On the homotopy classification of spaces

by

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Contents

Introduction. Consider a category K and denote by Ω the set of objects and by $\mathrm{Mor}[X, Y]$ — the set of corresponding morphisms (see [6]).

We define the quasiorder $\leqslant$ in Ω:

$$X \leqslant Y \underset{\mathrm{df}}{\Leftrightarrow} \bigvee_{f} f \in \mathrm{Mor}[Y, X].$$

We shall refer to this relation as the *natural order* in K.

The relation $\leqslant$ induces the equivalence relation $\equiv$:

$$X \equiv Y \underset{\mathrm{df}}{\Leftrightarrow} X \leqslant Y \wedge Y \leqslant X.$$

The decomposition of the set Ω into the equivalence classes with respect to the relation $\equiv$ will be called the *natural classification* in K.

It is clear that this concept is closely connected with the ideas of the famous Erlangen Program of Felix Klein (see [9] compare also [1]).

Examples. 1. Let G be an arbitrary group of transformations of a given space M onto itself. Consider the category K_G, the objects of which are subsets of M. Let $f \in \mathrm{Mor}[X, Y]$ whenever f maps X onto Y

and can be extended to $\bar{f} \in G$. Obviously the natural classification in K_G is a Klein's classification.

2. Let R be the category, which has the compact ANRs as objects and r-maps as morphisms. By means of the natural classification we get r-types (see [1] or [3], p. 17).

3. In the category of groups with r-homomorphisms as morphisms (see [3], p. 32 or Section 1 of this paper) the natural classification gives us r-types of groups, i.e. the classes of r-equal groups.

The natural order induces a relation $<$:

$$X < Y \underset{\mathrm{df}}{\Leftrightarrow} X \leqslant Y \wedge \sim(Y \leqslant X).$$

Moreover, we can introduce the notion of *neighbours* in K, i.e. such two objects $X, Y \in \Omega$ that $X < Y$ and no object $Z \in \Omega$ satisfies the condition: $X < Z < Y$. Then X is said to be the *left neighbour* of Y, in symbols: $X \dot{<} Y$.

In particular, in the category K_G (ex. 1) there are no neighbours at all; in the category R (ex. 2) we get r-neighbours (see [3], p. 200).

In this paper we are interested in some categories of topological spaces with h-maps as morphisms. The notion of h-map is due to J. H. C. Whitehead. The map $f\colon X \to Y$ is said to be an h-map provided that it has a right homotopy inverse, i.e. such a map $g\colon Y \to X$ that $fg \simeq 1_Y$ (1_Y denoting an identity map of the space Y). The natural order in such categories is called *h-domination* or *h-order* and is denoted by $\underset{h}{\leqslant}$; the relations $\equiv, <, \dot{<}$ are denoted by $\underset{h}{\equiv}, \underset{h}{<}, \underset{h}{\dot{<}}$ respectively. The natural classification in such categories is called *h-classification*; the equivalence class of a given space X with respect to the relation $\underset{h}{\equiv}$ is said to be an *h-type* of X and is denoted by $[X]_h$.

These notions are closely related to the notion of homotopy type introduced by Hurewicz. Two spaces X and Y are said to be *homotopically equivalent* (in symbols $X \simeq Y$) provided that there is a map $f\colon X \to Y$ having a two-sided homotopy inverse. Given a space X we shall denote by $[X]$ the *homotopy type* of X, i.e. the class of spaces homotopically equivalent to X. Obviously $[X] \subset [X]_h$, but the converse inclusion does not hold necessarily (see [5] and [15]). Some positive results are formulated in Section 3.

Section 1 deals with the theory of groups. The notion of admissible class of groups is defined there; it makes possible to distinguish the largest class of spaces which can be h-classified and h-ordered by means of Whitehead's theorems (see [16] or Section 3 of this paper).

The main results can be found in Sections 4-10.

1. Admissible classes of groups. This section is of auxiliary character. It contains some simple group-theoretical statements which will be usefull in the sequel. The author did not find them quoted explicitely in the literature.

Let us recall that, given two groups $\mathfrak{A}, \mathfrak{B}$, a homomorphism $\varphi\colon \mathfrak{A} \to \mathfrak{B}$ is said to be an *r-homomorphism* whenever there is a homomorphism $\psi\colon \mathfrak{B} \to \mathfrak{A}$ such that $\varphi\psi$ is an identity. If such an r-homomorphism does exist, we refer to the group $\mathfrak{B}$ as an *r-image* of $\mathfrak{A}$ (in symbols $\mathfrak{B} \underset{r}{\leqslant} \mathfrak{A}$).

To express that the groups $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic we write $\mathfrak{A} \approx \mathfrak{B}$.

Now, let us consider a class $\mathfrak{G}$ of groups satisfying the following three conditions:

(G–1) If $\mathfrak{A} \in \mathfrak{G}$ and $\mathfrak{B} \leqslant \mathfrak{A}$, then $\mathfrak{B} \in \mathfrak{G}$.

(G–2) If $\mathfrak{A}, \mathfrak{B} \in \mathfrak{G}$ and $\mathfrak{A} \underset{r}{\equiv} \mathfrak{B}$, then $\mathfrak{A} \approx \mathfrak{B}$.

(G–3) If $\mathfrak{A}, \mathfrak{B} \in \mathfrak{G}$, $\mathfrak{A} \approx \mathfrak{B}$ and $\varphi\colon \mathfrak{A} \to \mathfrak{B}$ is an r-homomorphism, then φ is an isomorphism.

We shall refer to such a class $\mathfrak{G}$ as an *admissible class* of groups.

Given any admissible class $\mathfrak{G}$ we shall also use the letter $\mathfrak{G}$ to denote such a category which has the elements of $\mathfrak{G}$ as objects and r-homomorphisms as morphisms. Let us observe that the natural classification in such a category $\mathfrak{G}$ is a decomposition of $\mathfrak{G}$ into classes of isomorphic groups.

EXAMPLE 1. Let $\mathfrak{G}_A$ be the class of finitely generated Abelian groups. It is known that

(1.1) An arbitrary group $\mathfrak{A} \in \mathfrak{G}_A$ is a direct sum of cyclic indecomposable groups: $\mathfrak{A} = \mathfrak{N}^k \times \mathfrak{N}_1^{k_1} \times \ldots \times \mathfrak{N}_\mu^{k_\mu}$, $\mathfrak{N}$ denoting a cyclic infinite group, $k \geqslant 0$ and $\mathfrak{N}_i$ $(i = 1, \ldots, \mu)$ being a group of order $p_i^{\alpha_i}$, where p_i are prime numbers and $(p_i, \alpha_i) \neq (p_{i'}, \alpha_{i'})$ for $i \neq i'$. Moreover, such a decomposition is unique (up to isomorphism) (see [10]).

(1.2) If $\mathfrak{A}, \mathfrak{B}$ are two arbitrary Abelian groups and $\varphi\colon \mathfrak{A} \to \mathfrak{B}$ is an r-homomorphism, then $\mathfrak{A} \approx \mathfrak{B} \times \ker\varphi$; (see [3], p. 33).

Applying (1.1) and (1.2) one can easily verify that

(1.3) *The class $\mathfrak{G}_A$ is admissible.*

EXAMPLE 2. Let $\mathfrak{G}_F$ be a class of finitely generated free groups. Observe that

(1.4) *The class $\mathfrak{G}_F$ is admissible.*

The proof of this remark is based on the following four statements:

(1) An r-image of any group $\mathfrak{A}$ is isomorphic to some of its subgroups ([3], p. 32).

(2) Any subgroup of a free group is again a free group ([7], p. 96).

(3) The rank $\varrho(\mathfrak{A})$ of a group $\mathfrak{A} \in \mathfrak{G}_F$ determines this group uniquely up to isomorphism and a free group of rank k is freely generated by any set of k elements which generate it (see [7], p. 109).

(4) If $\mathfrak{A} \in \mathfrak{G}_F$ and $\mathfrak{B} \underset{r}{\leqslant} \mathfrak{A}$, then $\varrho(\mathfrak{B}) \leqslant \varrho(\mathfrak{A})$.

The last property of the class $\mathfrak{G}_F$ is an immediate consequence of (3) and of the following argument:

For any group $\mathfrak{A}$ an arbitrary epimorphism $\varphi\colon \mathfrak{A} \to \mathfrak{B}$ transforms any system of generators of the group $\mathfrak{A}$ onto a system of generators of $\mathfrak{B}$.

Remark. The class of all free groups with the countable set of generators is not admissible, since it does not satisfy the condition (G–3).

By the properties (1.1) and (1.2) of the class $\mathfrak{G}_A$ we obtain the following propositions concerning the r-order and r-neighbours in the category $\mathfrak{G}_A$.

(1.5) *If* $\mathfrak{A}, \mathfrak{B} \in \mathfrak{G}_A$, *then*

$$\mathfrak{A} \underset{r}{\leqslant} \mathfrak{B} \Leftrightarrow \textit{there is such } \mathfrak{C} \in \mathfrak{G}_A \textit{ that } \mathfrak{B} \approx \mathfrak{A} \times \mathfrak{C}.$$

(1.6) *If* $\mathfrak{A}, \mathfrak{B} \in \mathfrak{G}_A$ *and* $\varphi\colon \mathfrak{B} \to \mathfrak{A}$ *is* r-*homomorphism, then*

$$\mathfrak{A} \underset{r}{\dot{<}} \mathfrak{B} \Leftrightarrow \ker\varphi \textit{ is a cyclic indecomposable and non-trivial group.}$$

(1.7) *If* $\mathfrak{A}, \mathfrak{B}, \mathfrak{B}' \in \mathfrak{G}_A$, *then*

$$\mathfrak{A} \times \mathfrak{B} \underset{r}{\leqslant} \mathfrak{A} \times \mathfrak{B}' \Leftrightarrow \mathfrak{B} \underset{r}{\leqslant} \mathfrak{B}', \qquad \mathfrak{A} \times \mathfrak{B} \approx \mathfrak{A} \times \mathfrak{B}' \Leftrightarrow \mathfrak{B} \approx \mathfrak{B}',$$

$$\mathfrak{A} \times \mathfrak{B} \underset{r}{<} \mathfrak{A} \times \mathfrak{B}' \Leftrightarrow \mathfrak{B} \underset{r}{<} \mathfrak{B}', \qquad \mathfrak{A} \times \mathfrak{B} \underset{r}{\dot{<}} \mathfrak{A} \times \mathfrak{B}' \Leftrightarrow \mathfrak{B} \underset{r}{\dot{<}} \mathfrak{B}'.$$

The theorem on uniqueness quoted above ([7], p. 109) implies the following proposition on the category $\mathfrak{G}_F$:

(1.8) *If* $\mathfrak{A}, \mathfrak{B} \in \mathfrak{G}_F$, *then* $\mathfrak{A}, \mathfrak{B}$ *are* r-*comparable and*

$$\mathfrak{A} \underset{r}{\dot{<}} \mathfrak{B} \Leftrightarrow \varrho(\mathfrak{A}) + 1 = \varrho(\mathfrak{B}).$$

Finally, let us formulate a simple

(1.9) Lemma on homomorphisms. *Let* $h_i\colon \mathfrak{A}_i \to \mathfrak{B}_i$ *be a homomorphism of a group* $\mathfrak{A}_i$ *into* $\mathfrak{B}_i$ *for* $i = 1, 2$.

If $h\colon \mathfrak{A}_1 \times \mathfrak{A}_2 \to \mathfrak{B}_1 \times \mathfrak{B}_2$ *is defined by the formula* $h(a_1, a_2) \underset{\mathrm{df}}{=} \big(h_1(a_1), h_2(a_2)\big)$ *for* $a_i \in \mathfrak{A}_i$, $i = 1, 2$, *then*

h *is epimorphism* $\Leftrightarrow$ h_i *are epimorphisms for* $i = 1, 2$,

h *is monomorphism* $\Leftrightarrow$ h_i *are monomorphisms for* $i = 1, 2$,

h *is isomorphism* $\Leftrightarrow$ h_i *are isomorphisms for* $i = 1, 2$.

2. Some classes of topological spaces. According to the notation used by Whitehead in [16] let (a) be the class of connected topological spaces which are h-dominated by CW-complexes. A class of all simply-connected spaces of class (a) will be denoted by (a_0) (¹).

Given a sequence $\{\mathfrak{G}_n\}$ of admissible classes of groups we shall denote by $(a^{\Pi,N})_{\mathfrak{G}_n}$ a class of such spaces in (a) that their homotopy groups π_n belong respectively to $\mathfrak{G}_n$ for $n = 1, 2, \ldots, N$ (N being an integer or $N = \infty$). Analogically, $(a_0^H)_{\mathfrak{G}_n}$ will be the class of such spaces in (a_0) that their homology group H_n belongs to $\mathfrak{G}_n$ for $n = 1, 2, \ldots$ Finally, the symbol (a_1) will be used for the class of spaces which are homotopically equivalent to some connected and simply connected polyhedrons.

Let us observe that

(2.1) *All the classes introduced above are closed under arbitrary* h-*maps.*

In fact, it is obvious for the class (a). The classes (a_0), $(a^{\Pi,N})_{\mathfrak{G}_n}$ and $(a_0^H)_{\mathfrak{G}_n}$ are also closed under h-maps, since h-maps induce r-homomorphisms of homology and homotopy groups and since $\mathfrak{G}_n$ are assumed to be admissible. By the theorem of de Lyra ([11], p. 58) the spaces of the class (a_1) coincide with the connected and simply connected spaces h-dominated by polyhedrons. Hence (a_1) is also closed under h-maps.

(2.2) *Every compact and connected* ANR-*space belongs to the class* (a).

In fact, by the Borsuk theorem ([2], p. 97) such a space is h-dominated by a polyhedron.

As an immediate consequence of both the Borsuk theorem ([2], p. 97) and the theorem of de Lyra ([11], p. 58), we obtain the following statement:

(2.3) *Every compact connected and simply connected* ANR-*space belongs to the class* (a_1).

Now let us establish some connections between the classes defined above:

(2.4) $(a_1) \subset (a_0^H)_{\mathfrak{G}_A} \subset (a_0) \subset (a)$,

(2.5) $(a_0^H)_{\mathfrak{G}_A} \subset (a^{\Pi,N})_{\mathfrak{G}_A} \subset (a)$.

The condition (2.4) is obvious. By the Serré theorem ((5.1) [11], p. 50), if connected and simply connected CW-complex has finitely generated homology groups, then it has finitely generated homotopy groups as well. Hence (2.5) is verified.

We have also

(2.6) *If* $\mathfrak{G}_n \subset \mathfrak{G}'_n$ *for* $n = 1, 2, \ldots$, *then* $(a_0^H)_{\mathfrak{G}_n} \subset (a_0^H)_{\mathfrak{G}'_n}$ *and* $(a^{\Pi,N})_{\mathfrak{G}_n} \subset (a^{\Pi,N})_{\mathfrak{G}'_n}$ *for every* N.

(2.7) *If* $N \leqslant N'$, *then* $(a^{\Pi,N'})_{\mathfrak{G}_n} \subset (a^{\Pi,N})_{\mathfrak{G}_n}$ *for an arbitrary sequence* $\{\mathfrak{G}_n\}$ *of admissible classes.*

(¹) In [16] the symbol (a_0) denotes some larger class.

Sometimes we shall use the notation $X \in (a^{\Pi,N})$ $\big((a_0^H)\big)$ to express that there exists such a sequence $\{\mathfrak{G}_n\}$ that $X \in (a^{\Pi,N})_{\mathfrak{G}_n}$ $\big((a_0^H)_{\mathfrak{G}_n}\big)$.

3. Consequences of two Whitehead's theorems. The further arguments are based upon two Whitehead's theorems (see [16], p. 215). The first one (denoted here by (π)) deals with the class (a), the second one (denoted by (H)) — with the class (a_0). According to the notation used in [16], let ΔX be the minimum of dimension of all CW-complexes which dominate X — whenever such CW-complexes do exist — and let $\Delta X = \infty$ otherwise. For any two spaces $X, Y \in (a)$ let $N = N(X, Y) = \max(\Delta X, \Delta Y)$.

THEOREM (Π). *If $X, Y \in (a)$ and the map $f\colon X \to Y$ induces isomorphisms $f_n\colon \Pi_n(X) \to \Pi_n(Y)$ for $n = 1, 2, \ldots, N$, then f is a homotopy equivalence.*

THEOREM (H). *If $X, Y \in (a_0)$ and the map $f\colon X \to Y$ induces isomorphisms $f_n\colon H_n(X) \to H_n(Y)$ for $n = 1, 2, \ldots$, then f is a homotopy equivalence.*

The above two theorems imply the following five propositions:

(3.1) *If $X \in (a^{\Pi,\Delta X})$ or $X \in (a_0^H)$, then $[X] = [X]_h$.*

The inclusion $[X] \subset [X]_h$ holds for arbitrary space X. The converse inclusion $[X]_h \subset [X]$ follows from both the condition (G–3) and one of Theorems (Π) and (H) respectively.

(3.2) *If $X, Y \in (a^{\Pi,N})_{\mathfrak{G}_n}$, the class $\mathfrak{G}_n$ being admissible for $n = 1, \ldots, N = \max(\Delta X, \Delta Y)$, then*

$$[X \underset{h}{<} Y] \Rightarrow [\bigwedge_n \Pi_n(X) \underset{r}{\leqslant} \Pi_n(Y) \wedge \bigvee_{k \leqslant N} \Pi_k(X) \underset{r}{<} \Pi_k(Y)]\,.$$

In fact, any h-map $f\colon Y \to X$ induces r-homomorphisms $f_n\colon \Pi_n(Y) \to \Pi_n(X)$. If $\Pi_k(X) \underset{r}{=} \Pi_k(Y)$ for every $k \leqslant N$, then $\Pi_k(X) \approx \Pi_k(Y)$ by (G–2) and f_k $(k = 1, 2, \ldots, N)$ are isomorphisms by (G–3). Hence by Theorem (Π) f is proved to be a homotopy equivalence, contrary to our assumption.

(3.3) *Let $X, Y \in (a^{\Pi,N})_{\mathfrak{G}_n}$, the class $\mathfrak{G}_n$ being admissible for $n = 1, \ldots, N = \max(\Delta X, \Delta Y)$. If there exists an integer k, such that $\Pi_k(X) \overset{\bullet}{\underset{r}{<}} \Pi_k(Y)$ and $\Pi_n(X) \approx \Pi_n(Y)$ whenever $k \neq n \leqslant N$, then*

$$X \leqslant Y \Rightarrow X \overset{\bullet}{\underset{h}{<}} Y\,.$$

If $X \underset{h}{=} Y$, then $\Pi_k(X) \underset{r}{=} \Pi_k(Y)$. Let $X \underset{h}{<} Y$ and suppose that Y is not a neighbour of X; i.e. there is a space $Z \in (a^{\Pi,N})_{\mathfrak{G}_n}$ such that $X \underset{h}{<} Z \underset{h}{<} Y$. So $\Pi_n(X) \underset{r}{\leqslant} \Pi_n(Z) \underset{r}{\leqslant} \Pi_n(Y)$ for $n = 1, 2, \ldots$

By hypothesis $\Pi_n(X) \approx \Pi_n(Y)$ for $N \geqslant n \neq k$, hence it follows from (G–2) that $\Pi_n(X) \approx \Pi_n(Z) \approx \Pi_n(Y)$ for $N \geqslant n \neq k$. By the statement (3.2) there are two integers $k', k'' \leqslant N$ such that $\Pi_{k'}(X) \underset{r}{<} \Pi_{k'}(Z)$ and $\Pi_{k''}(Z) \underset{r}{<} \Pi_{k''}(Y)$. Hence $k' = k'' = k$ and $\Pi_k(X) \underset{r}{<} \Pi_k(Z) \underset{r}{<} \Pi_k(Y)$, which contradicts our assumption.

Now let us formulate two statements (3.4) and (3.5). The proofs are analogous to that of (3.2) and (3.3); instead of Theorem (Π) we use now Theorem (H).

(3.4) *If $X, Y \in (a_0^H)_{\mathfrak{G}_n}$, the class $\mathfrak{G}_n$ being admissible for $n = 1, 2, \ldots$, then*

$$[X \underset{h}{<} Y] \Rightarrow [\bigwedge_n H_n(X) \underset{r}{\leqslant} H_n(Y) \ \wedge\ \bigvee_k H_k(X) \underset{r}{<} H_k(Y)].$$

(3.5) *Let $X, Y \in (a_0^H)_{\mathfrak{G}_n}$, the class $\mathfrak{G}_n$ being admissible for $n = 1, 2, \ldots$ If there exists an integer k such that $H_k(X) \overset{\bullet}{\underset{r}{<}} H_k(Y)$ and $H_n(X) \approx H_n(Y)$ for $n \neq k$, then*

$$X \underset{h}{\leqslant} Y \Rightarrow X \overset{\bullet}{\underset{h}{<}} Y\,.$$

Of course, the last two propositions concern in particular all compact connected and simply connected ANRs.

4. Natural order and Cartesian products. One can easily verify the following assertion:

(4.1) *Let X, Y, Y' be three topological spaces and let $f\colon Y' \to Y$ be an h-map which has a map $f'\colon Y \to Y'$ as the right homotopy inverse. If a map $g\colon X \times Y' \to X \times Y$ is defined by the formula*

$$g(x, y') \underset{\mathrm{df}}{=} \big(x, f(y')\big) \quad \text{for} \quad (x, y') \in X \times Y'\,,$$

then setting

$$g'(x, y) \underset{\mathrm{df}}{=} \big(x, f'(y)\big) \quad \text{for} \quad (x, y) \in X \times Y$$

we obtain the right homotopy inverse $g'\colon X \times Y \to X \times Y'$ of g.

This statement implies three corollaries:

(4.2) *For any spaces X, Y, Y',*

$$Y \underset{h}{\leqslant} Y' \Rightarrow X \times Y \underset{h}{\leqslant} X \times Y'\,.$$

(4.3) *For any spaces X, Y, Y'*

$$Y \simeq Y' \Rightarrow X \times Y \simeq X \times Y'\,.$$

(4.4) *The class (a_1) is closed under the Cartesian multiplication.*

Let us notice that the last proposition fails as regards the class (a), since the Cartesian product of two CW-complexes is not necessarily CW-complex again (see [4]).

The assertion (4.2) holds for arbitrary topological spaces. Under some additional assumptions we shall prove the similar property of the relation $\underset{h}{<}$:

(4.5) THEOREM. *If $X\times Y$, $X\times Y'\in(a^{\Pi,N})_{\mathfrak{G}_n}$, the class $\mathfrak{G}_n$ being admissible for $n\leqslant N=\max(\Delta Y, \Delta Y')$, then*

$$Y\underset{h}{<}Y'\Rightarrow X\times Y\underset{h}{<}X\times Y'.$$

Proof. Take the h-map $f\colon Y'\to Y$. By (4.1), setting $g(x, y')\underset{\mathrm{df}}{=}(x, f(y'))$ for every $(x, y')\in X\times Y'$, we obtain a h-map $g\colon X\times Y'\to X\times Y$. Let us suppose that $X\times Y\underset{h}{=}X\times Y'$. Then, for any point $(x_0, y_0')\in X\times Y'$, we have

$$\Pi_n(X\times Y', (x_0, y_0'))\approx\Pi_n(X\times Y, (x_0, y_0)),$$

where $y_0=f(x_0)$. Since $\Pi_n(X\times Y)$, $\Pi_n(X\times Y')$ belong to some admissible class for $n=1,\ldots,N$, the r-homomorphisms $g_n\colon\Pi_n(X\times Y', (x_0, y_0'))\to\Pi_n(X\times Y, (x_0, y_0))$ induced by g are isomorphisms for $n\leqslant N$.

Let I^n denote the n-dimensional Euclidean cube, and let $\dot{I}^n$ be its boundary. Any map $\varphi\colon(I^n, \dot{I}^n)\to(X\times Y, (x_0, y_0))$ is of the form

$$\varphi=(\varphi_x, \varphi_y),\quad (\varphi_x\colon(I^n, \dot{I}^n)\to(X, x_0),\quad \varphi_y\colon(I^n, \dot{I}^n)\to(Y, y_0));$$

analogically, any $\varphi'\colon(I^n, \dot{I}^n)\to(X\times Y', (x_0, y_0'))$ is of the form $\varphi'=(\varphi_x', \varphi_y')$.

Define the homomorphisms $h_n\colon\Pi_n(X\times Y, (x_0, y_0))\to\Pi_n(X, x_0)\times\Pi_n(Y, y_0)$ and $h_n'\colon\Pi_n(X\times Y', (x_0, y_0'))\to\Pi_n(X, x_0)\times\Pi_n(Y', y_0')$ by the formulae:

$$h_n([\varphi])\underset{\mathrm{df}}{=}([\varphi_x], [\varphi_y]),\qquad h_n'([\varphi'])\underset{\mathrm{df}}{=}([\varphi_x'], [\varphi_y']).$$

The homomorphisms h_n, h_n' are isomorphisms (see [8], p. 144).

Consider the following diagram

$$\begin{array}{ccc}\Pi_n(X\times Y', (x_0, y_0')) & \xrightarrow{g_n} & \Pi_n(X\times Y, (x_0, y_0))\\ \downarrow{\scriptstyle h_n'} & & \downarrow{\scriptstyle h_n}\\ \Pi_n(X, x_0)\times\Pi_n(Y', y_0') & \xrightarrow[g_n^*]{} & \Pi_n(X, x_0)\times\Pi_n(Y, y_0)\end{array}$$

Setting $g_n^*\underset{\mathrm{df}}{=}h_ng_nh_n'^{-1}$ for $n=1,2,\ldots$ we obtain isomorphisms

$$g_n^*\colon\Pi_n(X, x_0)\times\Pi_n(Y', y_0')\to\Pi_n(X, x_0)\times\Pi_n(Y, y_0)$$

(since each of g_n^* is a superposition of the isomorphisms for $n=1,\ldots,N$).

Let us observe that

$$g_n^*(a, b')=(a, f_n(b'))\ \text{for every}\ a\in\Pi_n(X, x_0),\ b'\in\Pi_n(Y', y_0')\quad(n=1,2,\ldots),$$

f_n being the homomorphism induced by f. In fact, denoting $a=[\varphi_x]$, $b'=[\varphi_y']$ (where $\varphi_x\colon(I^n, \dot{I}^n)\to(X, x_0)$, $\varphi_y'\colon(I^n, \dot{I}^n)\to(Y', y_0')$), we have

$$g_n^*(a, b')=h_ng_nh_n'^{-1}([\varphi_x], [\varphi_y'])=h_ng_n(\varphi_x, \varphi_y')=h_n[g(\varphi_x, \varphi_y')]$$
$$=h_n[(\varphi_x, f\varphi_y')]=([\varphi_x], [f\varphi_y'])=([\varphi_x], f_n[\varphi_y'])=(a, f_n(b')).$$

Hence the homomorphisms g_n^* satisfy the assumptions of Lemma (1.9) and therefore f_n are isomorphisms for $n=1,\ldots,N$. But $Y, Y'\in(a)$, since they are retracts of $X\times Y$ and $X\times Y'$ respectively. Thus it follows from Theorem (Π) that f is a homotopy equivalence, contrary to our assumption.

(4.6) Remark. The assumptions of Theorem (4.5) can not be omitted.

In fact, let us consider the following example [2]:

Define

$$X=\mathop{\mathsf{P}}_{n=1}^{\infty}A_n,\quad\text{where}\quad A_n=S^1\vee S^1\quad\text{for}\quad n=1,2,\ldots$$

(i.e. A_n is a one-point union of two circles). Then X is homeomorphic to $X\times\{y_0\}$ and we have

$$X\times\{y_0\}\underset{h}{=}X\times S^1\quad\text{although}\quad\{y_0\}\underset{h}{<}S^1.$$

5. Construction of h-neighbours by means of Cartesian multiplication. The question arises, what assumptions are to be done in order to prove that

$$Y\underset{h}{\dot{<}}Y'\Rightarrow X\times Y\underset{h}{\dot{<}}X\times Y'.$$

The answer is given in the following

(5.1) THEOREM. *Let k be an arbitrary positive integer. Let $X\times Y$, $X\times Y'\in(a^{\Pi,N})_{\mathfrak{G}_n}$, the class $\mathfrak{G}_n$ being admissible for $n=1,\ldots,N=\max[\Delta(X\times Y), \Delta(X\times Y')]$ and $\mathfrak{G}_k=\mathfrak{G}_\Delta$. If $Y\underset{h}{\leqslant}Y'$, $\Pi_k(Y)\underset{r}{\dot{<}}\Pi_k(Y')$ and $\Pi_n(Y)\approx\Pi_n(Y')$ for $N\geqslant n\neq k$, then $X\times Y\underset{h}{\dot{<}}X\times Y'$.*

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[2] This example was used by Fox ([5]) in order to prove that the h-type and the homotopy type for some spaces do not coincide. Let us notice that X fails to be compact ANR against the supposition due to Fox in [5].

Proof. By (4.1) we obtain the condition: $X\times Y \underset{h}{\leqslant} X\times Y'$. By (3.3) it suffices to verify that

$$\Pi_k(X\times Y)\underset{r}{\dot{<}}\Pi_k(X\times Y') \quad \text{and} \quad \Pi_n(X\times Y)\approx\Pi_n(X\times Y') \text{ for } N\geqslant n\neq k.$$

Since $\Pi_n(X\times Y)\approx\Pi_n(X)\times\Pi_n(Y)$ and $\Pi_n(X\times Y')\approx\Pi_n(X)\times\Pi_n(Y')$, we have $\Pi_n(X\times Y)\approx\Pi_n(X\times Y')$ for $n\neq k$. By the Remark (1.7) the condition $\Pi_k(Y)\underset{r}{\dot{<}}\Pi_k(Y')$ implies

$$\Pi_k(X)\times\Pi_k(Y)\underset{r}{\dot{<}}\Pi_k(X)\times\Pi_k(Y') .$$

Hence $\Pi_k(X\times Y)\underset{r}{\dot{<}}\Pi_k(X\times Y')$, which completes the proof.

Now, let us recall the notion of homotopy type in the sense used, for example, in [8], p. 198. Given a group $\mathfrak{A}$ and a positive integer k, we say the space Y to be of the *homotopy type* $(\mathfrak{A}, k)$ provided that

$$\Pi_n(Y)\approx\begin{cases}\mathfrak{A} & \text{for} \quad n=k ,\\ \{0\} & \text{for} \quad n\neq k .\end{cases}$$

Theorem (5.1) implies the following

(5.2) COROLLARY. *Let k be the positive integer, and let $\mathfrak{A}$ be some cyclic indecomposable group. If the space Y' is of the homotopy type $(\mathfrak{A}, k)$ and $X, X\times Y'\in(\alpha^{\Pi,N})_{\mathfrak{G}_n}$, the class $\mathfrak{G}_n$ being admissible for $n=1,\ldots,N$ $=\max(\Delta X, \Delta(X\times Y'))$, then $X\underset{h}{\dot{<}}X\times Y'$.*

Proof. Setting in Theorem (5.1) $Y=\{y_0\}$ we obtain $\Pi_n(Y)\approx\Pi_n(Y')$ for $n\neq k$ and $\Pi_k(Y)=\{0\}\underset{r}{\dot{<}}\mathfrak{A}\approx\Pi_k(Y')$ (by (1.6)). Hence $X\times Y\underset{h}{\dot{<}}X\times Y'$ and therefore $X\underset{h}{\dot{<}}X\times Y'$.

(5.3) EXAMPLE. In (5.2) set $Y'=S^1$ ($k=1$, $\mathfrak{A}=\mathfrak{N}$ — the group of integers, $\mathfrak{G}_n=\mathfrak{G}_A$ for $n=1,2,\ldots$). We obtain the following sequences of h-neighbours:

$$\{x_0\}\underset{h}{\dot{<}}S^1\underset{h}{\dot{<}}S^1\times S^1\underset{h}{\dot{<}}\ldots$$

$$S^m\underset{h}{\dot{<}}S^m\times S^1\underset{h}{\dot{<}}S^m\times S^1\times S^1\underset{h}{\dot{<}}\ldots , \qquad m\geqslant 2 .$$

The question remains open, whether in (5.1) the assumption concerning the homotopy groups can be omitted or not. In particular, the author does not know, if $S^m\underset{h}{\dot{<}}S^m\times S^m$ for $m\geqslant 2$, although the relation $\{x_0\}\underset{h}{\dot{<}}S^m$ does hold (as we shall prove in Section 7).

6. Homotopy properties of h-neighbours. Some condition sufficient for two given spaces of the class (α) to be h-neighbours, was formulated in Section 3 (the statement (3.3)). Assuming the spaces under consideration to be of given homotopy types (in the sense determined in the previous section), we can establish a connection between the relations $\underset{h}{\leqslant}, \underset{h}{<}, \underset{h}{\dot{<}}$ (with respect to these spaces) and the relations $\underset{r}{\leqslant}, \underset{r}{<}, \underset{r}{\dot{<}}$ (with respect to their homotopy groups).

(6.1) THEOREM. *Let $X, Y\in(\alpha)$, $\mathfrak{A}, \mathfrak{B}\in\mathfrak{G}$, where $\mathfrak{G}=\mathfrak{G}_A$ or $\mathfrak{G}=\mathfrak{G}_F$, and let m, n be two positive integers. If X and Y are of the homotopy type $(\mathfrak{A}, m)$ and $(\mathfrak{B}, n)$ respectively, then*

(a) $X\underset{h}{\leqslant}Y \Leftrightarrow m=n\wedge\mathfrak{A}\underset{r}{\leqslant}\mathfrak{B}$,

(b) $X\underset{h}{<}Y \Leftrightarrow m=n\wedge\mathfrak{A}\underset{r}{<}\mathfrak{B}$,

(c) $X\underset{h}{\dot{<}}Y \Leftrightarrow m=n\wedge\mathfrak{A}\underset{r}{\dot{<}}\mathfrak{B}$.

Proof. 1°. Let $\mathfrak{G}=\mathfrak{G}_A$.

Firstly we shall verify all the implications $\Rightarrow$.

(a) If $X\underset{h}{\leqslant}Y$, then $\Pi_k(X)\underset{r}{\leqslant}\Pi_k(Y)$ for $k=1,2,\ldots$; hence $m=n$ and $\mathfrak{A}\underset{r}{\leqslant}\mathfrak{B}$.

(b) Assuming $X\underset{h}{<}Y$, let us take a h-map $f\colon Y\to X$ and suppose that $\mathfrak{A}\underset{r}{\equiv}\mathfrak{B}$. Since $\mathfrak{A}, \mathfrak{B}\in\mathfrak{G}_A$, we have $\mathfrak{A}\approx\mathfrak{B}$ and $f_n\colon \mathfrak{B}\to\mathfrak{A}$ is an isomorphism. But $\Pi_k(X)=\{0\}=\Pi_k(Y)$ for $k\neq n$ so $f_k\colon \Pi_k(Y)\to\Pi_k(X)$ is isomorphism for $k=1,2,\ldots$ Then, by Theorem (Π), we have $X\simeq Y$, contrary to our assumption. Hence $\mathfrak{A}\underset{r}{<}\mathfrak{B}$.

(c) If $X\underset{h}{\dot{<}}Y$, then $\mathfrak{A}\underset{r}{<}\mathfrak{B}$ by the assertion above. In order to prove that $\mathfrak{A}\underset{r}{\dot{<}}\mathfrak{B}$, let us suppose that there exists a group $\mathfrak{C}\in\mathfrak{G}_A$ satisfying the condition: $\mathfrak{A}\underset{r}{<}\mathfrak{C}\underset{r}{<}\mathfrak{B}$. Let $\varphi\colon \mathfrak{B}\to\mathfrak{C}$ and $\psi\colon \mathfrak{C}\to\mathfrak{A}$ be the r-homomorphisms. By (1.2) we have $\mathfrak{B}\approx\mathfrak{C}\times\ker\varphi$ and $\mathfrak{C}\approx\mathfrak{A}\times\ker\psi$, where $\ker\varphi\neq 0$ and $\ker\psi\neq 0$. By the Whitehead theorem on realizability ([17], p. 261) there exist two locally finite CW-complexes Z' and Z'' of the homotopy types $(\ker\varphi, n)$ and $(\ker\psi, n)$ respectively. Then $X\times Z'\times Z''$ is of the homotopy type $(\mathfrak{A}\times\ker\varphi\times\ker\psi, n)$, i.e. of the type $(\mathfrak{B}, n)$. Since $X, Y\in(\alpha)$ and their homotopy groups are countable, it follows from theorem of de Lyra ([11], p. 48) that there exist two locally finite CW-complexes K, L such that $X\simeq K$ and $Y\simeq L$. According to the statement in [16], p. 227, the space $K\times Z'\times Z''$ is a CW-complex as well.

Moreover, by (4.3), $X\times Z'\times Z''\simeq K\times Z'\times Z''$. Since any two CW-complexes of the same homotopy type $(\mathfrak{B}, n)$ are homotopically equivalent ([8], p. 199), so we have $K\times Z'\times Z''\simeq L$ and therefore $X\times Z'\times Z''\simeq Y$. Hence $X \underset{h}{<} X\times Z' \underset{h}{<} Y$, contrary to the assumption.

Now let us prove $\Leftarrow$.

(a) If $\mathfrak{A}\leqslant\mathfrak{B}$, then $\mathfrak{B}\simeq\mathfrak{A}\times\ker\zeta$, the homomorphism ζ being an r-homomorphism. Take a locally finite CW-complex Z of the homotopy type $(\ker\zeta, n)$. Since the spaces Y and $X\times Z$ are both of the homotopy type $(\mathfrak{B}, n)$, we infer that $Y\simeq X\times Z$. Hence $X \underset{h}{\leqslant} Y$.

(b) Moreover, if $\mathfrak{A} \underset{r}{<} \mathfrak{B}$, then $\ker\zeta\neq 0$ and therefore $X \underset{h}{<} Y$.

(c) Assume $\mathfrak{A} \underset{r}{\dot{<}} \mathfrak{B}$. According to (1.6), we have $\mathfrak{B}\approx\mathfrak{A}\times\mathfrak{C}'$, the group $\mathfrak{C}'$ being cyclic indecomposable. Let Y' be a locally finite CW-complex of the homotopy type $(\mathfrak{C}', n)$. Then $X\times Y'\simeq Y$ and hence, by (5.2), we obtain the desired relation $X \underset{h}{\dot{<}} Y$.

2°. Let $\mathfrak{G}=\mathfrak{G}_F$.

If $n>1$ and $\mathfrak{A}, \mathfrak{B}\in\mathfrak{G}_F$, then $\mathfrak{A}\approx\mathfrak{B}\approx\mathfrak{N}$ (the group of integers), so $\mathfrak{A}, \mathfrak{B}\in\mathfrak{G}_A$. Therefore we can assume $n=1$. By the arguments similar to those used in the case 1°, we infer that X and Y are homotopically equivalent to some locally finite CW-complexes, and the homotopy type of such CW-complexe determines it uniquely up to a homotopy equivalence. Then

$$X\simeq S^1_{(1)}\vee\ldots\vee S^1_{(k)},\quad Y\simeq S^1_{(1)}\vee\ldots\vee S^1_{(l)},\quad \text{where } k=\varrho(\mathfrak{A}),\ l=\varrho(\mathfrak{B}).$$

Hence, applying (1.8), we obtain:

$$X \underset{h}{\leqslant} Y \Leftrightarrow k\leqslant l \Leftrightarrow \mathfrak{A} \underset{r}{\leqslant} \mathfrak{B},$$

$$X \underset{h}{<} Y \Leftrightarrow k< l \Leftrightarrow \mathfrak{A} \underset{r}{<} \mathfrak{B},$$

$$X \underset{h}{\dot{<}} Y \Leftrightarrow k+1= l \Leftrightarrow \mathfrak{A} \underset{r}{\dot{<}} \mathfrak{B},$$

Thus the proof is complete.

There is a question, whether the implication

$$X \underset{h}{\dot{<}} Y \Rightarrow \bigwedge_k [\Pi_k(X)\approx\Pi_k(Y)]\vee[\Pi_k(X) \underset{h}{\dot{<}} \Pi_k(Y)]$$

holds.

In Section 7 we construct an example (see (7.4)) which gives the negative answer even in the case of the simply connected polyhedrons.

7. Construction of h-neighbours by means of one-point addition. Let us recall, that the space Y is said to be of the *homology type* $(\mathfrak{A}, k)$ (where $k>1$), provided that

$$\Pi_1(Y)=\{0\} \text{ and } H_n(Y)=\begin{cases}\mathfrak{A} & \text{for } n=k,\\ \{0\} & \text{for } n\neq k.\end{cases}$$

As proved in Section 5 (Corollary (5.2)) for spaces satisfying certain conditions, we can construct an h-neighbour Y of X by means of multiplying X by the space Y' of homotopy type $(\mathfrak{A}, n)$, the group $\mathfrak{A}$ being cyclic indecomposable. Now, we are going to prove the similar statement on the one-point addition of a space of the homology type $(\mathfrak{A}, n)$.

(7.1) THEOREM. *Let $k>1$ and let $X, X\vee Y\in(a_0^H)_{\mathfrak{G}_n}$, the class $\mathfrak{G}_n$ being admissible for every n, and $\mathfrak{G}_k=\mathfrak{G}_A$. If Y is of the homology type $(\mathfrak{A}, k)$, the group $\mathfrak{A}$ being cyclic indecomposable one, then*

$$X \underset{h}{\dot{<}} X\vee Y.$$

Proof. Obviously $X \underset{h}{\leqslant} X\vee Y$. Moreover, $\Pi_1(X\vee Y)=\{0\}=\Pi_1(X)\ast\ast\Pi_1(Y)$ ([8], p. 146), and

$$H_n(X\vee Y)\approx H_n(X) \quad \text{for} \quad n\neq k, \quad H_k(X\vee Y)\approx H_k(X)\times\mathfrak{A};$$

then, by (1.6), $H_k(X) \underset{r}{\dot{<}} H_k(X\vee Y)$. Hence from (3.5) if follows that $X \underset{h}{\dot{<}} X\vee Y$.

EXAMPLES.

(7.2) If $\mathfrak{A}\approx\mathfrak{N}$ — the additive group of integers and Y is of the homology type $(\mathfrak{A}, k)$, then $Y\simeq S^k$. Assuming X to be of the class $(a_0^H)_{\mathfrak{G}_n}$, where $\mathfrak{G}_k=\mathfrak{G}_A$, we obtain $X \underset{h}{\dot{<}} X\vee S^k$.

For instance,

$$\{x_0\} \underset{h}{\dot{<}} S^{k_1} \underset{h}{\dot{<}} S^{k_1}\vee S^{k_2} \underset{h}{\dot{<}} \ldots, \quad \text{for} \quad k_i\geqslant 2,\ i=1,2,\ldots$$

(7.3) If $\mathfrak{A}=\mathfrak{N}_m$ — a cyclic group of order m, then $m=p^\beta$, where p is a prime number. Then $Y\simeq P_m^{k+1}$ (on pseudoprojective space P_m^{k+1} see [8], p. 321), and we have

$$X \underset{h}{\dot{<}} X\vee P_m^{k+1} \quad \text{for} \quad X\in(a_0^H)_{\mathfrak{G}_n},\ \mathfrak{G}_k=\mathfrak{G}_A.$$

For instance,

$$\{x_0\} \underset{h}{\dot{<}} P_{m_1}^{n_1} \underset{h}{\dot{<}} P_{m_1}^{n_1}\vee P_{m_2}^{n_2} \underset{h}{\dot{<}} \ldots,$$

the integers m_j being powers of some prime numbers, $m_j \geqslant 2$ and $n_j \geqslant 3$;

$$P_m^k \overset{\bullet}{\underset{h}{<}} P_m^k \vee S^n \ (n \geqslant 2,\ k \geqslant 3).$$

Remarks.

(7.4) As concerns the question set at the end of Section 6 let us notice that $S^2 \overset{\bullet}{\underset{h}{<}} S^2 \vee S^3$, but $\Pi_6(S^2)$ and $\Pi_6(S^2 \vee S^3)$ are neither isomorphic or r-neighbours.

In fact, since $\Pi_6(S^2) = \mathfrak{N}_{12}$ (a finite group of order 12), and $\Pi_6(S^2 \vee S^3)$ has a divisor $\mathfrak{N}_{12} \times \mathfrak{N}_{12}$ (see [8]), so

$$\Pi_6(S^2) \underset{r}{<} \mathfrak{N}_{12} \times \mathfrak{N}_3 \underset{r}{<} \Pi_6(S^2 \vee S^3)\ .$$

(7.5) The question arises, whether $X \overset{\bullet}{\underset{h}{<}} X \vee S^1$. We are going to show that, as regards locally finite polytopes, the answer is negative.

For the purpose, given any sequence $\{X_n\}$ of spaces, we define the space X:

$$X = \bigcup_{n=1}^{\infty} Y_n\,, \quad \text{where } Y_1 \underset{\mathrm{df}}{=} X_1\,, \quad Y_{n+1} = Y_n \vee X_{n+1} \text{ for } n = 1, 2, \ldots,$$

and Y_n is assumed to be a subset of Y_{n+1}.

This space X will be denoted in symbols: $\bigvee_{n=1}^{\infty} X_n$.

Now, let us consider the following example (³).

Take a sequence $\{T_n\}$, the space T_n being a torus for $n = 1, 2, \ldots$ The spaces

$$X = \bigvee_{n=1}^{\infty} T_n \quad \text{and} \quad Y = X \vee S_1$$

are both locally finite polytopes. Besides, like in [15], $X \underset{r}{\equiv} Y$, then all the more $X \underset{h}{\equiv} Y$. Hence, really X and $X \vee S^1$ are not r-neighbours. As regards compact ANRs, the question is open.

8. Construction of h-neighbours by means of topological division. In contrast with Cartesian multiplication and one-point addition, a result of a topological division — as defined in [13] — is, in general, a space h-incomparable with a given one; for example, if $X = S^1 \times S^1$ and $A = S^1 \times \{x_0\}$, then the spaces X and $X|A$ are h-incomparable. However, Theorem (4.1) of [13] implies the following two statements.

(³) This is a slight modification of Stewart's example of two r-equal spaces which are not homotopically equivalent (see [15]).

(8.1) Theorem. *If X and A are both compact and connected* ANR-*spaces, and there exists an* AR-*set D such that $A \subset D \subset X$ then*

$$X \underset{h}{\leqslant} X|A\ .$$

(8.2) Theorem. *If X is compact and connected* ANR-*space and there exists an* AR-*set D such that $S^n \subset D \subset X$ (S^n denoting n-dimensional sphere, $n > 0$), then*

$$X \overset{\bullet}{\underset{h}{<}} X|S^n\,.$$

The first of these two statements is an immediate consequence of Theorem (4.1) of [13] mentioned above. The second one fol rom Theorem (4.1) of [13] and (7.1), (2.3).

9. Homological properties of h-neighbours. Some condition sufficient for two given spaces of the class (a_0) to be h-neighbours, was formulated in Section 3 (the statement (3.5)). Assuming the considered spaces to be of given homology types, we can establish a connection between the relations $\underset{h}{\leqslant}, \underset{h}{<}, \overset{\bullet}{\underset{h}{<}}$ (with respect to these spaces) and the relations $\underset{r}{\leqslant}, \underset{r}{<}, \overset{\bullet}{\underset{r}{<}}$ (with respect to their homology groups).

(9.1) Theorem. *Let $X, Y \in (a_0)$, $\mathfrak{A}, \mathfrak{B} \in \mathfrak{G}_A$ and let m, n be two positive integers. If X and Y are of the homology type $(\mathfrak{A}, m)$ and $(\mathfrak{B}, n)$ respectively, then*

(a) $X \underset{h}{\leqslant} Y \Leftrightarrow m = n \wedge \mathfrak{A} \underset{r}{\leqslant} \mathfrak{B}$,

(b) $X \underset{h}{<} Y \Leftrightarrow m = n \wedge \mathfrak{A} \underset{r}{<} \mathfrak{B}$,

(c) $X \overset{\bullet}{\underset{h}{<}} Y \Leftrightarrow m = n \wedge \mathfrak{A} \overset{\bullet}{\underset{r}{<}} \mathfrak{B}$.

(An analogy to Theorem (6.1) is evident).

Proof. We shall use the following two results:

(1) If two spaces $X, Y \in (a_0)$ are both of the same homology type, then $X \simeq Y$ ([12]).

(2) For any group $\mathfrak{A} \in \mathfrak{G}_A$ and arbitrary integer $n > 1$ there exists a finite polytope of homology type $(\mathfrak{A}, n)$ ([11], p. 60 or [14], p. 262).

Firstly let us verify the implications $\Rightarrow$.

(a) is obvious.

(b) Let $X \underset{h}{<} Y$. By (a) we have $m = n$ and $\mathfrak{A} \underset{r}{\leqslant} \mathfrak{B}$. Suppose $\mathfrak{A} \underset{r}{\equiv} \mathfrak{B}$; by the properties (G–2) and (G–3) of the class $\mathfrak{G}_A$ and by Whitehead's Theorem (H) we obtain $X \simeq Y$, contrary to our assumption.

(c) Let $X \underset{h}{\overset{\bullet}{<}} Y$. By (b) we have $m = n$ and $\mathfrak{A} \underset{r}{<} \mathfrak{B}$. Suppose there exists a group $\mathfrak{C} \in \mathfrak{G}_A$ such that $\mathfrak{A} \underset{r}{<} \mathfrak{C} \underset{r}{<} \mathfrak{B}$ and let $\varphi\colon \mathfrak{B} \to \mathfrak{C}$, $\psi\colon \mathfrak{C} \to \mathfrak{A}$ be the r-homomorphisms. Then $\mathfrak{B} \approx \mathfrak{C} \times \ker\varphi$, $\mathfrak{C} \approx \mathfrak{A} \times \ker\psi$, where $\ker\varphi \neq 0$, $\ker\psi \neq 0$. Take two finite polytopes Z' and Z'' of the homology type $(\ker\varphi, n)$ and $(\ker\psi, n)$ respectively (by (2) such polytopes do exist). By (1) and (2) we can assume X to be also a finite polytope. Then $X \vee Z' \vee Z'' \in (\alpha_0)$. Moreover, $X \vee Z' \vee Z''$ is of the homology type $(\mathfrak{A} \times \ker\varphi \times \ker\psi, n)$, i.e. of the type $(\mathfrak{B}, n)$, as well as the space Y. Hence it follows from (1), that $X \vee Z' \vee Z'' \simeq Y$ and then $X \underset{h}{<} X \vee Z' \underset{h}{<} Y$, contrary to our assumption.

Now let us prove $\Leftarrow$.

(a) If $m = n$ and $\mathfrak{A} \underset{r}{\leqslant} \mathfrak{B}$, then $\mathfrak{B} \approx \mathfrak{A} \times \ker\zeta$, the function $\zeta\colon \mathfrak{B} \to \mathfrak{A}$ being an r-homomorphism. Take a finite polytope Z of the homology type $(\ker\zeta, n)$ and assume again X to be a finite polytope as well. Then the polytope $X \vee Z$ is of the homology type $(\mathfrak{A} \times \ker\zeta, n)$, i.e. of the type $(\mathfrak{B}, n)$, as well as the space Y. Hence $X \vee Z \simeq Y$ and therefore $X \underset{h}{\leqslant} Y$.

(b) If $m = n$ and $\mathfrak{A} \underset{r}{<} \mathfrak{B}$, then the r-homomorphism ζ mentioned in (a) has a non-trivial kernel; hence $X \underset{h}{<} Y$.

(c) Let $m = n$ and $\mathfrak{A} \underset{r}{\overset{\bullet}{<}} \mathfrak{B}$. Then, by (1.6), $\ker\zeta$ is a cyclic indecomposable group. Hence, by (7.1), we have $X \underset{h}{\overset{\bullet}{<}} X \vee Z$ and therefore $X \underset{h}{\overset{\bullet}{<}} Y$, which completes the proof.

10. Decreasing sequences of spaces. The sequence $\{X_\mu\}_{\mu=1,2,\ldots}$ of spaces will be said *h-decreasing* (*h-increasing*), provided that $X_{\mu+1} \underset{h}{<} X_\mu$ $(X_\mu \underset{h}{<} X_{\mu+1})$ for $\mu = 1, 2, \ldots$

The examples given in Section 7 show that

(10.1) *In the class (α_1) there exist h-increasing sequences.*

(10.2) *There exists such a space $X \in (\alpha_1)$, which has an infinite family of h-incomparable left h-neighbours.*

We can prove a little more (see (10.3)). To this effect, let us define a homotopical dimension of the space X (in symbols $\mathrm{dh}\,X$) as follows:

$$\mathrm{dh}\,X \underset{\mathrm{df}}{=} \min_{X' \simeq X} \dim X' .$$

(10.3) *There exists a space $X \in (\alpha_1)$, which has an infinite family $\{X_\nu\}_{\nu=1,2,\ldots}$ of h-incomparable h-neighbours all of one homotopical dimension.*

Proof. Let us observe that, for any integer n, $\mathrm{dh}\,P_m^{n+1} = n+1$, the space P_m^{n+1} being a pseudoprojective space (see [8]). In fact, if $Y \simeq P_m^{n+1}$ then $H_n(Y) = \mathfrak{N}_m$ — a cyclic group of order m; hence, by theorem on universal coefficients, $H_{n+1}(Y, \mathfrak{N}_m) \neq 0$, and therefore $\dim Y \geqslant n+1$.

Now, take a sequence $\{m_\nu\}_{\nu=0,1,2,\ldots}$, where m_i is a power of a prime number, $i = 0, 1, 2, \ldots$ and $m_\nu \neq m_{\nu'}$ for $\nu \neq \nu'$. Let $X \underset{\mathrm{df}}{=} P_m^{n+1}$ and $X_\nu \underset{\mathrm{df}}{=} X \vee P_{m_\nu}^{n+1}$ for $\nu \geqslant 1$. As showed in (7.3), we have $X \underset{h}{\overset{\bullet}{<}} X_\nu$ for $\nu = 1, 2, 3, \ldots$; moreover, each two of the spaces X_ν are incomparable, since by (1.1) their n-th homology groups $H_n(X_\nu) = \mathfrak{N}_{m_0} \times \mathfrak{N}_{m_\nu}$ are r-incomparable. Hence X and $\{X_\nu\}_{\nu=1,2,\ldots}$ are proved to be the desired ones.

There is a question, whether there exist a space X and infinite sequence $\{X_\mu\}$, $\mu = 1, 2, \ldots$ such that $X_\mu \underset{h}{<} X$ for $\mu = 1, 2, \ldots$ and no one of X_μ is homotopically equivalent to another. In particular

1° Does there exist a h-decreasing sequence?

2° Do there exist a space X and a sequence $\{X_\mu\}$ of h-incomparable spaces such that $X_\mu \underset{h}{<} X$ for every μ?

As regards the class (α_1), the answer to the question 1° is given in the following

(10.4) Theorem. *There is no one h-decreasing infinite sequence in the class (α_1).*

Proof. In order to prove our theorem, it suffices to verify the following condition:

(10.5) *If $X, X_\mu \in (\alpha_1)$, $X_\mu \underset{h}{\leqslant} X$ for $\mu = 1, 2, \ldots$ and no one of X_μ is homotopically equivalent to another, then there exists μ_0 such that any two X_{μ_1}, X_{μ_2} are incomparable for $\mu_1, \mu_2 > \mu_0$.*

Take X and $\{X_\mu\}$ satisfying the assumptions of (10.5), and observe that

$$\bigvee_{k \geqslant 2} [H_k(X) \neq 0] \wedge [H_i(X) = 0 \text{ for } i > k]$$

and

$$\bigwedge_{\mu} \bigvee_{k_\mu \geqslant 2} [H_{k_\mu}(X_\mu) \neq 0] \wedge [H_i(X_\mu) = 0 \text{ for } i > k_\mu] .$$

Since $X_\mu \underset{h}{<} X$, we have $k_\mu < k$ for $\mu = 1, 2, \ldots$ Given $i \leqslant k$, let us consider the sequence $\{H_i(X_\mu)\}_{\mu=1,2,\ldots}$. Since $\{X_\mu\} \subset (\alpha_1)$, all the groups $H_i(X_\mu)$ belong to $\mathfrak{G}_A$ and therefore almost all of these groups are isomorphic, i.e.

$$\bigvee_{\mu_i} \bigwedge_{\mu, \mu' > \mu_i} H_i(X_\mu) \approx H_i(X_{\mu'}) .$$

Now, setting

$$\mu_0 \underset{\mathrm{df}}{=} \max_{2 \leqslant i \leqslant k} \mu_i$$

we obtain the following condition:

$$\bigwedge_{\mu,\mu'>\mu_0} H_i(X_\mu)\approx H_i(X_{\mu'}) \quad \text{for} \quad i=1,2,\ldots$$

Let us suppose that

$$\bigwedge_{\nu} \bigvee_{\mu,\mu'>\nu} X_\mu, X_{\mu'} \text{ are } h\text{-comparable}$$

and take $\nu=\mu_0$. Then the spaces $X_\mu, X_{\mu'}$ are h-comparable and $H_i(X_\mu)\approx H_i(X_{\mu'})$ for $i=1,2,\ldots$; hence, by Whitehead's Theorem (H), we obtain $X_\mu\simeq X_{\mu'}$, contrary to the assumption. This completes the proof.

As regards the class (α_1), the question 2° remains open. It is closely related to the following problem concerning the class (α_1):

Given a sequence of groups $\{\mathfrak{A}_n\}\subset\mathfrak{G}_A$ such that $\mathfrak{A}_n=(0)$ for almost all n, do these groups $\mathfrak{A}_n$ can be realized as n-th homology groups for only finite number of homotopy types?

The positive answer to this problem would imply the negative answer to the question 2° for the class (α_1).

Concerning the class (α_0), the answer to both questions 1° and 2° is positive. Moreover, we shall prove the following

(10.6) THEOREM. (a) *There exists a h-decreasing sequence $\{X_\mu\}_{\mu=1,2,\ldots}$ of h-neighbours in (α_0);*

(b) *There exist in (α_0) a space Y and a sequence $\{Y_\nu\}$ of h-incomparable spaces, such that $Y_\nu \underset{h}{<} Y$ for $\nu=1,2,\ldots$*

Proof. (a) Let $X_\mu \underset{\text{df}}{=} \bigvee_{n=1}^{\infty} S^{\mu+n}$ for $\mu=1,2,\ldots$, S^m denoting m-sphere. Since each of the spaces X_μ is a locally finite polytope, so it is a CW-complex ([16], p. 223). Besides X_μ is connected and simply connected; thus $X_\mu\in(\alpha_0)$.

Since $X_\mu=S^{\mu+1}\vee X_{\mu+1}$ and $H_k(X_\mu)\in\mathfrak{G}_A$ for every k, it follows from (7.1) that

$$X_{\mu+1} \underset{h}{\hat{<}} X_\mu .$$

(b) Let $Y\underset{\text{df}}{=}X_1$, i.e. $Y\underset{\text{df}}{=}\bigvee_{n=2}^{\infty} S^n$. Then we have

$$S^\nu \underset{h}{<} Y \quad \text{for} \quad \nu=2,3,\ldots$$

and $S^\nu, S^{\nu'}$ are incomparable for $\nu\neq\nu'$.

This completes the proof.

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