Constructive methods in probabilistic metric spaces

by

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0. Introduction. This paper initiates a development of the theory of probabilistic metric spaces in which the role of the t-norm is ancillary; indeed, the t-norms are considered only insofar as we wish to clarify the relationship between this and previous work in probabilistic metric spaces. Moreover, our principal interest here will not be in t-norms as defined in [1], but with t-norms satisfying a weaker set of conditions.

In section one of this paper we make some basic definitions along with a brief discussion of t-norms. The remaining sections will give constructive solutions of some problems in pseudo-metrically generated spaces, metrization and completion of spaces.

1. Preliminaries. We shall be concerned here with a family

\[ \mathfrak{F} = \{ F_{pq} : p, q \in S \} \]

of one-dimensional probability distribution functions \( F_{pq} \) satisfying the following conditions: for each pair \( p, q \in S \),

1. \( F_{pq} \) is left-continuous,
2. \( F_{pq} = F_{qp} \),
3. \( F_{pq}(0) = 0 \),
4. \( F_{pq} = H \) if, and only if, \( p = q \),

where \( H \) is the function defined by

\[ H(x) = \begin{cases} 0 & \text{for } x \leq 0, \\ 1 & \text{for } x > 0. \end{cases} \]

If the family \( \mathfrak{F} \) satisfies the additional condition

5. \( F_{pq}(a) = 1 \) and \( F_{pq}(y) = 1 \Rightarrow F_{pq}(x+y) = 1 \)

for all \( p, q, r \in S \) and \( x, y > 0 \), then the pair \( (S, \mathfrak{F}) \) is a probabilistic metric space in the sense of Schweizer and Sklar [1].

In his original paper [2], Menger required the members of the family \( \mathfrak{F} \) to satisfy (instead of condition 5) the condition:

6. \( F_{pq}(x+y) \geq T(F_{pq}(x), F_{pq}(y)) \)\( ^{(*)} \)

for all \( p, q \in S \), where \( T \) is a t-norm.
for all \( p, q, r \in S \) and \( x, y > 0 \), for some function
\[
T : [0, 1] \times [0, 1] \to [0, 1]
\]
satisfying:
\[
\begin{align*}
(1) & \quad T(a, b) \leq T(c, d) \text{ for } a \leq c \text{ and } b \leq d, \\
(2) & \quad T(a, b) = T(b, a), \\
(3) & \quad T(1, 1) = 1, \\
(4) & \quad T(a, 1) > 0 \text{ for } a > 0;
\end{align*}
\]
and called a statistical metric a pair \((S, T)\) satisfying (1)-(4) and (5m).

Schweizer and Sklar, in [1], replaced these conditions by the requirements:
\[
\begin{align*}
(1) & \quad T(a, b) \leq T(c, d) \text{ for } a \leq c \text{ and } b \leq d, \\
(2) & \quad T(a, b) = T(b, a), \\
(3) & \quad T(a, 1) = a \text{ and } T(0, 0) = 0, \\
(4) & \quad T(T(a, b), c) = T(a, T(b, c))
\end{align*}
\]
and called a Menger Space a probabilistic metric space for which there exists a t-norm satisfying (5m), a t-norm being a function with properties (1)-(4).

For any probabilistic metric space \((S, \mathcal{D})\), there is a natural function
\[
T_{\mathcal{D}} : [0, 1] \times [0, 1] \to [0, 1]
\]
which has most of the properties of a t-norm. Namely,
\[
T_{\mathcal{D}}(a, b) = \inf\{F_{\mathcal{D}}(x+y) : F_{\mathcal{D}}(x) > a, F_{\mathcal{D}}(y) > b\}.
\]
It is easy to verify that (t1)-(t3) and (5m) are satisfied.

Since (t4) is not necessarily satisfied by \(T_{\mathcal{D}}\), it is not necessarily a t-norm; however, if there is a t-norm \(T\) for \((S, \mathcal{D})\), then \(T \leq T_{\mathcal{D}}\) in the sense of (1) since \(T_{\mathcal{D}}\) is clearly the strongest function having properties (t1)-(t3) and (5m). It is this "quasi t-norm" with which we shall mainly concern.

We shall also have occasion to make use of the family \(\mathcal{D} = \{d_a : 0 \leq a < 1\}\) of functions from \(S \times S\) to \([0, \infty]\) defined by
\[
(\mathcal{D}) \quad d_{a}(p, q) = \inf\{x : F_{\mathcal{D}}(x) > a\}.
\]
In view of the fact that each \(F_{\mathcal{D}}\) is increasing and left-continuous, we have
\[
(\mathcal{D}) \quad d_{a}(p, q) < x \iff F_{\mathcal{D}}(a) > a.
\]
Furthermore, the \(F_{\mathcal{D}}(a)\) can be recovered from the \(d_{a}(p, q)\); namely,
\[
(\mathcal{D}) \quad F_{\mathcal{D}}(a) = \sup\{x : d_{a}(p, q) < x\}.
\]

The family \(\mathcal{D}\) has the following properties: for each \(a\),
\[
\begin{align*}
(\mathcal{D}) & \quad d_{a}(p, q) \geq 0, \\
(\mathcal{D}) & \quad d_{a}(p, p) = 0, \\
(\mathcal{D}) & \quad d_{a}(p, q) = d_{a}(q, p), \quad \text{for all } p, q \in S. \\
\end{align*}
\]
If, moreover, we assume the family \(\mathcal{G}\) to satisfy the condition: for each \(a(0 < a < 1),\)
\[
(\mathcal{G}) \quad F_{\mathcal{G}}(a) > a \quad \text{and} \quad F_{\mathcal{G}}(a) = a \Rightarrow F_{\mathcal{G}}(a+y) > a
\]
for all \(p, q, r \in S\) and \(x, y > 0\), then the \(d_{a}(p, q)\) have the additional property: for each \(a,\)
\[
(\mathcal{G}) \quad d_{a}(p, r) \leq d_{a}(p, q) + d_{a}(q, r)
\]
for all \(p, q, r \in S\), i.e. each \(d_{a}\) is a pseudo-metric for \(S\). Conversely, (d) also implies (\(\mathcal{G}\)) so that we have

**Lemma 1.** \(D\) is a family of pseudo-metrics if, and only if, the family \(G\) satisfies (\(\mathcal{G}\)). For \(0 < a < 1\), \(d_{a}\) is a metric if, and only if each \(F_{\mathcal{G}}\) is continuous at 0.

**Proof.** It only remains to prove the last assertion, for which we need only note that \(d_{a}(p, q) = 0\) if, and only if, \(F_{\mathcal{G}}(a) > a\) for all \(x > 0\).

In subsequent parts of this paper, we shall make use of conditions similar to (\(\mathcal{D}\)). In order to emphasize the geometric and uniform character of these conditions, we make the following definitions for later use:
\[
(\mathcal{D}) \quad U_{\mathcal{D}}(x, a) = \{(y, q) : F_{\mathcal{D}}(x+y) > a\},
\]
and put
\[
\begin{align*}
U_{\mathcal{D}}(x, a) = U_{\mathcal{D}}(x, a) \quad (q) \quad & F_{\mathcal{D}}(a) > a; \\
U_{\mathcal{D}}(x, a) = U_{\mathcal{D}}(x, a) \quad a \quad & F_{\mathcal{D}}(a) = a; \\
U_{\mathcal{D}}(x, a) = U_{\mathcal{D}}(x, a) \quad & a \quad F_{\mathcal{D}}(a) < a;
\end{align*}
\]
Then condition (\(\mathcal{D}\)) becomes
\[
(\mathcal{D}) \quad U_{\mathcal{D}}(x, a) \cap U_{\mathcal{D}}(x, a) \subseteq U_{\mathcal{D}}(x+y, a)
\]
for all \(x, y > 0\) and \(0 < a < 1\).

2. Pseudo-metrically generated spaces. A probabilistic metric space is said to be pseudo-metrically generated if there is a probability space \((\mathcal{D}, S, \mu)\) satisfying:
\[
\begin{align*}
(\mathcal{D}) & \quad \mathcal{D} \text{ is a collection of pseudo-metrics for } S; \\
(\mathcal{D}) & \quad \text{for every real number } x \text{ and every pair } p, q \in S, \text{ the set } \{d \in \mathcal{D} : d(p, q) < x\} \text{ is } \mathcal{D}-\text{Measurable}; \\
(\mathcal{D}) & \quad F_{\mathcal{D}}(a) = \mu\{d \in \mathcal{D} : d(p, q) < a\}.
\end{align*}
\]
The space is metrically generated if the pseudo-metrics are metrics.
In [3], Stevens showed that if \((S, \mathcal{B})\) is a Menger space under the \(t\)-norm \(T = \min\) and if each \(F_{\nu}(p \neq q)\) is continuous, then \((S, \mathcal{B})\) is metrically generated.

The continuity of the \(F_{\nu}\) is not necessary for \((S, \mathcal{B})\) to be metrically generated, i.e.

**Theorem 1.** If \((S, \mathcal{B})\) is a Menger space under the \(t\)-norm \(T = \min\) then \((S, \mathcal{B})\) is pseudo-metrically generated. If, furthermore, the \(F_{\nu}(p \neq q)\) are continuous at 0, then \((S, \mathcal{B})\) is metrically generated.

**Proof.** This follows easily from Lemma 1 and the following

**Lemma 2.** If \((S, \mathcal{B})\) is a probabilistic metric space, \(T_{\mathcal{X}} \geq \min i, f, and only if \(i, (A)\) holds.

**Proof.** If \(F_{\nu}(a) > a\) and \(F_{\nu}(b) > b\), then from (A) it follows that \(F_{\nu}(a + b) > \min(a, b, a, b)\) and hence \(T_{\mathcal{X}}(a, b) \geq \min(a, b).\) On the other hand, if \(F_{\nu}(a) > a\) and \(F_{\nu}(b) > b\), then, for some \(b > a\), \(F_{\nu}(a) > b\) and \(F_{\nu}(b) > b\), so that \(F_{\nu}(a + b) > T_{\mathcal{X}}(b, b) > b > a\).

Since Lemmas 1 and 2 imply that the family \(\mathcal{D} = \{d_{\alpha}: 0 < \alpha < 1\}\), where \(d_{\alpha}\) is defined by (1), is a family of pseudo-metrics if \(T = \min\), the theorem follows if we put

\[
\mu(d_{\alpha}, d_{\alpha}(p, q) = x) = P(a, d_{\alpha}(p, q) < x),
\]

where \(P\) is Lebesgue measure on \((0, 1)\).

3. Metricization. Thorpe, in [4], has shown that \((S, \mathcal{U}_{\mathcal{B}})\) is a generalized topological space in the sense of Appert and Fan; and he showed that if \((S, \mathcal{B})\) is a probabilistic metric space and \(T\) is a function satisfying \((t_{1})\) and \((5m)\) for which

\[
\sup \{T_{\mathcal{X}}(a, a): 0 < a < 1\} = 1,
\]

then the generalized topological space \((S, \mathcal{U}_{\mathcal{B}})\) is metrizable.

We have the following

**Theorem 2.** Let \((S, \mathcal{B})\) be a probabilistic metric space. In order that a function \(T\) satisfying \((t_{1})\), \((5m)\), and \((V)\) exist, it is necessary and sufficient that:

\[
T(a, a'): U(y, a') \subset U(a + y, a)
\]

for all \(a, y > 0\).

**Proof.** Given \(a < 1\), choose \(a' < 1\) so that \(T(a', a') > a\), and suppose \(F_{\nu}(a) > a\) and \(F_{\nu}(a') > a\). Then \(F_{\nu}(a + y) = T(F_{\nu}(a), F_{\nu}(a')) > T(a', a') > a\).

On the other hand, for \(a < 1\), choose \(a' < 1\) according to (B). Then, if \(1 > b > a'\), we have, for \(F_{\nu}(a) > b\) and \(F_{\nu}(y) > b\) that \(F_{\nu}(a + y) > a\). Thus \(T_{\mathcal{X}}(b, b) > a\).

4. Completion. We begin this section with some definitions. Let \((S, \mathcal{B})\) be a probabilistic metric space. A sequence \((p_{n})\) in \(S\) is said to be Cauchy if, for each pair \((x, a)\), there is a positive integer \(N\) such that \((p_{m}, p_{n}) \in U(x, a)\) for all \(m, n \geq N\).

The following theorem is also clear.

**Theorem 3.** If \((S, \mathcal{B})\) is a probabilistic metric space, the family \(\mathcal{U}_{\mathcal{B}}\) is a basis for a separated uniformity for \(S\) if, and only if, for each pair \((x, a)\), there is a pair \((x', a')\) such that

\[
U(x', a') \cdot U(x', a') \subset U(x, a).
\]

Since the uniformity generated by \(\mathcal{U}_{\mathcal{B}}\) has a countable basis, this yields the

**Corollary.** \((S, \mathcal{U}_{\mathcal{B}})\) is metricizable if, and only if, (C) holds.

Now, (C) is formally weaker than (B) so that Thorpe's theorem is a consequence of Theorems 2 and 3. We exhibit an example of a probabilistic metric space satisfying (C) but not (B).

**Example.** Let \(M(t, a)\) be a continuous, real-valued function defined for all \(0 < a < 1, t > 0\) with the following properties: For each \(t > 0, M(t, a)^{2} + c = a^{2} + 1\). For each \(a, M(0, a) = 0\) and \(M(1, a) > 1, M(t, a)\) is linear for \(0 < t < 1\) and strictly decreasing for \(t > 1\) with \(lim M(t, a) = 0\).

Let \(S\) be the reals and put \(d_{\nu}(p, q) = M(|p - q|, a)\), then for each pair \((p, q), d_{\nu}(p, q)\) is a continuous, increasing function of a \((0 < a < 1)\) so that the family \(\mathcal{B}\) defined by (III) is a family of probability distribution functions satisfying conditions (1)-(4). \((S, \mathcal{B})\) is a probabilistic metric space for:

1. \(\mathcal{B}\) also satisfies (5), \(F_{\nu}(a) = 1\) if, and only if, \(M(|p - q|, a) < a\) for all \(0 < a < 1\), by (III). But this is true precisely when \(p = q\), so (5) holds.

2. \(\mathcal{B}\) does not satisfy (B). Given any pair \(0 < a, a' < 1\), we find \(p, q, r \in S\) and \(x, y > 0\) so that (B) is violated. Pick \(p, r\) so that \(|p - r| = 1\) and \(a = \frac{1}{2}\) and \(1 - 2x < M(1, a)\). Choose \(q\) so that both \(M(|p - q|, a') < x\) and \(M(|y - r|, a) < x\). Then we have \(F_{\nu}(2a) > a\) with \(F_{\nu}(a') > a\) and \(F_{\nu}(a') > a'\).

3. \(\mathcal{B}\) satisfies (C). It is sufficient to note that, for each \(0 < a < 1\), we have \(M(u, a) < M(1, a)M(u, a) + M(t, a)\) whenever \(u < 1\). For suppose \(0 < a < 1\) and \(x > 0\) given, then if \(M(|p - q|, a) < x/M(1, a)\) and \(M(|y - r|, a') < x/M(1, a)\), we have \(M(|p - r|, a') < x\).

It is easy to show that \(T_{\mathcal{X}}(a, b) = 0, 0 < a, b < 1, and T_{\mathcal{X}}(a, 1) = a, a > 0 < a < 1\), using the fact that \(F_{\nu}(a) = 1\) for some \(a\) if, and only if, \(p = q\). In other words, \(T_{\mathcal{X}}\) is the smallest \(t\)-norm \(T_{\mathcal{X}}\).
We say that the probabilistic metric spaces \((S, \mathcal{F})\) and \((S', \mathcal{F}')\) are isometric if there is a mapping \(\varphi: S \to S'\), one-one and onto, such that \(F_{m}\mathcal{F} = F_{m}\mathcal{F}'\) for every pair \(m, n \in S\). The mapping \(\varphi\) is called an isometry.

The space \((S, \mathcal{F})\) is complete if every Cauchy sequence converges. The space \((S^*, \mathcal{F}^*)\) is said to be a completion of \((S, \mathcal{F})\) if \((S^*, \mathcal{F}^*)\) is complete and \((S, \mathcal{F})\) is isometric to a dense subspace of \((S^*, \mathcal{F}^*)\). The Menger space \((S^*, \mathcal{F}^*, T^*)\) is a completion of \((S, \mathcal{F}, T)\) if \((S^*, \mathcal{F}^*)\) is a completion of \((S, \mathcal{F})\) and \(T^* = T\).

It is known (see [5]) that if \((S, \mathcal{F}, T)\) is a Menger space with \(T\) a continuous -n-norm, then there is a completion unique to within isometry.

For spaces \((S, \mathcal{F})\) in general, we can prove the following theorem.

**Theorem 4.** The space \((S, \mathcal{F})\) has a completion if

1. For each triple \((x, y, n)\), there exists an \(n'\) such that
   \[U(x, y, n) \cap U(x, y, n') = \emptyset\]
2. Whenever \((p, q) \in \mathcal{F}(x, y, n)\), there is a pair \((x', y')\) such that
   \[U(x', y', n') \times U(x', y', n') \subset U(x, y, n)\]

The first condition is a uniformity condition intermediate to conditions (B) and (O), and the second says the \(U(x, y, n)\) are open in the product topology.

**Proof of theorem.** Consider the set of all Cauchy sequences in \(S\). We define an equivalence relation among such sequences by \((p_n) \sim (q_n)\) if, for each pair \((x, y)\), there is a positive integer \(N\) such that \((p_m, q_n) \in U(x, y, n)\) for \(m > N\). The relation is clearly reflexive and symmetric. Transitivity follows easily from (i).

Let \(S^* = \{\pi, \varphi, y, \ldots\}\) be the collection of all equivalence classes of Cauchy sequences in \(S\), and for each pair \((x, y)\) define

\[
\bar{U}(x, y) = \{(\pi, \varphi) : \text{for each } (p_n) \in \pi \text{ and } (q_n) \in \varphi, \forall N \in \mathbb{N}, (p_m, q_n) \in U(x, y, n)\}.
\]

For each pair \(x, y \in S^*\), define \(F_{m}\mathcal{F}\) by

\[
F_{m}\mathcal{F} = \max\{a : (\pi, \varphi) \in \bar{U}(x, y)\}.
\]

Then the \(F_{m}\mathcal{F}\) are increasing functions, \(0 < F_{m}\mathcal{F} \leq 1\), satisfying conditions (2)-(4) of Section 0. Condition (5) is also satisfied. For suppose \(F_{m}\mathcal{F}(y) = 1\) and \(F_{m}\mathcal{F}(\pi) = 1\). Let \(a < 1\) be fixed and choose \(a'\) according to (i). If \((p_n) \in \pi\) and \((q_n) \in \varphi\) and \((r_n) \in \psi\), choose \(N\) so large that \((p_m, q_n, r_l) \in U(x, y, n)\) and \((q_m, r_n) \in U(x', y', n)\) for \(m, n, l > N\). Then \((p_m, q_n) \in U(x, y, n)\) for \(m, n > N\), i.e., \((x, y) \in U(x, y, n)\) for every \(n' < 1\).
that \((\bar{p}_a, \pi) \in \overline{U}(a, a)\) for \(a > N\). For let \((p_a, p_b, p_c) \in \pi\). Then there is a \(M\) such that \((p_a, p_b) \in U(a, a')\) for \(m > M\) and a \(K\) such that \((p_b, p_c) \in U(a, a')\) for \(k > K\). Then \((p_a, p_B) \in U(a, a)\) for \(m, k > \max(M, N, K)\).

To conclude the proof of the theorem, let \(m_a\) be Cauchy in \((S^*, \overline{F})\) and \((x, a)\) be given. Let \(m_a \geq 0\) and \(a_n \uparrow a\). For each pair \((m_a, a_n)\) there is a \(p_a, p_b, p_c\) such that \(\overline{U}(\pi, \pi) \in \overline{U}(m_a, a_n)\). Choose \(a'\) according to Lemma 3, then there exist a \(N\) such that \((p_a, p_b) \in U(a, a')\) for \(m, n > N\) and a \(M\) such that \((p_a, p_c) \in U(a, a)\) for \(m > M\). Thus \((\overline{p}_a, \overline{p}_b, \overline{p}_c) \in \overline{U}(a, a)\) for \(m, n > \max(M, N)\). Since \((p_a)\) is Cauchy, \((p_a) \in S^*\). Therefore \(\overline{p}_a \to \pi\) in \((S^*, \overline{F})\) and \(m_a \to a\). Q.E.D.

**Theorem 5.** If \((S, F, T)\) is a Measure space with a continuous \(t\)-norm, \(T\), then \((S, F)\) satisfies the hypotheses of Theorem 4.

Proof. If \(T\) is continuous, then \(\sup \{T(a, a) : 0 \leq a \leq 1\} = 1\), and hence \(\psi_T\) is also continuous, given \(\epsilon > 0\) there is a \(\delta < 1\) such that \(T(a, b) > a - \epsilon\), for \(a > \delta\), uniformly in \(a\). Using (50) and (60) we can show

\[ F_{\psi_T}(a) > \sup \{F_{\psi, \psi}(a - 2\alpha') : a \geq \alpha\} \]

Let \(F_{\psi, \psi}(a) \geq a\) be given. There is a \(a'\) such that \(F_{\psi, \psi}(a) > \alpha\) by left-continuity. Choose \(a' > a - \epsilon\) for this \(a'\), for \(a > \delta\), then we have for \(F_{\psi, \psi}(a') > a'\) and \(F_{\psi, \psi}(a') > a'\).

**Lemma 5.** For the family \(\overline{F}\) defined in the proof of Theorem 4,

\[ F_{\overline{F}}(a) = \inf \lim_{a \to a} F_{\psi_T}(a) \]

Remark. In the proof of Sherwood's completion theorem [5], it is shown that if \(T\) is continuous, then \(\lim_{a \to a} F_{\psi_T}\) exists, and is independent of the choice of \((p_a) \in \pi\) and \((p_b) \in \pi\); and hence \(S^*\) for the completion is determined by defining \(F_{\psi_T} = \lim_{a \to a} F_{\psi_T}\). The space \(S^*\) is the same in Sherwood's result and Theorem 4. Hence, under the hypothesis \(T\) is continuous, we have \(F_{\psi_T} = \lim_{a \to a} F_{\psi_T}(a)\) by the above lemma, so that the \(S^*\) of Theorem 4 and \(S^*\) of Sherwood's theorem are identical, and the two completions are the same.

**Theorem 6.** In general, the completion of a space \((S, F)\) is not unique unless it was completed originally.

Proof. Suppose \((S, F)\) is not complete and \((S^*, \overline{F}^*)\) is a completion of \((S, F)\). Then \(S^* \neq \emptyset\). Let \(\overline{F}^*\) be the class of all left-continuous probability distribution functions \(P\) for which \(P(0) = 0\). We make the following definitions:

\[ L(F) = \{p_a, g \in S^* : S^* \times S^* \times S : F_{\overline{F}^*} = F\} \]

\[ L_p(F) = \{g \in S^* : (p, g) \in L(F)\} \]

\[ a_p(F) = \inf \{(g : g(a) = 1) \leq \infty\} \]

Pick \(F \in \overline{A}\) for which \(L(F) \neq \emptyset\), and then \(\overline{F}^* \in \overline{A}\) such that \(F \neq F^*\) but \(a_p(F) = a_p(F^*)\). We now define \(\overline{F}^*\) by replacing \(F_{\overline{F}^*}\) by \(F_{\overline{F}^*} = F\), for each pair \((p, g) \in L(F)\) and leaving \(F_{\overline{F}^*}\) unchanged otherwise. We claim that the pair \((S^*, \overline{F}^*)\) is a completion of \((S, F)\), not isometric to \((S^*, \overline{F}^*)\).

To verify that \((S^*, \overline{F}^*)\) is a completion of \((S, F)\), we first see that the injection \((S, F) \to (S^*, \overline{F}^*)\) is an isometry since neither \(S\) nor \(\overline{F}^*\) were changed by the above. \((S^*, \overline{F}^*)\) is still a probabilistic metric space, for condition (5) still holds by virtue of \(a_p(F) = a_p(F^*)\). Thirdly, \((S, F)\) is dense in \((S^*, \overline{F}^*)\) for \(p \in S\) and \(L_p(F) = \emptyset\), then for any pair \((p, g)\) there is a \(q \in S^*\) for which \(q \in U_p(a, a) = U_p(a, a)\) since no \(F_{\overline{F}^*}\) have been changed. On the other hand, if \(L_p(F) \neq \emptyset\), choose \((q, a)\) such that \(F_{\overline{F}^*} < a\). Then for any \(y < b\) we have \(U_q(a, b) = U_q(a, b)\) and \(U_q(a, b) < \emptyset\). Finally, \((S^*, \overline{F}^*)\) is complete; for \(a = b\) are chosen as above, then \(U_q(a, b) = U_q(a, b)\) for \(0 < y < x\) and \(1 > b > a\). Hence the Cauchy sequences in \((S^*, \overline{F}^*)\) and \((S^*, \overline{F}^*)\) coincide.

Now, suppose there is an isometry \(\psi : (S^*, \overline{F}^*) \to (S^*, \overline{F}^*)\). We may assume \(\psi(S) = S\). If \(p, g \in L(F)\), then \(p \in S\) or \(q \in S\). Assume the first. Consider the pair \((\psi^*(p), \psi^*(q))\). We have \(F_{\psi^*(p)} = F_p\), which implies \(\psi^*(p), \psi^*(q) \in S\). But \(\psi^*\) maps \(S\) onto itself and hence is not one-one. We conclude with an example of a probabilistic metric space which has no continuous \(t\)-norm but satisfies the conditions of Theorem 4.

**Example.** Let \(f_a, 0 \leq a < 1\), be a family of non-negative, strictly convex functions on \([0, 1]\), satisfying \(f_a(0) = 0, f_a(1) = 1, f_a \leq \kappa\) for \(a < b\), and \(\lim f_a(b) = b\).

Define, for \(p, q \in (0, 1)\), \(a_p(q) = a_p(p - q)\); then \(F_{\psi_T}\) according to formula (III).

1. \((S, F)\) is a probabilistic metric space. To prove this, we need only verify (5). But this follows from the fact that \(F_{\psi_T}(a) > a\) if, and only if, \(|p - q| < f_a(a)\) and hence \(F_{\psi_T}(a) = 1\) if, and only if, \(|p - q| < a\).

2. \((S, F)\) satisfies condition (i). If for \((a, y; a)\) are given, choose \(a'\) so that \(F_{\psi_T}(a') < f_{a'}(a + y)\).
3. $(S', S)$ satisfies condition (ii). Suppose $|p - q| < f^{-1}_s(a)$. Choose $a' = a$ and $a''$ so that $0 < f^{-1}_s(a'') < f^{-1}_s(a) - |p - q|$. 

4. Finally, $\mathcal{T}(a, b) = 0$ for $0 < a, b < 1$. Suppose $a, b, c$ given. Choose $x$ so that $f^{-1}_s(2x) < f^{-1}_s(x) + f^{-1}_s(x)$ and then $p, q, r$ such that $|p - q| < f^{-1}_s(x)$, $|q - r| < f^{-1}_s(x)$ and $|p - r| > f^{-1}_s(2x)$.

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Sequents in many valued logic II

by

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The notions of validity in classical and intuitionistic logic may be defined semantically by the methods of Tarski [5] and Kripke [2] respectively. If we replace the two truth-values occurring in these definitions by a system of M truth-values, we obtain what may be referred to as classical M-valued logic and intuitionistic M-valued logic respectively. Gentzen [1] gives sequent calculi LA and LJ for classical and intuitionistic logic. The present work is concerned with the many valued analogues of these calculi. We shall limit our attention here to propositional logic; some remarks about predicate logic will be made at the end of the paper. We show that for each choice of M-valued truth-functions there exist corresponding sequent calculi LA, MA and LJ, MA for classical M-valued logic and intuitionistic M-valued logic respectively.

The relation between these calculi is similar to that between LA and LJ. We note that the calculus MA differs from the sequent calculus constructed in [3] (§1) in that the notion of sequent is more restricted.

We take $M = \{0, 1, ..., M - 1\}$ ($M > 2$) as the set of truth-values and consider a fixed system of M-valued truth-functions $f : \mathcal{M} \to \mathcal{M}$ ($k = 1, ..., w$). We also choose a set $\mathcal{S}$ of atomic statements and connectives $P_k$ of degree $k$ ($k = 1, ..., w$), thus determining the set $\mathcal{S}$ of statements. We denote statements by the letters $\alpha, \beta, \gamma, ..., \phi$, and finite sets of statements by $\Gamma$, $\Delta$, ...

A sequent is an expression of the form

(1) $\Gamma \vdash |\Gamma_1| \ldots |\Gamma_{M-1}| P_{M-1},$

where for each $\alpha \in \mathcal{S}$ the set $\{w : \alpha \in \Gamma_w\}$ is the complement of an interval of M. Thus if $\alpha \in \Gamma_m$ then either $\alpha \in \Gamma_{m'}$ for all $m' < m$ or $\alpha \in \Gamma_{m'}$ for all $m' > m$. Sequents will be denoted by the letters $\Pi, \Sigma, ..., \Omega$. We observe that the notion of sequent as here defined coincides with that used in [3] only in the case $M = 2$.

* This paper is a sequel to [3]. We note that p. 32 line 18 of [3] should read: $\alpha, f^{\alpha}_{\phi} \gamma = (\alpha = \phi \gamma) \circ f^{\phi}_{\phi}$. It is simpler however to make the correction in the way suggested in [4].