Constructive methods in probabilistic metric spaces

by

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0. Introduction. This paper initiates a development of the theory of probabilistic metric spaces in which the role of the t-norm is ancillary; indeed, the t-norms are considered only insofar as we wish to clarify the relationship between this and previous work in probabilistic metric spaces. Moreover, our principal interest here will be not in t-norms as defined in [1], but with t-norms satisfying a weaker set of conditions.

In section one of this paper we make some basic definitions along with a brief discussion of t-norms. The remaining sections will give constructive solutions of some problems in pseudo-metrically generated spaces, metrization and completion of spaces.

1. Preliminaries. We shall be concerned here with a family

$$\mathfrak{F} = \{F_{pq}\colon\ p, q \in S\}$$

of one-dimensional probability distribution functions F_{pq} satisfying the following conditions: for each pair $p, q \in S$,

(1) F_{pq} is left-continuous,

(2) $F_{pq} = F_{qp}$,

(3) $F_{pq}(0) = 0$,

(4) $F_{pq} = H$ if, and only if, $p = q$;

where H is the function defined by

$$H(x) = \begin{cases} 0 & \text{for } x \leqslant 0\,, \\ 1 & \text{for } x > 0\,. \end{cases}$$

If the family $\mathfrak{F}$ satisfies the additional condition

(5) $F_{pq}(x) = 1$ and $F_{qr}(y) = 1 \Rightarrow F_{pr}(x+y) = 1$

for all p, q, $r \in S$ and $x, y > 0$, then the pair $(S, \mathfrak{F})$ is a *probabilistic metric space* in the sense of Schweizer and Sklar [1].

In his original paper [2], Menger required the members of the family $\mathfrak{F}$ to satisfy (instead of condition 5) the condition:

(5m) $F_{pr}(x+y) \geqslant T\big(F_{pq}(x), F_{qr}(y)\big)$

for all $p, q, r \in S$ and $x, y > 0$, for some function

$$T\colon [0,1]\times[0,1]\to[0,1]$$

satisfying:

(t_1) $T(a, b) \leqslant T(c, d)$ for $a \leqslant c$ and $b \leqslant d$,

(t_2) $T(a, b) = T(b, a)$,

(t_3) $T(1, 1) = 1$,

(t_4) $T(a, 1) > 0$ for $a > 0$;

and called a *statistical metric* a pair $(S, \mathfrak{F})$ satisfying (1)–(4) and (5m).

Schweizer and Sklar, in [1], replaced these conditions by the requirements:

(t_1') $T(a, b) \leqslant T(c, d)$ for $a \leqslant c$ and $b \leqslant d$,

(t_2') $T(a, b) = T(b, a)$,

(t_3') $T(a, 1) = a$ and $T(0, 0) = 0$,

(t_4') $T(T(a, b), c) = T(a, T(b, c))$

and called a *Menger Space* a probabilistic metric space for which there exists a t-norm satisfying (5m), a *t-norm* being a function with properties (t_1')–(t_4').

For any probabilistic metric space $(S, \mathfrak{F})$, there is a natural function $T_{\mathfrak{F}}\colon [0,1]\times[0,1]\to[0,1]$ which has most of the properties of a t-norm. Namely,

$$T_{\mathfrak{F}}(a, b) = \inf\{F_{pr}(x+y)\colon F_{pq}(x) \geqslant a, F_{qr}(y) \geqslant b\}.$$

It is easy to verify that (t_1)–(t_3) and (5m) are satisfied.

Since (t_4) is not necessarily satisfied by $T_{\mathfrak{F}}$, it is not necessarily a t-norm; however, if there is a t-norm T for $(S, \mathfrak{F})$, then $T \leqslant T_{\mathfrak{F}}$ in the sense of [1] since $T_{\mathfrak{F}}$ is clearly the strongest function having properties (t_1)–(t_3) and (5m). It is this "quasi t-norm" with which we shall be mainly concerned.

We shall also have occasion to make use of the family $\mathfrak{D} = \{d_a\colon 0 \leqslant a < 1\}$ of functions from $S\times S$ to $[0, \infty)$ defined by

(I) $$d_a(p, q) = \inf\{x\colon F_{pq}(x) > a\}.$$

In view of the fact that each F_{pq} is increasing and left-continuous, we have

(II) $$d_a(p, q) < x \Leftrightarrow F_{pq}(x) > a.$$

Furthermore, the $F_{pq}(x)$ can be recovered from the $d_a(p, q)$; namely,

(III) $$F_{pq}(x) = \sup\{a\colon d_a(p, q) < x\}.$$

The family $\mathfrak{D}$ has the following properties: for each a.

(a) $d_a(p, q) \geqslant 0$,

(b) $d_a(p, p) = 0$.

(c) $d_a(p, q) = d_a(q, p)$.

If, moreover, we assume the family $\mathfrak{F}$ to satisfy the condition: for each a $(0 \leqslant a < 1)$,

(IV) $$F_{pq}(x) > a \quad\text{and}\quad F_{qr}(y) > a \Rightarrow F_{pr}(x+y) > a$$

for all $p, q, r \in S$ and $x, y > 0$, then the $d_a(p, q)$ have the additional property: for each a,

(d) $d_a(p, r) \leqslant d_a(p, q) + d_a(q, r)$

for all $p, q, r \in S$, i.e. each d_a is a pseudo-metric for S. Conversely, (d) also implies (IV) so that we have

LEMMA 1. *D is a family of pseudo-metrics if, and only if, the family $\mathfrak{F}$ satisfies* (IV). *For $0 < a < 1$, d_a is a metric if, and only if each F_{pq} is continuous at* 0.

Proof. It only remains to prove the last assertion, for which we need only note that $d_a(p, q) = 0$ if, and only if, $F_{pq}(x) > a$ for all $x > 0$.

In subsequent parts of this paper, we shall make use of conditions similar to (IV). In order to emphasize the geometric and uniform character of these conditions, we make the following definitions for later use:

$$U(x, a) = \{(p, q)\colon F_{pq}(x) > a\},$$
$$U_p(x, a) = \{q\colon F_{pq}(x) > a\};$$

and put

$$\mathfrak{U} = \{U(x, a)\colon x > 0, 0 \leqslant a < 1\},$$
$$\mathfrak{U}_S = \{U_p(x, a)\colon p \in S, x > 0, 0 \leqslant a < 1\}.$$

Then condition (IV) becomes

(A) $$U(x, a)\cdot U(x, a) \subset U(x+y, a)$$

for all $x, y > 0$ and $0 \leqslant a < 1$.

2. Pseudo-metrically generated spaces. A probabilistic metric space is said to be *pseudometrically generated* if there is a probability space $(\mathfrak{D}, \mathfrak{B}, \mu)$ satisfying:

(1) $\mathfrak{D}$ is a collection of pseudo-metrics for S;

(2) for every real number x and every pair $p, q \in S$, the set $\{d \in \mathfrak{D}\colon d(p, q) < x\}$ is $\mathfrak{B}$-Measurable;

(3) $F_{pq}(x) = \mu\{d \in \mathfrak{D}\colon d(p, q) < x\}$.

The space is *metrically generated* if the pseudo-metrics are metrics.

In [3], Stevens showed that if $(S, \mathfrak{F})$ is a Menger space under the t-norm $T = \min$ and if each $F_{pq}(p \neq q)$ is continuous, then $(S, \mathfrak{F})$ is metrically generated.

The continuity of the F_{pq} is not necessary for $(S, \mathfrak{F})$ to be metrically generated, i.e.

THEOREM 1. *If $(S, \mathfrak{F})$ is a Menger space under the t-norm $T = \min$ then $(S, \mathfrak{F})$ is pseudo-metrically generated. If, furthermore, the $F_{pq}(p \neq q)$ are continuous at* 0, *then $(S, \mathfrak{F})$ is metrically generated.*

Proof. This follows easily from Lemma 1 and the following

LEMMA 2. *If $(S, \mathfrak{F})$ is a probabilistic metric space, $T_{\mathcal{F}} \geqslant \min$ if, and only if,* (A) *holds.*

Proof. If $F_{pq}(x) > a$ and $F_{qr}(y) > b$, then from (A) it follows that $F_{pr}(x+y) > \min(a, b)$ and hence $T_{\mathcal{F}}(a, b) \geqslant \min(a, b)$. On the other hand, if $F_{pq}(x) > a$ and $F_{qr}(y) > a$, then, for some $b > a$, $F_{pq}(x) > b$ and $F_{qr}(y) > b$, so that $F_{pr}(x+y) \geqslant T_{\mathcal{F}}(b, b) \geqslant b > a$.

Since Lemmas 1 and 2 imply that the family $\mathfrak{D} = \{d_a\colon 0 < a < 1\}$, where d_a is defined by (I), is a family of pseudo-metrics if $T = \min$, the theorem follows if we put

$$\mu\{d_a\colon d_a(p, q) < x\} = P\{a\colon d_a(p, q) < x\},$$

where P is Lebesgue measure on $(0, 1)$.

3. Metrization. Thorpe, in [4], has shown that $(S, \mathfrak{U}_S)$ is a generalized topological space in the sense of Appert and Fan; and he showed that if $(S, \mathfrak{F})$ is a probabilistic metric space and T is a function satisfying (t_1) and (5m) for which

$$\text{(V)} \qquad \sup\{T(a, a)\colon 0 \leqslant a < 1\} = 1,$$

then the generalized topological space $(S, \mathfrak{U}_S)$ is metrizable.

We have the following

THEOREM 2. *Let $(S, \mathfrak{F})$ be a probabilistic metric space. In order that a function T satisfying* (t_1), (5m), *and* (V) *exist, it is necessary and sufficient that: for each a, there is an a' such that*

$$\text{(B)} \qquad U(x, a') \cdot U(y, a') \subset U(x+y, a)$$

for all $x, y > 0$.

Proof. Given $a < 1$, choose $a' < 1$ so that $T(a', a') > a$, and suppose $F_{pq}(x) > a$ and $F_{qr}(y) > a'$. Then $F_{pr}(x+y) \geqslant T\big(F_{pq}(x), F_{qr}(y)\big) \geqslant T(a', a') > a$.

On the other hand, for $a < 1$, choose $a' < 1$ according to (B). Then, if $1 > b > a'$, we have, for $F_{pq}(x) \geqslant b$ and $F_{qr}(y) \geqslant b$ that $F_{pr}(x+y) > a$. Thus $T_{\mathcal{F}}(b, b) \geqslant a$.

The following theorem is also clear.

THEOREM 3. *If $(S, \mathfrak{F})$ is a probabilistic metric space, the family $\mathfrak{U}$ is a basis for a separated uniformity for S if, and only if, for each pair (x, a), there is a pair (x', a') such that*

$$\text{(C)} \qquad U(x', a') \cdot U(x', a') \subset U(x, a).$$

Since the uniformity generated by $\mathfrak{U}$ has a countable basis, this yields the

COROLLARY. *$(S, \mathfrak{U}_S)$ is metrizable if, and only if,* (C) *holds.*

Now, (C) is formally weaker than (B) so that Thorpe's theorem is a consequence of Theorems 2 and 3. We exhibit an example of a probabilistic metric space satisfying (C) but not (B).

EXAMPLE. Let $M(t, a)$ be a continuous, real-valued function defined for all $0 \leqslant a < 1$, $t > 0$ with the following properties: For each $t > 0$, $M(t, a) \nearrow +\infty$ as $a \nearrow 1$. For each a, $M(0, a) = 0$ and $M(1, a) = 1$, $M(t, a)$ is linear for $0 \leqslant t \leqslant 1$ and strictly decreasing for $t > 1$ with $\lim\limits_{t\to\infty} M(t, a) = \frac{1}{2}$.

Let S be the reals and put $d_a(p, q) = M(|p-q|, a)$, then for each pair (p, q), $d_a(p, q)$ is a continuous, increasing function of a $(0 \leqslant a < 1)$ so that the family $\mathfrak{F}$ defined by (III) is a family of probability distribution functions satisfying conditions (1)–(4). $(S, \mathfrak{F})$ is a probabilistic metric space, for:

1. *$\mathfrak{F}$ also satisfies* (5). $F_{pq}(x) = 1$ if, and only if, $M(|p-q|, (a) < x$ for all $0 \leqslant a < 1$, by (III). But this is true precisely when $p = q$, so (5) holds.

2. *$\mathfrak{F}$ does not satisfy* (B). Given any pair $0 \leqslant a$, $a' < 1$, we find $p, q, r \in S$ and $x, y > 0$ so that (B) is violated. Pick p, r so that $|p-r| = 1$ and x so that $1 < 2x < M(1, a)$. Choose q so that both $M(|p-q|, a') < x$ and $M(|q-r|, a') < x$. Then we have $F_{pr}(2x) \leqslant a$ with $F_{pq}(x) > a'$ and $F_{qr}(x) > a'$.

3. *$\mathfrak{F}$ satisfies* (C). It is sufficient to note that, for each $0 \leqslant a < 1$, we have $M(u, a) \leqslant 2M(1, a)[M(s, a) + M(t, a)]$ whenever $u \leqslant s+t$. For suppose $0 \leqslant a < 1$ and $x > 0$ given, then if $M(|p-q|, a) < x/4M(1, a)$ and $M(|q-r|, a) < x/4M(1, a)$, we have $M(|p-r|, a) < x$.

It is easy to show that $T_{\mathcal{F}}(a, b) = 0$, for $0 \leqslant a$, $b < 1$, and $T_{\mathcal{F}}(a, 1) = a$, for $0 \leqslant a < 1$, using the fact that $F_{pq}(x) = 1$ for some x if, and only if, $p = q$. In other words, $T_{\mathcal{F}}$ is the smallest t-norm T_∞.

4. Completion. We begin this section with some definitions. Let $(S, \mathcal{F})$ be a probabilistic metric space. A sequence (p_n) in S is said to be *Cauchy* if, for each pair (x, a), there is a positive integer N such that $(p_m, p_n) \in U(x, a)$ for all $m, n > N$.

We say that the probabilistic metric spaces $(S, \mathcal{F})$ and $(S', \mathcal{F}')$ are *isometric* if there is a mapping $\varphi\colon S \to S'$, one-one and onto, such that $F_{pq} = F_{\varphi(p)\varphi(q)}$ for every pair $p, q \in S$. The mapping φ is called an *isometry*.

The space $(S, \mathcal{F})$ is *complete* if every Cauchy sequence converges. The space $(S^*, \mathcal{F}^*)$ is said to be a *completion* of $(S, \mathcal{F})$ if $(S^*, \mathcal{F}^*)$ is complete and $(S, \mathcal{F})$ is isometric to a dense subspace of $(S^*, \mathcal{F}^*)$. The Menger space $(S^*, \mathcal{F}^*, T^*)$ is a *completion* of $(S, \mathcal{F}, T)$ if $(S^*, \mathcal{F}^*)$ is a completion of $(S, \mathcal{F})$ and $T^* = T$.

It is known (see [5]) that if $(S, \mathcal{F}, T)$ is a Menger space with T a continuous t-norm, then there is a completion unique to within isometry.

For spaces $(S, \mathcal{F})$ in general, we can prove the following

THEOREM 4. *The space $(S, \mathcal{F})$ has a completion if*

(i) *for each triple $(x, y;\ a)$, there exists an a' such that*

$$U(x, a') \cdot U(y, a') \subset U(x+y;\ a);$$

(ii) *whenever $(p, q) \in U(x, a)$, there is a pair (x', a') such that*

$$U_p(x', a') \times U_q(x', a') \subset U(x, a)\,.$$

The first condition is a uniformity condition intermediate to conditions (B) and (C), and the second says the $U(x, a)$ are open in the product topology.

Proof of theorem. Consider the set of all Cauchy sequences in S. We define an equivalence relation among such sequences by $(p_n) \sim (q_n)$ if, for each pair (x, a), there is a positive integer N such that $(p_n, q_n) \in U(x, a)$ for $n > N$. The relation is clearly reflexive and symmetric. Transitivity follows easily from (i).

Let $S^* = \{\pi, \varphi, \psi \ldots\}$ be the collection of all equivalence classes of Cauchy sequences in S, and for each pair (x, a) define

$$\widetilde{U}(x, a) = \{(\pi, \varphi)\colon \text{for each } (p_n) \in \pi \text{ and } (q_n) \in \varphi,\ \exists N \in (p_m, q_n) \in U(x, a) \text{ for } m, n > N\}\,.$$

For each pair $\pi, \varphi \in S^*$, define $\widetilde{F}_{\pi\varphi}$ by

$$\widetilde{F}_{\pi\varphi}(x) = \sup\{a\colon (\pi, \varphi) \in \widetilde{U}(x, a)\}\,,$$

then the $\widetilde{F}_{\pi\varphi}$ are increasing functions, $0 \leqslant \widetilde{F}_{\pi\varphi} \leqslant 1$, satisfying conditions (2)-(4) of Section 0. Condition (5) is also satisfied. For suppose $\widetilde{F}_{\pi\varphi}(x) = 1$ and $F_{\varphi\psi}(y) = 1$. Let $a < 1$ be fixed and choose a' according to (i). If $(p_n) \in \pi$, $(q_n) \in \varphi$ and $(r_n) \in \psi$, choose N so large that $(p_m, q_n) \in U(x, a')$ and $(q_n, r_l) \in U(y, a')$ for $m, n, l > N$. Then $(p_m, r_l) \in U(x+y, a)$ for $m, l > N$, i.e. $(\pi, \psi) \in \widetilde{U}(x+y, a)$ for every $a < 1$.

However, the $\widetilde{F}_{\pi\varphi}$ are not necessarily left-continuous, nor is it necessarily true that $\lim\limits_{X\to\infty} \widetilde{F}_{\pi\varphi}(x) = 1$. We therefore define a new family $\mathcal{F}^*$ from the old $\widetilde{\mathcal{F}} = \{\widetilde{F}_{\pi\varphi}\colon \pi, \varphi \in S^*\}$ as follows. Let $F'_{\pi\varphi}$ be $\widetilde{F}_{\pi\varphi}$ changed at the discontinuity points to be left-continuous. If $\lim\limits_{X\to\infty} F'_{\pi\varphi}(x) < 1$,

$$F^*_{\pi\varphi}(x) = \begin{cases} F'_{\pi\varphi}(x)\,, & x \leqslant 1\,,\\ G_{\pi\varphi}(x)\,, & x > 1\,,\end{cases}$$

where $G_{\pi\varphi} = G_{\varphi\pi}$ is increasing, left-continuous, $G_{\pi\varphi}(1) \geqslant F'_{\pi\varphi}(1)$, $G_{\pi\varphi}(x) < 1$ and $\lim\limits_{x\to\infty} G_{\pi\varphi}(x) = 1$. Otherwise, $F^*_{\pi\varphi} = F'_{\pi\varphi}$. Then $(S^*, \mathcal{F}^*)$ is a probabilistic metric space.

Now, for $p \in S$, let $\widetilde{p}$ denote the class of all sequences in S, converging in $(S, \mathcal{F})$ to p, and denote by $\widetilde{S}$ the collection of these equivalence classes.

The mapping $p \to \widetilde{p}$ is an isometry. This will follow if we prove that $(p, q) \in U(x, a)$ if, and only if, $(\widetilde{p}, \widetilde{q}) \in \widetilde{U}(x, a)$. Suppose $(p, q) \in U(x, a)$. Let $(p_n) \in \widetilde{p}$ and $(q_n) \in \widetilde{q}$. Then $p_n \to p$ and $q_n \to q$. Choose (x', a') according to (ii), then there exists N for which $(p_m, p) \in U(x', a')$ and $(q_n, q) \in U(x', a')$ for $m, n > N$. Thus, $(p_m, p_n) \in U(x, a)$ for $m, n > N$.

On the other hand, suppose $(\widetilde{p}, \widetilde{q}) \in \widetilde{U}(x, a)$. Since $(p_n = p) \in \widetilde{p}$ and $(q_n = q) \in \widetilde{q}$, we have $(p, q) \in U(x, a)$. Finally, $F_{pq}(x) = \sup\{a\colon (p, q) \in U(x, a)\} = \sup\{a\colon (\widetilde{p}, \widetilde{q}) \in \widetilde{U}(x, a)\} = \widetilde{F}_{\widetilde{p}\widetilde{q}}(x)$.

Let $U^*(x, a) = \{(\pi, \varphi)\colon F^*_{\pi\varphi}(x) > a\}$ and put

$$\mathfrak{U}^* = \{U^*(x, a)\colon x > 0,\ 0 \leqslant a < 1\}\,, \qquad \widetilde{\mathfrak{U}} = \{\widetilde{U}(x, a)\colon x > 0,\ 0 \leqslant a < 1\}\,.$$

Since $\widetilde{U}(y, a) \subset U^*(x, a) \subset \widetilde{U}(x, a)$ for $0 < y < x < 1$, it follows that $\widetilde{\mathfrak{U}}$ and $\mathfrak{U}^*$ are equivalent bases for a uniform structure (property (i) is inherited by $\widetilde{\mathfrak{U}}$, from $\mathfrak{U}$); and hence to show that $(S^*, \mathcal{F}^*)$ is complete and that $\widetilde{S}$ is dense in $(S^*, \mathcal{F}^*)$, we need only show these to be true for $(S^*, \widetilde{\mathcal{F}})$.

$\widetilde{S}$ is dense in $(S^*, \widetilde{\mathcal{F}})$. For suppose $\pi \in S^*$ and let $(p_n) \in \pi$. Let (x, a) be given and choose a' according to (i) for the triple $(x/2, x/2;\ a)$. There is N such that $(p_m, p_n) \in U(x/2, a')$ for $m, n > N$. Let $(p'_n) \in \widetilde{p}_{N+1}$, then there is M such that $(p'_n, p_{N+1}) \in U(x/2, a')$ for $n > M$. Thus, $(p_m, p'_n) \in U(x, a)$ for $m, n > \max(M, N)$. In other words, $(\pi, \widetilde{p}_{N+1}) \in \widetilde{U}(x, a)$.

$(S^*, \widetilde{\mathcal{F}})$ is complete. To see this, we first state two lemmas.

LEMMA 3. *If property* (i) *holds,*

$$U(x/3, a') \cdot U(x/3, a') \cdot U(x/3, a') \subset U(x, a) \quad \textit{for some } a'.$$

LEMMA 4. *If $(p_n) \in \pi$, then $\widetilde{p}_n \to \pi$ in $(S^*, \widetilde{\mathcal{F}})$.*

Proof. Let (x, a) be given. Pick a' according to Lemma 3. Then there exists a N such that $(p_m, p_n) \in U(x/3, a')$ for $m, n > N$. Claim that $(\tilde{p}_n, \pi) \in \tilde{U}(x, a)$ for $n > N$. For let $(q_m) \in \tilde{p}_n$ and $(p'_k) \in \pi$. Then there is a M such that $(q_m, p_n) \in U(x/3, a')$ for $m > M$ and a K such that $(p_k, p'_k) \in U(x/3, a')$ for $k > K$. Then $(q_m, p'_k) \in U(x, a)$ for $m, k > \max(M, N, K)$.

To conclude the proof of the theorem, let (π_n) be Cauchy in $(S^*, \tilde{\mathcal{F}})$ and (x, a) be given. Let $x_n \searrow 0$ and $a_n \nearrow 1$. For each pair (x_n, a_n) there is a $p_n \in S$ such that $(\tilde{p}_n, \pi_n) \in \tilde{U}(x_n, a_n)$. Choose a' according to Lemma 3, then there exist a N such that $(\pi_m, \pi_n) \in \tilde{U}(x/3, a')$ for $m, n > N$ and a M such that $(\tilde{p}_n, \pi_n) \in \tilde{U}(x_n, a_n) \subset \tilde{U}(x/3, a')$ for $n > M$. Thus $(\tilde{p}_m, \tilde{p}_n) \in \tilde{U}(x, a)$ for $m, n > \max(M, N)$ yields $(p_m, p_n) \in U(x, a)$ for $m, n > \max(M, N)$. Since (p_n) is Cauchy, $(p_n) \in \pi \in S^*$. Therefore $\tilde{p}_n \to \pi$ in $(S^*, \tilde{\mathcal{F}})$ and $\pi_n \to \pi$. Q.E.D.

THEOREM 5. *If $(S, \mathcal{F}, T)$ is a Menger space with a continuous t-norm, T, then $(S, \mathcal{F})$ satisfies the hypotheses of Theorem 4.*

Proof. If T is continuous, then $\sup\{T(a, a)\colon 0 \leqslant a < 1\} = 1$, and hence by Theorem 2 (i) is satisfied.

Since T is uniformly continuous, given $\varepsilon > 0$ there is $\delta < 1$ such that $T(a, b) > a - \varepsilon$, for $b > \delta$, uniformly in a. Using (5m) and (t'_4) we can show

$$F_{p'q'}(x) \geqslant T\big[F_{pq}(x-2x'),\, T\big(F_{pp'}(x'), F_{qq'}, (x')\big)\big]\,.$$

Let $F_{pq}(x) > a$ be given. There is a x' such that $F_{pq}(x-2x') > a$ by left-continuity. Choose ε so that $F_{pq}(x-2x') - \varepsilon > a$ and, for this ε, δ as above. Pick an a' so that $T(a', a') > \delta$, then we have for $F_{pp'}(x') > a'$ and $F_{qq'}(x') > a'$,

$$F_{p'q'}(x) \geqslant T\big(F_{pq}(x-2x'),\, T(a', a')\big) > F_{pq}(x-2x') - \varepsilon > a\,.$$

LEMMA 5. *For the family $\tilde{\mathcal{F}}$ defined in the proof of Theorem 4,*

$$F_{\pi\varphi}(x) = \inf_{\substack{(p_n)\in\pi\\(q_n)\in\varphi}} \liminf_{m,n\to\infty} F_{p_m q_n}(x)\,.$$

Remark. In the proof of Sherwood's completion theorem [5], it is shown that if T is continuous, then $\lim\limits_{n\to\infty} F_{p_n q_n}$ exists, and is independent of the choice of $(p_n) \in \pi$ and $(q_n) \in \varphi$; and hence $\mathcal{F}^*$ for the completion is determined by defining $F^*_{\pi\varphi} = \lim\limits_{n\to\infty} F_{p_n q_n}$. The space S^* is the same in Sherwood's result and Theorem 4. Hence, under the hypothesis T is continuous, we have $\tilde{F}_{\pi\varphi}(x) = \lim\limits_{m,n\to\infty} F_{p_m q_n}(x)$ by the above lemma, so that the $F^*_{\pi\varphi}$ of Theorem 4 and $F^*_{\pi\varphi}$ of Sherwood's theorem are identical, and the two completions are the same.

THEOREM 6. *In general, the completion of a space $(S, \mathcal{F})$ is not unique unless it was complete originally.*

Proof. Suppose $(S, \mathcal{F})$ is not complete and $(S^*, \mathcal{F}^*)$ is a completion of $(S, \mathcal{F})$. Then $S^* - S \neq \emptyset$. Let Δ be the class of all left-continuous probability distribution functions F for which $F(0) = 0$. We make the following definitions:

$$L(F) \;= \{(p, q) \in S^* \times S^* - S \times S\colon F^*_{pq} = F\}\,,$$

$$L_p(F) = \{q \in S^*\colon (p, q) \in L(F)\}\,,$$

$$d_1(F) \;= \inf\{x\colon F(x) = 1\} \leqslant \infty\,.$$

Pick $F \in \Delta$ for which $L(F) \neq \emptyset$, and then $F' \in \Delta$ such that $F' \neq F$ but $d_1(F') = d_1(F)$. We now define $\mathcal{F}'$ by replacing $F^*_{pq} = F$ by $F'_{pq} = F'$, for each pair $(p, q) \in L(F)$ and leaving F^*_{pq} unchanged otherwise. We claim that the pair $(S^*, \mathcal{F}')$ is a completion of $(S, \mathcal{F})$, not isometric to $(S^*, \mathcal{F}^*)$.

To verify that $(S^*, \mathcal{F}')$ is a completion of $(S, \mathcal{F})$, we first see that the injection $(S, \mathcal{F}) \to (S^*, \mathcal{F}')$ is an isometry since neither S nor $\mathcal{F}$ were changed by the above. $(S^*, \mathcal{F}')$ is still a probabilistic metric space, for condition (5) still holds by virtue of $d_1(F) = d_1(F')$. Thirdly, $(S, \mathcal{F})$ is dense in $(S^*, \mathcal{F}')$ for if $p \in S^*$ and $L_p(F) = \emptyset$, then for any pair (x, a) there is a $q \in S$ for which $q \in U^*_p(x, a) = U'_p(x, a)$ since no F^*_{pq} have been changed. On the other hand, if $L_p(F) \neq \emptyset$, choose (x, a) so that $F(x) \leqslant a$ and $F'(x) \leqslant a$. Then for any $y \leqslant x$ and $b \geqslant a$ we have $U^*_p(y, b) = U'_p(y, b)$ and thus $U'_p(y, b) \cap S \neq \emptyset$. Finally, $(S^*, \mathcal{F}')$ is complete; for if (x, a) are chosen as above, then $U^*(y, b) = U'(y, b)$ for $0 < y \leqslant x$ and $1 > b \geqslant a$. Hence the Cauchy sequences in $(S^*, \mathcal{F}^*)$ and $(S^*, \mathcal{F}')$ coincide.

Now, suppose there is an isometry $\varphi\colon (S^*, \mathcal{F}^*) \to (S^*, \mathcal{F}')$. We may assume $\varphi(S) = S$. If $(p, q) \in L(F)$, then $p \notin S$ or $q \notin S$. Assume the first. Consider the pair $\big(\varphi(p), \varphi(q)\big)$. We have $F'_{\varphi(p)\varphi(q)} = F_{pq} = F$, which implies $\varphi(p), \varphi(q) \in S$. But φ maps S onto itself and hence is not one-one.

We conclude with an example of a probabilistic metric space which has no continuous t-norm but satisfies the conditions of Theorem 4.

EXAMPLE. Let f_a, $0 \leqslant a < 1$, be a family of non-negative, strictly convex functions on $[0, 1]$, satisfying $f_a(0) = 0$, $f_a(1) = 1$, $f_a < f_b$ for $a < b$, and $\lim\limits_{a\to 1} f_a(t) = t$.

Define, for $p, q \in (0, 1)$, $d_a(p, q) = f_a(|p-q|)$; then F_{pq} according to formula (III).

1. *$(S, \mathcal{F})$ is a probabilistic metric space.* To prove this, we need only verify (5). But this follows from the fact that $F_{pq}(x) > a$ if, and only if, $|p-q| < f_a^{-1}(x)$ and hence $F_{pq}(x) = 1$ if, and only if, $|p-q| \leqslant x$.

2. *$(S, \mathcal{F})$ satisfies condition* (i). For if $(x, y;\, a)$ are given, choose a' so that $f_{a'}^{-1}(x) + f_{a'}^{-1}(y) < f_a^{-1}(x+y)$.

3. *$(S, \mathcal{F})$ satisfies condition* (ii). Suppose $|p-q| < f_a^{-1}(x)$. Choose $a' = a$ and x' so that $0 < 2f_a^{-1}(x') < f_a^{-1}(x) - |p-q|$.

4. *Finally, $T(a, b) = 0$ for $0 \leq a, b < 1$.* Suppose a, b, c given. Choose x so that $f_c^{-1}(2x) < f_a^{-1}(x) + f_b^{-1}(x)$ and then p, q, r such that $|p-q| < f_a^{-1}(x)$, $|q-r| < f_b^{-1}(x)$ and $|p-r| \geq f_c^{-1}(2x)$.

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Sequents in many valued logic II *

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The notions of validity in classical and intuitionistic logic may be defined semantically by the methods of Tarski [5] and Kripke [2] respectively. If we replace the two truth-values occurring in these definitions by a system of M truth-values, we obtain what may be referred to as classical M-valued logic and intuitionistic M-valued logic respectively. Gentzen [1] gives sequent calculi LK and LJ for classical and intuitionistic logic. The present work is concerned with the many valued analogues of these calculi. We shall limit our attention here to propositional logic; some remarks about predicate logic will be made at the end of the paper. We show that for each choice of M-valued truth-functions there exist corresponding sequent calculi LK_M and LJ_M for classical M-valued logic and intuitionistic M-valued logic respectively. The relation between these calculi is similar to that between LK and LJ. We note that the calculus LK_M differs from the sequent calculus constructed in [3] (§1) in that the notion of sequent is more restricted.

We take $\boldsymbol{M} = \{0, 1, \ldots, M-1\}$ $(M \geq 2)$ as the set of truth-values and consider a fixed system of M-valued truth-functions $f_k\colon \boldsymbol{M}^{r_k} \to \boldsymbol{M}$ $(k = 1, \ldots, u)$. We also choose a set $\mathfrak{A}$ of atomic statements and connectives F_k of degree r_k $(k = 1, \ldots, u)$, thus determining the set $\mathfrak{S}$ of statements. We denote statements by the letters $\alpha, \beta, \gamma, \ldots$, and finite sets of statements by $\Gamma, \Delta, \ldots$

A sequent is an expression of the form

(1) $$\Gamma_0|\Gamma_1| \ldots |\Gamma_{M-2}|\Gamma_{M-1},$$

where for each $\alpha \in \mathfrak{S}$ the set $\{m\colon \alpha \in \Gamma_m\}$ is the complement of an interval of $\boldsymbol{M}$. Thus if $\alpha \in \Gamma_{m'}$ then either $\alpha \in \Gamma_{m'}$ for all $m' < m$ or $\alpha \in \Gamma_{m'}$ for all $m' > m$. Sequents will be denoted by the letters $\Pi, \Sigma, \ldots, \Omega$. We observe that the notion of sequent as here defined coincides with that used in [3] only in the case $M = 2$.

* This paper is a sequel to [3]. We note that p. 32 line 18 of [3] should read: $\alpha, \Gamma^{[m]}\gamma = ((J_m\alpha \supset \gamma) \supset \Gamma^{[m]}\gamma)$. It is simpler however to make the correction in the way suggested in [4].