieved by $\Omega$ and $\Delta^\Sigma$ with respect to a different sequent $[F_k a_1, ..., a_n] \subseteq \Omega'$; continuing in this way we obtain a sequent $\Omega, \Omega', \Omega'', ..., $ which must terminate after a finite number of steps in a sequent $\Omega''$ with the desired properties (9) and (10).

Let $\Pi$ be the sequent whose $i$-th place is $\Omega(0) \cap \cap \Omega(i)$ for each $i \in IM$; since $\Omega$ is unprovable and $\Pi F_k a_1, ..., a_n \subseteq \Omega'$ it follows that $\Pi^\Sigma F_k a_1, ..., a_n \subseteq \Omega'$ is unprovable; but then since $\Pi$ satisfies the restrictions, we see that $\Pi^\Sigma F_k a_1, ..., a_n$ is unprovable for some $i \in I^+$; the sequent $\Delta^\Sigma = \Pi^\Sigma F_k a_1, ..., a_n$ has the desired properties (9) and (10).

Let $\Pi$ be an unprovable sequent. We construct a “tree” $A$ and a mapping $\lambda \rightarrow \Pi_i$ which associates with each node $\lambda$ an unprovable sequent $\Pi_i$. The construction proceeds by levels: at the 0-th level we place a single node $\lambda_0$ with $\Pi_0 = \Pi^*$; if $\lambda$ is at the $k$-th level and $\Sigma = [F_k a_1, ..., a_n] \subseteq \Pi_i$ then we connect $\lambda$ to a node $\mu = (\Sigma, (k+1))$ at the $(k+1)$-th level and set $\Pi_\mu = \Pi^\Sigma$. The set $A$ is thus partially ordered in the obvious way.

By (7) and (9) we see that for all $a \in \mathbb{S}$ and $m \in IM$

$$\text{if } [a] \subseteq \Pi_\mu \text{ then } [a] \subseteq \Pi_m \text{ whenever } \lambda < \mu.$$
No $H_a$ is fundamental and so it is possible for each $a \in \mathbb{M}$ to define $v(a)$ as the least $m$ such that $a \in H_a[m]$. The family $(v(a))_{a \in \mathbb{M}}$ is an intuitionistic valuation in view of (11).

We shall prove that for all $a \in \mathbb{M}$

\[(12) \quad \text{if } a \in H_a[m] \quad \text{then } v(a) \neq m \quad \lambda \in A.\]

This holds for $a \in \mathbb{M}$ by construction. Suppose (12) holds for $a_1, \ldots, a_r$ and consider $a = F_{a_1 \ldots a_r}$. If $a \in H_a[m]$ then either $[a]_{\lambda} \subseteq H_\lambda$ or $[a]_{\lambda} \subseteq H_\lambda$.

In the first case we have by (11)

\[|F_{a_1 \ldots a_r}| \subseteq H_\lambda \quad \text{for all } \mu \geq \lambda.\]

Hence, by (8), for all $\mu \geq \lambda$ we have $H_\mu(a_1, \ldots, a_r) \subseteq H_\mu$ for some $i \in I_\mu$.

Thus by inductive hypothesis we have for each $\mu \geq \lambda$

\[\bigvee_{i \in I_\mu} \left[ v(a_i) \notin R_i^{j} \wedge \ldots \wedge v(a_{i+1}) \notin R_{i+1}^{j} \right].\]

We deduce by (4) that for each $\mu \geq \lambda$

\[f_\mu(v(a_1), \ldots, v(a_r)) > m.\]

Thus by (2) we have

\[v(F_{a_1 \ldots a_r}) > m.\]

In the second case $|F_{a_1 \ldots a_r}| \subseteq H_\lambda$, so by (10) we have for suitable $\mu \geq \lambda$

\[H_{\mu}(a_1, \ldots, a_r) \subseteq H_\mu \quad \text{for some } i \in I_\mu.\]

Thus by inductive hypothesis

\[\bigvee_{i \in I_\mu} \left[ v(a_i) \notin R_i^{j} \wedge \ldots \wedge v(a_{i+1}) \notin R_{i+1}^{j} \right].\]

Hence by (5) we have

\[f_\mu(v(a_1), \ldots, v(a_r)) < m,\]

and so by (2)

\[v(F_{a_1 \ldots a_r}) < m.\]

Thus in either case we have $v(a) \neq m$, and this completes the proof of (12). If $H_a$ were valid then $H_a$ would be valid; hence for some $a \in \mathbb{M}$ and $m \in \mathbb{M}$ we would have

\[v(a) = m \quad \lambda \in A.\]

which contradicts (12). We see therefore that every unprovable sequent is invalid, which was to be shown.

Theorems 1 and 2 solve the problem of constructing sequent calculi for classical and intuitionistic propositional logic. We may consider the same problem for predicate logic. From [3] it follows that for each choice of $M$-valued truth-functions and quantifiers there exists a calculus of sequents for the corresponding classical $M$-valued predicate logic. However it remains open whether a similar result holds for intuitionistic $M$-valued predicate logic. In certain cases the result does hold — e.g. for the quantifiers $\exists X = \sup X$ and $\forall X = \inf X$.

References


