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Projective potencies and multiplicative extension operators

by

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Introduction. In the first section of this paper we introduce a notion of the *p-th projective potency* $Y^{[p]}$ of a space Y (unless otherwise stated, by a space we shall mean a compact, metric space). Given a point $y \in Y$ we introduce a natural embedding $j_p^{Y,y}\colon Y^{[p]} \to Y^{[p+1]}$.

Let S_1 be a circumference. Let P_n be an n-dimensional real projective space, embedded as an "improper hyperplane" in P_{n+1} and let $r_n\colon P_n \to P_{n+1}$ denote this embedding.

The term "projective potency" is justified by the following

THEOREM 1. *$S_1^{[n]}$ is homeomorphic to P_n; moreover there are homeomorphisms $h_n\colon S_1^{[n]} \xrightarrow{\text{onto}} P_n$ such that*

$$r_n h_n = h_{n+1} j_n \quad \textit{for } n = 1, 2, \ldots,$$

where $j_n = j_n^{S_1,t}$ for any fixed point $t \in S_1$.

For $n \geqslant 2$ we have no satisfactory topological description for the projective potencies of S_n (by S_n we denote the n-dimensional Euclidean unit sphere). We know however that the homotopical type of $S_n^{[p]}$ is not trivial. Precisely, we have

THEOREM 2. *The embedding $j^p\colon S_n \to S_n^{[p]}$ is not homotopically trivial for $n, p = 1, 2 \ldots$, where $j^p = j_{p-1}^{S_n,s} \circ \ldots \circ j_1^{S_n,s}$ for any fixed s of S_n.*

In the second section of the paper Theorem 2 is applied to the problem of the existance of multiplicative extension operators.

Let $C(Y)$ denote the space of all real-valued maps on a space Y with the uniform convergence topology. Let $Y \subset X$, by a *meo* (*multiplicative extension operator*) we mean a map (= continuous transformation) $M\colon C(Y) \to C(X)$ such that

$$M(fg) = Mf \cdot Mg \quad \text{for } f, g \in C(Y),$$

$$(Mf)(y) = f(y) \quad \text{for } y \in Y \text{ and } f \in C(Y).$$

The main result of Section 2 is the following

THEOREM 3. *There is no meo from $C(S_n)$ into $C(K_{n+1})$ for $n = 1, 2, \ldots$, (K_{n+1} denotes the unit Euclidean ball).*

This theorem solves Problem 2 of [1].

It is worth emphasising that, by Corollary 3.2 in [2], there exist multiplicative extension operators from $C_+(S_n)$ to $C_+(K_{n+1})$ (here $C_+(X)$ denotes the cone of non negative functions in $C(X)$).

Notation. R denotes the set of real numbers, I — the unit interval; for $x \in R^n$: $\|x\| = \sqrt{x_1^2 + \ldots + x_n^2}$, we write

$$K_n = \{x \in R^n : \|x\| \leqslant 1\}, \quad S_{n-1} = \{x \in R^n : \|x\| = 1\}.$$

1. Projective potencies. A $y \in Y$ is said to be an *essential coordinate* of a point $x = (x_1, \ldots, x_p) \in Y^p$ if card $\{j : x_j = y\}$ is an odd number(¹). The number of different essential coordinates of a point is called its *range*.

The *p-th projective potency* $Y^{[p]}$ is the quotient space Y^p/E_p^Y where the equivalence relation E_p^Y is defined by the following condition:

$$xE_p^Y y \quad \text{if } x \text{ and } y \text{ have the same essential coordinates (}^2\text{)}.$$

By ψ_p^Y we denote the quotient map from Y^p onto $Y^{[p]}$. Let $y \in Y$, $x \in Y^p$. The formula

$$j_p^{Y,y}(\psi_p^Y(x)) = \psi_{p+1}^Y(x, y), \quad \text{where } (x, y) = (x_1, \ldots, x_p, y),$$

defines an embedding $j_p^{Y,y}: Y^{[p]} \to Y^{[p+1]}$.

By the projective space P_n we mean the quotient space $K_n/\approx$ where

$$x \approx y \quad \text{if } x = y \text{ or } x = -y \in S_{n-1}.$$

The formula $k_n(x) = (x, \sqrt{1 - \|x\|^2})$ defines a natural embedding $k_n: K_n \to S_n$. The embedding $r_n: P_n \to P_{n+1}$ is defined by the formula $r_n(p_n(x)) = p_{n+1}(k_n(x))$, where p_n is the quotient map from K_n onto P_n.

Proof of Theorem 1. First we shall investigate the space $I^{[n]}$. We shall prove the following

1.1. PROPOSITION. *There are homeomorphisms $q_n: I^{[n]} \xrightarrow{\text{onto}} K_n$ such that*

$$(0) \qquad q_{n+1} j_n^{I,0} q_n^{-1} = -q_{n+1} j_n^{I,1} q_n^{-1} = k_n.$$

Proof. Write $\psi_n = \psi_n^I$ and $E_n = E_n^I$. Let $\sigma_n = \{x \in I^n : x_1 \geqslant x_2 \geqslant \ldots \geqslant x_n\}$. It is clear that σ_n is a simplex with the vertices $e_i = \{\underbrace{1, \ldots, 1}_{i\text{-times}}, 0, \ldots, 0\}$.

(¹) e. g. if $x = (yzywz)$ or $x = (zzzwz)$, then only w is an essential coordinate of x.

(²) e. g. $(yzzzyzw)\, E_7^Y\, (zyzywtt)\, E_7^Y\, (mkmkwpp)$ but not $xyyyy\, E_5^Y\, (xxyyy)$.

A point $x = \sum a_i e_i \in \sigma_n$ will be identified with its baricentric coordinates $a = (a_0, a_1, \ldots, a_n)$ where $a_i \geqslant 0$, $\sum_{i=0}^{n} a_i = 1$ (³).

To prove the proposition it is enough to construct maps $f_n: \sigma_n \xrightarrow{\text{onto}} K_n$ such that

$$(1) \qquad f_n(a) = f_n(b) \quad \text{iff } aE_n b,$$

$$(2) \qquad f_n(a, 0) = -f_n(0, a) = k_{n-1}(f_{n-1}(a)) \quad \text{for each } a \in \sigma_{n-1} \text{ (}^4\text{)}$$

(then we put $q_n = f_n \psi_n^{-1}$).

To simplify the argumentation we define a (discontinuous) function $R: \sigma_n \to \sigma_n$ which is a "selection function" for E_n, i.e. R satisfies the following two conditions

$$(3) \qquad Ra = Rb \quad \text{iff } aE_n b,$$

$$(3') \qquad aE_n Ra \text{ (or, equivalently, } R(Ra) = Ra) \quad \text{for each } a \in \sigma_n.$$

Given $x = a \in \sigma_n$, we put $Ra = y = (y_1, \ldots, y_k, 0, \ldots, 0) \in \sigma_n$ where k is the range of x, and $y_1 > y_2 > \ldots > y_k$ are all essential coordinates of x. It is obvious that R is a "selection function".

The following description of R will be more useful

$$R = R_1 \circ R_2 \circ \ldots \circ R_n \text{ where } R_n a = a \text{ and for } i = 1, 2, \ldots, n-1,$$

$$R_i a = \begin{cases} (a_0, \ldots, a_{i-2}, a_{i-1} + a_{i+1}, a_{i+2}, \ldots, a_n, 0, 0) & \text{if } a_i = 0, \\ a & \text{if } a_i \neq 0. \end{cases}$$

Given $a \in \sigma_n$, put $\bar{a} = a_1 + a_3 + a_5 + \ldots$ We see that

$$(4) \qquad \bar{a} = \overline{Ra}, \text{ hence if } aE_n b, \text{ then } \bar{a} = \bar{b}.$$

Let $T_n = \{a \in \sigma_n : \bar{a} = \frac{1}{2}\}$.

1.2. LEMMA. *There is a natural homeomorphism between $\psi_n(T_n)$ and $\psi_{n-1}(\sigma_{n-1}) = I^{[n-1]}$.*

Proof. It is enough to construct a map $g_n: T_n \xrightarrow{\text{onto}} \sigma_{n-1}$ such that

(5) if $aE_n b$, then $(g_n a) E_n (g_n b)$, i.e. there is a map $\gamma: \psi_n(T_n) \xrightarrow{\text{onto}} I^{[n-1]}$ such that the following diagram commutes

$$\begin{array}{ccc} T_n & \xrightarrow{g_n} & \sigma_{n-1} \\ \downarrow{\scriptstyle \psi_n} & & \downarrow{\scriptstyle \psi_{n-1}} \\ \psi_n(T_n) & \xrightarrow{\gamma} & I^{[n-1]} \end{array}$$

(6) if $g_n a = g_n b$, then $Ra = Rb$, i.e. the above map γ is a 1-1 map (and, by the compactness, is a homeomorphism).

(³) we use letters x, y to denote Euclidean coordinates and a, b — baricentric ones.

(⁴) if $a = (a_0, \ldots, a_{n-1})$, then $(a, 0) = (a_0, \ldots, a_{n-1}, 0)$ and $(0, a) = (0, a_0, \ldots, a_{n-1})$.

We see that for $a \in \sigma_1$ we have

$$(f_2(a,0))_1 = -(f_2(0,a))_1 = (f_1 a)_1 ,$$
$$\|f_2(a,0)\| = \|f_2(0,a)\| = 1 ,$$
$$\operatorname{sgn}(f_2(a,0))_2 = -\operatorname{sgn}(f_2(0,a))_2 = 1 .$$

Since for $a \in \sigma_n$ we have

$$\overline{(a,0)} = 1-\overline{(0,a)} = \bar{a} ,$$
$$g_n P(a,0) = (g_{n-1}Pa, 0) ,$$
$$g_n P(0,a) = (0, g_{n-1}Pa) ,$$

an easy induction implies that

$$(f_n(a,0))_i = -(f_n(0,a))_i = (f_{n-1}a)_i \quad \text{for } i = 1, \ldots, n-1 ,$$
$$\|f_n(a,0)\| = \|f_n(0,a)\| = 1 ,$$
$$\operatorname{sgn}(f_n(a,0))_n = -\operatorname{sgn}(f_n(0,a))_n = 1 .$$

Three last equations imply (2). This completes the proof of 1.1.

Our Theorem 1 is an easy consequence of 1.1. Indeed, let $S_1 = I/R_1$ where xR_1y if $x = y$ or $x = 0, y = 1$; it is clear that $S_1^{[n]} = I^{[n]}/R_n$ where xR_ny if $x = y$ or there is a $z \in I^{[n-1]}$ such that $x = j_{n-1}^{I,0}(z)$, $y = j_{n-1}^{I,1}(z)$.

By (0), this completes the proof of Theorem 1. ∎

In the sequel we shall use the following notion of a (cellular) polyhedron: by a *polyhedron* we mean a compact metric space P with its finite disjoint triangulation $\mathcal{C}$. The elements of $\mathcal{C}$ are called *cells*. Each cell Γ satisfies the following conditions.

A. There is a number $k = \dim\Gamma$ and a map φ_Γ: $I^k \to P$ (a characteristic map for Γ) such that φ_Γ maps $\operatorname{Int} I^k$ onto Γ homeomorphically.

B. The set $\dot{\Gamma} = \overline{\Gamma} - \Gamma$ is a union of cells of lower dimensions.

If a cell $\Delta \subset \dot{\Gamma}$ and $\dim\Delta = \dim\Gamma - 1$, then Δ is called a *face* of Γ.

We shall use Z_2 as the coefficient group for homology groups of P. The nth homology group of P will be denoted by $H_n(P)$.

Let Δ be an $(n+1)$-dimensional cell and let Γ be a face of Δ. A point $y \in \Gamma$ will be called *normal for* Δ if for each $x \in \varphi_\Delta^{-1}(y)$ (we have $x \in \dot{I}^{n+1}$) there is a neighbourhood U_x of x in $\dot{I}^{n+1}$ such that the restriction $\varphi_\Delta | U_x$ is a homeomorphism. We make use of the following well known (cf. for instance [4], pp. 56, 57 and 14, 19) fact concerning the boundary operation ∂:

1.3. Proposition. *If for each face Γ of Δ there is a point $y_\Gamma \in \Gamma$, which is normal for Δ, then*

$$\partial\Delta = \Gamma_1 + \ldots + \Gamma_k$$

where Γ_i, $i = 1, \ldots, k$ are all faces of Δ such that $\operatorname{card}\varphi_\Delta^{-1}(y_{\Gamma_i})$ is an odd number.

1.4. Theorem. *The space $S_n^{[p]}$ is a polyhedron for $p, n = 1, 2, \ldots$; the induced homomorphism j^p: $H_n(S_n) \to H_n(S_n^{[p]})$, where $j^p = j_{p-1}^{S_n,S} \circ \ldots \circ j_1^{S_n,S}$ for an arbitrary point $S \in S_n$, is not trivial.*

Proof. Let $t \in I^{pn}$. We write

$$t = (t^1, \ldots, t^p) = (t_1^1, \ldots, t_n^1, t_1^2, \ldots, t_n^2; \ldots, t_1^p, \ldots, t_n^p) ,$$

where $t^j = (t_1^j, \ldots, t_n^j) \in I^n$. Put

$$\pi_t(i,j) = \pi_i(j),\ \chi_t(i,j) = \begin{cases} \operatorname{sgn}(1-t_i^{\pi_i(1)}) & \text{if } j = 0 , \\ \operatorname{sgn}(t_i^{\pi_i(j)} - t_i^{\pi_i(j+1)}) & \text{if } 1 \leqslant j < p , \\ \operatorname{sgn}(t_i^{\pi_i(j)}) & \text{if } j = p , \end{cases}$$

where $\pi_1, \ldots, \pi_n$ are permutations of numbers $1, \ldots, p$ such that

$$(*) \qquad t_i^{\pi_i(1)} \geqslant t_i^{\pi_i(2)} \geqslant \ldots \geqslant t_i^{\pi_i(p)} ,$$

$$(**) \qquad \text{if } t_i^{\pi_i(j)} = t_i^{\pi_i(k)} \text{ and } j < k, \quad \text{then } \pi_i(j) < \pi_i(k) .$$

The pair of functions (π_t, χ_t) is uniquely determined by t; we shall call it a *characteristic pair of* t. The set of all characteristic pairs will be denoted by $\mathcal{K}$. Every pair $(\pi, \chi) \in \mathcal{K}$ will be identified with the set $\{t \in I^{pn}$: $(\pi_t, \chi_t) = (\pi, \chi)\}$; thus $\mathcal{K}$ is a disjoint covering of I^{pn}. The covering $\mathcal{K}$ may be obtained by "cutting" the cube I^{pn} by all hyperplanes

$$t_i^j = t_i^k \quad \text{for } 1 \leqslant i \leqslant n, 1 \leqslant j < k \leqslant p .$$

The set (π, χ) is convex and open in the hyperplane

$$(8) \qquad t_i^{\pi(i,j)} = \begin{cases} 1, & \text{if } j = 1 \text{ and } \chi(i,0) = 0 , \\ t_i^{\pi(i,j+1)} & \text{if } 1 \leqslant j < p \text{ and } \chi(i,j) = 0 , \\ 0 & \text{if } j = p \text{ and } \chi(i,p) = 0 . \end{cases}$$

Observe that (8) is a system of $(p+1)n - \sum\chi(i,j)$ independent equations. Thus we have

$$(9) \qquad \dim(\pi, \chi) = \sum\chi(i,j) - n .$$

The set $\overline{(\pi, \chi)}$ consists of all points, satisfying (8) and the following condition

$$\text{if } j < l, \text{ then } 0 \leqslant t_j^{\pi(i,j)} \leqslant t_i^{\pi(i,l)} \leqslant 1 .$$

Hence the set $\overline{(\pi, \chi)}$ is the union of all sets (π', χ') such that

(10) if $\chi'(i,j) = 1$, then $\chi(i,j) = 1$ and numbers $\pi'(i,1), \ldots, \pi'(i,j-1)$ form a permutation of numbers $\pi(i,1), \ldots, \pi(i,j-1)$.

Let φ be a natural map from I^n onto S_n such that $\varphi(\partial I^n) = s$ and φ maps homeomorphically $\operatorname{Int} I^n$ onto $S_n - \{s\}$. Then the formula

$$f(t^1, \ldots, t^p) = \psi_p^{S_n}(\varphi t^1, \ldots, \varphi t^p)$$

defines a map $f\colon I^{pn} \xrightarrow{\text{onto}} S_n^{[p]}$.

For $u = (u_1, \ldots, u_n) \in I^n$ and $v = (v_1, \ldots, v_n) \in I^n$ we shall write $u < v$ if there is a number q such that $u_1 = v_1, \ldots, u_q = v_q$ and $u_{q+1} < v_{q+1}$. Furthermore, let Z be the set of all points $t \in I^{pn}$ such that

(11) there is an r such that

1° $t^1 > t^2 > \ldots > t^r = t^{r+1} = \ldots = t^p = (0, \ldots, 0)$,

2° $t_j^i \neq 0$ nor 1 for $1 \leqslant i < r$, $1 \leqslant j \leqslant n$.

We see that

(12) the restriction $f_{|Z}$ is a one to one map.

Let us notice, that (11) may be expressed by some conditions concerning only χ_t and π_t. Thus, if $t \in Z$, then $(\pi_t, \chi_t) \subset Z$.

Let $\mathcal{M}$ be the family of all characteristic pairs contained in Z. Let $\mathcal{C} = \{f((\pi, \chi))\colon (\pi, \chi) \in M\}$. We will show that

(13) the family $\mathcal{C}$ satisfies conditions A and B; the restriction $f_{|\overline{(\pi,\chi)}}$ is a characteristic map for a cell $f((\pi, \chi)) \in \mathcal{C}$.

Let $t = (t^1, \ldots, t^p) \in I^{pn}$ and let π be an arbitrary permutation of numbers $1, \ldots, p$. We define

$$a_\pi(t) = \big((at)^1, \ldots, (at)^p\big), \quad b(t) = \big((bt)^1, \ldots, (bt)^p\big), \quad c(t) = \big((ct)^1, \ldots, (ct)^p\big)$$

where

$$(at)^j = t^{\pi(j)},$$

$$(bt)^j = \begin{cases} t^j & \text{if } t_i^j \neq 0 \text{ nor } 1 \text{ for } i \neq 1, \ldots, n, \\ (0, \ldots, 0) & \text{otherwise}, \end{cases}$$

$$(ct)^j = \begin{cases} t^j & \text{if } t^j \text{ is an essential element of } (t^1, \ldots, t^p) \text{ and } t^i \neq t^j \text{ for } i < j, \\ (0, \ldots, 0) & \text{otherwise}. \end{cases}$$

1.5. Lemma. *If $\bar{t} = a_\pi(t)$ or $b(t)$ or $c(t)$, then $f((\pi_t, \chi_t)) = f((\pi_{\bar{t}}, \chi_{\bar{t}}))$ and $\dim(\pi_{\bar{t}}, \chi_{\bar{t}}) \leqslant \dim(\pi_t, \chi_t)$.*

Proof. Put respectively $a(u) = a_\pi(u)$ or $b(u)$ or $c(u)$. Obviously in all three cases $a((\pi_t, \chi_t)) \subset (\pi_{a(t)}, \chi_{a(t)})$.

We will show that the opposite inclusion also holds. Let $u \in (\pi_{a(t)}, \chi_{a(t)})$. In the case $a = a_\pi$ put $w^j = u^{\pi^{-1}(j)}$. Next observe that in the cases $a = b$ and $a = c$ the system

$$\begin{aligned} w_i^j &= u_i^j && \text{if } \bar{t}^j = t^j, \\ w_i^j &= t_i^j && \text{if } t_i^j = 0, 1, \\ w_i^j &= w_i^k && \text{if } t_i^j = t_i^k, \\ w_i^j &< w_i^k && \text{if } t_i^j < t_i^k \end{aligned}$$

is consistent, thus it has a solution w (and $w \in (\pi_t, \chi_t)$). It is clear that $a(w) = u$, hence a maps (π_t, χ_t) onto $(\pi_{\bar{t}}, \chi_{\bar{t}})$.

Since $a(u+v) = a(u) + a(v)$, we infer that a does not enlarge the dimension. Therefore $\dim(\pi_{\bar{t}}, \chi_{\bar{t}}) \leqslant \dim(\pi_t, \chi_t)$.

We have $fa(u) = f(u)$, because in all cases the sequences $\varphi(t^1), \ldots, \varphi(t^p)$ and $\varphi(\bar{t}^1), \ldots, \varphi(\bar{t}^p)$ have the same essential elements. Hence $f((\pi_t, \chi_t)) = f((\pi_{\bar{t}}, \chi_{\bar{t}}))$. ■

Let $t \in (\pi, \chi)$. Choosing appropriately a permutation π, we get $\bar{t} = a_\pi cb(t) \in Z$. Put $(\pi', \chi') = (\pi_{\bar{t}}, \chi_{\bar{t}})$. Then $(\pi', \chi') \in \mathcal{M}$ and, by 1.5, we have

(14) for each (π, χ) in $\mathcal{K}$ there is a $(\pi', \chi') \in \mathcal{M}$ such that

$$f((\pi, \chi)) = f((\pi', \chi')) \quad \text{and} \quad \dim(\pi', \chi') \leqslant \dim(\pi, \chi).$$

If $(\pi', \chi') \in \mathcal{M}$, then, by (12), $\dim(\pi', \chi') = \dim f((\pi', \chi'))$. Therefore, by (14), we get

(15) $$\dim f((\pi, \chi)) \leqslant \dim(\pi, \chi) \quad \text{for each } (\pi, \chi) \in \mathcal{K}.$$

Also by (12) and (14) we have

(16) if (π, χ) and $(\pi', \chi') \in \mathcal{K}$, then either $f((\pi, \chi)) = f((\pi', \chi'))$ or $f((\pi, \chi)) \cap f((\pi', \chi')) = \emptyset$.

Now let $\Gamma = (\pi, \chi) \in \mathcal{M}$, $\dot{\Gamma} = \bigcup \Delta_i$ with $\Delta_i \in \mathcal{K}$. Since $\overline{\Gamma}$ is compact, we have

$$\overline{f(\Gamma)} = f(\overline{\Gamma}) = f(\Gamma \cup \bigcup \Delta_i) = f(\Gamma) \cup \bigcup f(\Delta_i).$$

By (15) and (12)

$$\dim f(\Delta_i) \leqslant \dim \Delta_i < \dim \Gamma = \dim f(\Gamma).$$

Hence, by (16), the sets $f(\Gamma)$ and $f(\Delta_i)$ are disjoint. Thus

(17) $$(f(\Gamma))^{\cdot} = \bigcup f(\Delta_i) = f(\bigcup \Delta_i) = f(\dot{\Gamma}).$$

Hence f maps Γ on $f(\Gamma)$ homeomorphically, and $\mathcal{C}$ satisfies the condition A. Also, by (14) and (17), condition B is satisfied. This completes the proof of (13).

Now let $\tau = S_n - \{s\}$, $\Gamma = j^p(\tau)$, i.e., $\Gamma = f\big((\Sigma, \Omega)\big)$ where

$$\Sigma(i,j) = j\,, \qquad \Omega(i,j) = \begin{cases} 1 & \text{for } j = 0, 1\,, \\ 0 & \text{for } j > 1. \end{cases}$$

Let $(\pi, \chi) \in M$, $\Delta = f\big((\pi, \chi)\big)$ and $\dim\Delta = n+1$. By (11):

$$\chi(i, 0) = 1 \qquad \text{for } i = 1, \ldots, n$$

and there is a k (equal to $q-1$ or p) such that $\chi(i,k) = 1$ for $i = 1, \ldots, n$. Since $\dim(\pi, \chi) = n+1$, by (10) and (11), there is exactly one pair (l, r) with $0 < r < k$ such that $\chi(l, r) = 1$. Let $t \in (\pi, \chi)$, thus $1 > t_i^1 = \ldots = t_i^k > t_i^{k+1} = \ldots = t_i^p = 0$ for $i \neq l$ and

$$1 > t_l^1 = \ldots = t_l^r > t_l^{r+1} = \ldots = t_l^k > t_l^{k+1} = \ldots = t_l^p = 0$$

(it is easy to see that, by (*), (**), and (11), $\pi(i,j) = j$ for $i = 1, \ldots, n$).

If $r > 2$, then $t^1 = t^2$; if $r = 1$ and $k > 2$, then $t^2 = t^3$. Hence, by (11), we have $r = 1$, $k = 2$ and thus (π, χ) is determined by the number l and χ is of the form

$$\chi = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \ldots 0 \\ 1 & 0 & 1 & 0 & 0 \ldots 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot\;\cdot\;\cdot \\ 1 & 0 & 1 & 0 & 0 \ldots 0 \\ 1 & 1 & 1 & 0 & 0 \ldots 0 \\ 1 & 0 & 1 & 0 & 0 \ldots 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot\;\cdot\;\cdot \\ 1 & 0 & 1 & 0 & 0 \ldots 0 \end{bmatrix} l\text{th row}\,.$$

Let $\Xi = f(\pi', \chi')$ be a face of Δ. Then $\dim(\pi', \chi') = n$ and, by (17), (π', χ') is a face of (π, χ).

Hence, by (9) and (10), there is exactly one pair (i,j) such that $\chi(i,j) = 1$ and $\chi'(i,j) = 0$. If $i \neq l$, then for every $t \in (\pi', \chi')$ we have $t_i^k = 0$ or 1 for $1, 2, \ldots, p$. Hence $\varphi(t^1) = \ldots = \varphi(t^p) = s$ and thus $\Xi = f\big((\pi', \chi')\big) = \{\psi_p^{T_n}(s)\}$. Therefore $\dim\Xi = 0 < n$ and Ξ is not a face of Δ. Similarly if $i = l$, $j = 1$, then for $t \in (\pi', \chi')$ we have $t^1 = t^2 > t^3 = \ldots = t^p = (0, \ldots, 0)$, hence also $\Xi = \{\psi_p^{S_n}(s)\}$.

Finally, let $i = l$, $j = 0$ or 2, i.e. $(\pi', \chi') = (\pi_1, \chi_1)$ or $(\pi', \chi') = (\pi_2, \chi_2)$ where

$$(\pi_1, \chi_1) = \{t \in I^{pn}\colon 1 = t_l^1 > t_l^2 > 0,\ 1 > t_i^1 = t_i^2 > 0 \text{ for } i \neq l,\ t^j = 0 \text{ for } j \geqslant 3\}$$

and

$$(\pi_2, \chi_2) = \{t \in I^{pn}\colon 1 > t_l^1 > t_l^2 = 0,\ 1 > t_i^1 = t_i^2 > 0 \text{ for } i \neq l,\ t^j = 0 \text{ for } j \geqslant 3\}\,.$$

We see that

$$\varphi_\Delta(t) = \begin{cases} f(t^2, 0, 0, \ldots, 0) & \text{for } t \in (\pi_1, \chi_1)\,, \\ f(t^1, 0, 0, \ldots, 0) & \text{for } t \in (\pi_2, \chi_2) \end{cases}$$

and obviously the restriction of φ_Δ either to (π_1, χ_1) or to (π_2, χ_2) is a homeomorphism onto Γ. Hence every point $y \in \Gamma$ is regular for Δ and $\operatorname{card}\varphi_\Delta^{-1}(y) = 2$. Therefore, by 1.3, the coefficient of Γ in $\partial\Delta$ is equal to 0. Thus Γ, being obviously a cycle, is not a boundary of a chain. This implies that $j_*^p\colon H_n(S_n) \to H_n(S_n^{[p]})$ is not a trivial homomorphism. ■

Theorem 2 is an immediate corollary of Theorem 1.4.

2. Multiplicative extension operators. A map $m\colon C(Y) \to R$ is said to be a *multiplicative functional* if $m(f \cdot g) = m(f) \cdot m(g)$ for every $f, g \in C(Y)$. If m is a multiplicative functional, then the restriction

$$|m| = m_{|C_+(Y)}$$

is a multiplicative functional on $C_+(Y)$. We shall write $Sm = S|m|$ (cf. 2.1 in [2]). It is easy to see that

(18) if $f(y) = g(y)$ for every $y \in Sm$, then $m(f) = m(g)$.

We have the following (cf. Theorem 2.2 in [2])

2.1. THEOREM. *Let $M\colon C(Y) \to C(X)$ be a function such that for every x in X the functional M_x defined by*

$$M_x(f) = Mf(x) \qquad \text{for } f \in C(Y)\,,$$

is multiplicative and non constant. Then M is a multiplicative operator (i.e. a continuous function from $C(Y)$ to $C(X)$ such that $M(fg) = Mf \cdot Mg$ for $f, g \in C(Y)$).

Proof. The multiplicavity of M is obvious. We will show that M is continuous. Let $f_n \rightrightarrows f$. We have to show that $M(f_n) \rightrightarrows Mf$ or equivalently, that $M_{x_n}(f_n) \to M_x(f)$ whenever $x_n \to x$.

Since $|f_n| \rightrightarrows |f|$, by theorem 2.2 in [2], we have $|M_{x_n}(f_n)| \to |M_x(f)|$. The case $M_x(f) = 0$ is trivial, let us assume that $M_x(f) \neq 0$ and thus $f(y) \neq 0$ for $y \in SM_x$. Since SM_x is compact, there is an open set $U \supset SM_x$ such that $|f(y)| > 0$ for $y \in \overline{U}$. Hence

$$(19) \qquad \frac{f_n}{f} \rightrightarrows 1 \quad \text{on } \overline{U}\,.$$

It follows from the formula (3) in [2] that $\operatorname{dist}(SM_{x_n}, SM_x) \to 0$. Thus we may assume without loss of generality that

$$(20) \qquad SM_{x_n} \subset U \quad \text{for } n = 1, 2, \ldots \quad \text{and } SM_x \subset U\,.$$

Let us define a map $M'\colon C(\bar U)\to C(\{x\}\cup\bigcup\{x_n\})$ by

$$(M'g)(z)=M_z(\bar g)\quad \text{for } z\in\{x\}\cup\bigcup\{x_n\},\ g\in C(\bar U).$$

Here $\bar g\in C(Y)$ is an arbitrary extension of g (the definition makes sens, by (18) and (20)). By (19), we may assume that $\frac{f_n}{f}>0$ and thus $M'\frac{f_n}{f}=|M'|\frac{f_n}{f}$. By 2.2 in [2], M' is continuous and thus, by (19), we have $M'\frac{f_n}{f}\rightrightarrows M'1=1$ (we have $M'1=1$ because M'_x is not constant for any x). Thus, by the multiplicativity of M', we have $M'f_n\rightrightarrows M'f$ on $\{x\}\cup\bigcup x_n$. Hence $M'_{x_n}(f_n)\to M'_x(f)$ and, by (18), $M_{x_n}(f_n)\to M_x(f)$. ■

Let Y be a space. By $\mathcal{K}(Y)$ we shall denote the family of all closed at most countable subsets of Y with the dist metric, i.e.

$$\operatorname{dist}(A,B)=\sup_{x\in A}d(x,B)+\sup_{y\in B}d(y,A)\quad \text{for } A,B\in\mathcal{K}(Y).$$

$\mathcal{K}_p(Y)$ will denote the subspace of $\mathcal{K}(Y)$ consisting of all at most p-point subsets of Y.

Suppose that $M\colon C(Y)\to C(X)$ is a meo. By 3 in [2], it is easy to see that the formula

$$\mathcal{S}_M(x)=SM_x$$

defines a map $\mathcal{S}_M\colon X\to\mathcal{K}(Y)$.

If there exists a number p such that $\mathcal{S}_M(X)\subset\mathcal{K}_p(Y)$, then M is called *p-fold*. We shall prove the following

2.2. THEOREM. *Let Y be a simplicial polyhedron and let $M\colon C(Y)\to C(X)$ be a meo. Then there exists a p-fold meo from $C(Y)$ into $C(X)$ for some integer p.*

The proof of Theorem 2.2 will require some notation and lemmas. Let $Y\subset X$ be arbitrary spaces, let $M\colon C(Y)\to C(X)$ be a meo and let φ be a map from $\mathcal{S}_M(X)\times Y$ into Y. We shall denote $\varphi_x(y)=\varphi(\mathcal{S}_M(x),y)$.

2.3. LEMMA. *If $\varphi_y(y)=y$ for $y\in Y$, then the formula*

$$N_x(f)=M_x(f\circ\varphi_x)$$

defines a meo $N\colon C(Y)\to C(X)$. If additionally card $\varphi_x(\mathcal{S}_M(x))\leqslant p$ *for every $x\in X$, then N is p-fold.*

Proof: The second part of the lemma is obvious. We shall prove the first one. We see that $N_y(f)=f(y)$ for $y\in Y$, and thus, by Theorem 2.1, it is sufficient to show that if $f\in C(Y)$, then $Nf\in C(X)$.

Let $x_n\to x$. Let us notice that

$$f\circ\varphi_{x_n}\rightrightarrows f\circ\varphi_x. \tag{21}$$

Indeed, let $y_n\to y$ in Y. We have $(f\circ\varphi_{x_n})(y_n)=f[\varphi(\mathcal{S}_M(x_n),y_n)]$. Since f, φ and $\mathcal{S}_M$ are continuous, $(f\circ\varphi_{x_n})(y_n)\to f[\varphi(\mathcal{S}_M(x),y)]=(f\circ\varphi_x)(y)$. This implies (20).

It follows from (21) and the continuity of M that if $n\to\infty$, then $N_{x_n}(f)=M_{x_n}(f\circ\varphi_{x_n})\to M_x(f\circ\varphi_x)=N_x(f)$. ■

By a *cubic polyhedron* we shall mean any union of faces of the n-cube I^n.

2.4. LEMMA. *Every simplicial polyhedron is homeomorphic to a cubic polyhedron.*

Proof. Every polyhedron is a subpolyhedron of a simplex with its natural triangulation. Thus it is sufficient to prove that for every n there is a homeomorphism h of an n-dimensional simplex σ onto I^n such that for every wall τ of σ, $h(\tau)$ is a union of faces of I^n.

We may assume that $\sigma=\{(t_1,\dots,t_n)\in I^n\colon \sum_{i=1}^n t_i\leqslant 1\}$. Let $t=(t_1,\dots,t_n)\in\sigma$, put

$$h(t)=\Big(\sum_{i=1}^n t_i\Big)\cdot\big(\max_{1\leqslant i\leqslant n}t_i\big)^{-1}\quad\text{if } t\neq 0 \text{ and } h(0)=0.$$

It is a routine matter to check that h is the desired homeomorphism. ■

2.5. LEMMA. *If $\mathcal{A}$ is a compact subspace of $\mathcal{K}(I)$, then there exists $a=a_{\mathcal{A}}>0$ such that for each $A\in\mathcal{A}$, the complement $I-A$ contains an interval with length greater than a.*

Proof. Put for $A\in\mathcal{K}(I)$:

$$g(A)=\sup\{b\colon \text{there is an interval } L\subset I-A \text{ such that } |L|\geqslant b\}$$

(here $|L|$ denotes the length of an interval L).

It is easy to see that g is a positive continuous function on $\mathcal{K}(I)$ because $I\notin\mathcal{K}(I)$. Since $\mathcal{A}$ is compact, there is a positive number a such that $g(A)\geqslant a$ for $A\in\mathcal{A}$. ■

Proof of Theorem 2.2. By 2.4, we may assume that Y is a cubic polyhedron in I^n. We define for $x=(x_1,\dots,x_n)\in I^n$: $\pi_i(x)=x_i$ and for $A\in\mathcal{K}(I^n)$: $\pi_i(A)=\{\pi_i(x)\colon x\in A\}$. Since π_i are continuous, the set $\mathcal{A}=\bigcup_i\pi_i(\mathcal{S}_M(x))$ is a compact subspace of $\mathcal{K}(I)$. Denote $b=a_{\mathcal{A}}/2$.

For $A\in\mathcal{A}$ we shall define a function $f_A\colon I\to I$.

Let $A^1,A^2,\dots$ be different components of the set $\operatorname{conv}A-A$, ordered in such a way that:

$$|A^1|\geqslant|A^2|\geqslant\dots\quad\text{and}\quad\bigcup_{i\geqslant1}A^i=\operatorname{conv}A-A.$$

Let $A^0=I-\operatorname{conv}A$. We define

$$g(t)=\begin{cases}t & \text{for } t\geqslant 0\\ 0 & \text{for } t\leqslant 0.\end{cases}$$

Put

$$f_A(t) = \Big[\sum\Big(\frac{|0, t\cap A^i|}{|A^i|}\, g(|A^i|-b)\Big)\Big]\cdot\Big[\sum g(|A^i|-b)\Big]^{-1}$$

(this formula defines a function, because $g(|A^i|-b) = 0$ for almost all i and there is such i that $|A^i| \geqslant a_{\mathcal{A}} > b$).

We shall prove that the function $f\colon \mathcal{A}\times I\to I$, defined by

$$f(A, t) = f_A(t) \quad \text{for } A\in\mathcal{A}$$

is continuous.

Observe that $|A^i| < b$ for $i > b^{-1}$. Let $A_n\in A$ and $t\in I$, let $\operatorname{dist}(A_n, A)\to 0$ and $t_m\to t$. Let $A^i = (b^i, c^i)$. Denote for $1\leqslant i\leqslant b^{-1}$:

$$b_n^i = \sup\Big\{v\in A_n\colon v < \frac{b^i+c^i}{2}\Big\};\qquad c_n^i = \inf\Big\{\tau\in A_n\colon \frac{b^i+c^i}{2} < \tau\Big\}$$

and $B_n^0 = A_n^0$; $B_n^i = (b_n^i, c_n^i)$.

It is easy to see that

(22) $$\lim_{n=\infty} b_n^i = b^i,\qquad \lim_{n=\infty} c_n^i = c^i \quad \text{for}\quad 0\leqslant i\leqslant b^{-1},$$

(22′) for each $0\leqslant i\leqslant b^{-1}$ and $n = 1, 2\ldots$ there is an index j such that $B_n^i = A_n^j$; for almost all n, all sets A_n^j with $|A_n^j| > b$ appear among B_n^i for $i\leqslant b^{-1}$.

Thus, by (22′), we have for $t\in I$

$$f_{A_n}(t) = \Big[\sum_{i\leqslant b^{-1}}\frac{|(0, t)\cap B_n^i|}{|B_n^i|}\, g(|B_n^i|-b)\Big]\cdot\Big[\sum_{i\leqslant b^{-1}} g(|B_n^i|-b)\Big]^{-1}$$

for almost all n.

By (22), $\lim\limits_{n=\infty}|B_n^i| = |A^i|$ and, since $t_m\to t$, $\lim\limits_{m,n=\infty}|(0, t_m)\cap B_n^i| = |(0, t)\cap A^i|$. Thus $\lim\limits_{m,n=\infty} f_{A_n}(t_m) = f_A(t)$, hence f is continuous.

Observe that f has the following properties:

(23) $f(A, 0) = 0,\ f(A, 1) = 1$ for every $A\in\mathcal{A}$,

(24) $f(\{t\}, u) = u$ for $t, u\in I$,

(25) $\operatorname{card} f_A(A)\leqslant b^{-1}$ for every $A\in\mathcal{A}$.

Now for $B\in S_M(X)$ and $y = (y_1, \ldots, y_n)\in I^n$ put:

$$\varphi^i = \varphi^i(B, y) = f_{\pi_i(B)}(y_i)\ \text{ and }\ \varphi(B, y) = (\varphi^1, \ldots, \varphi^n).$$

By (23), if $y\in\sigma$ where σ is a face of I^n, then $\varphi(B, y)\in\sigma$ for every $B\in S_M(X)$. Thus φ is a map from $S_M(X)\times Y$ into Y.

By (24), $\varphi(\{y\}, y) = y$ for $y\in Y$.

Finally, by (25), $\operatorname{card}\varphi_B(B)\leqslant p = \operatorname{entier}(b^{-1})^n$. This, by Lemma 2.3, completes the proof of Theorem 2.2. ■

2.6. Proposition. *If there exists a p-fold meo $M\colon C(Y)\to C(X)$, then there exists a map $r\colon X\to Y^{[2p+1]}$ such that*

$$r(y) = j^{2p+1}(y)\quad \textit{for } y\in Y$$

where $j^{2p+1} = j_{2p}^{Y,y_0}\circ j_{2p-1}^{Y,y_0}\circ\ldots\circ j_1^{Y,y_0}$ and y_0 is any fixed point of Y.

Proof. For $x\in X$, M_x is a multiplicative functional on $C(Y)$. Hence, by a theorem of Turowicz [3] on the representation of multiplicative functionals, there is a sequence $\{\beta_x(y)\}_{y\in SM_x}$, where $\beta_x(y) = 1$ or 2 such that

$$M_x(f) = M_x(|f|)\cdot\prod_{y\in SM_x}\operatorname{sgn} f(y)^{\beta_x(y)}\quad \text{for } f\in C(Y).$$

Let $SM_x = \{y_1, \ldots, y_k\}$, we define $r(x)$ by

$$r(x) = \varphi_{2p+1}^{Y}(z_1, z_2, \ldots, z_{2p+1})$$

where

$$z_i = y_j\quad \text{for}\quad \sum_{\nu=1}^{j-1}\beta_x(y_\nu) < i\leqslant\sum_{\nu=1}^{j}\beta_x(y_\nu),$$

$$z_i = y_0\quad \text{for}\quad \sum_{\nu=1}^{k}\beta_x(y_\nu) < i\leqslant 2p+1.$$

Let $x_m\to x$. Denote $SM_{x_m} = A_m$ and $SM_x = A$. Let $A = \{y^1, \ldots, y^n\}$. Pick open sets $K_i\subset Y$ for $i\leqslant n$ so that $y^i\in K_i$ and $\overline{K}_i\cap\overline{K}_j = \emptyset$ for $i\neq j$. Since $\operatorname{dist}(A_m, A)\to 0$, we may assume without loss of generality that $A_m\subset\bigcup\limits_{i\leqslant n} K_i$ for $m = 1, 2\ldots$ Denote

$$A_{i,m} = K_i\cap S_m\ \text{ and }\ a_{i,m} = \sum_{y\in A_{i,m}}\beta_{x_m}(y).$$

Since $\sum\limits_{i\leqslant n} a_{i,m}\leqslant 2p$ for every m, we may divide the sequence $\{A_m\}$ into a finite number of subsequences so that A_k and A_m belong to the same subsequence if $a_{i,k} = a_{i,m}$ for each $i\leqslant n$.

Without loss of generality one may assume (replacing if necessary the sequence $\{A_m\}$ by a suitable subsequence) that $\{A_m\}$ coincides with one of these subsequences, i.e. there are a_i for $i\leqslant n$ such that $a_{i,m} = a_i$ for $m = 1, 2, \ldots$ Let in the sequence $\{y_1^{i,m}, \ldots, y_{a_i}^{i,m}\}$ every $y\in A_{i,m}$ appear $\beta_{x_m}(y)$ times. We put

$$a_m = (z_{1,m}, z_{2,m}, \ldots, z_{2p+1,m});\qquad a = (z_1, z_2, \ldots, z_{2p+1})$$

where

$$z_{i,m} = y_i^{j,m} \text{ and } z_i = y^j \text{ if } i = a_1 + a_2 + \ldots + a_{j-1} + l, \text{ with } 1 \leqslant l \leqslant a_j$$

$$z_{i,m} = z_i = y_0 \text{ if } i > \sum_{\nu=1}^{n} a_\nu .$$

Since dist $(A_m, A) \to 0$, we have

$$\lim_{m=\infty} y_j^{i,m} = y^i \quad \text{for } i = 1, \ldots, n, \ j = 1, \ldots, a_i .$$

Hence $a_m \to a$ in Y^{2p+1}, thus $\psi_{2p+1}^{Y}(a_m) \to \psi_{2p+1}^{Y}(a)$ in $Y^{[2p+1]}$.

Now we show that $\psi_{2p+1}^{Y}(a) = r(x)$. Let $f_i \in C(Y)$ be such that

$$f_i|K_j = \begin{cases} -1 & \text{if } j = i \\ 1 & \text{if } j \neq i \end{cases} \quad \text{for } i = 1, \ldots, n .$$

We have:

$$(Mf_i)(x_m) = (-1)^{a_{i,m}} \text{ for } m = 1, 2 \ldots \quad \text{and} \quad (Mf_i)(x) = (-1)^{\beta_x(y^i)} .$$

Since $\lim_{m=\infty} (Mf_i)(x_m) = (Mf_i)(x)$, we have

$$a_i = a_{i,m} = \beta_x(y^i) \ (\mathrm{mod}\ 2) \text{ for almost all } m .$$

Hence

$$\psi_{2p+1}^{Y}(a) = \psi_{2p+1}^{Y}(b)$$

where

$$b = (z^1, z^2, \ldots, z^{2p+1})$$

with

$$z^i = y^j \text{ for } \sum_{\nu=1}^{j-1} \beta_x(y^\nu) < i \leqslant \sum_{\nu=1}^{j} \beta_x(y^\nu)$$

and

$$x^i = y_0 \text{ for } i > \sum_{\nu=1}^{n} \beta_x(y^\nu)$$

i.e., $\psi_{2p+1}^{Y}(a) = r(x)$.

Obviously $\psi_{2p+1}^{Y}(a_m) = r(x_m)$, thus $r(x_m) \to r(x)$. This proves the continuity of r.

The second part of the theorem is trivial. ■

Theorem 3 is an easy consequence of 2.2, 2.6 and Theorem 2. Indeed, suppose to the contrary that there is a meo M: $C(S_n) \to C(K^{n+1})$. Then, by 2.2, there is a p-fold meo M': $C(S_n) \to C(K^{n+1})$. This, by 2.6, implies the existence of a map r: $K^{n+1} \to S_n^{[2p+1]}$ such that

$$r(y) = j^{2p+1}(y) \quad \text{for } y \in S_n .$$

This *ex definitione* means that j^{2p+1} is a homotopically trivial map, a contradiction with Theorem 2.

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