$C^*(X)$ with cardinality 2^{ω_0} and with $\|f_A - f_B\| \geq 1$ for $A \neq B$ subsets of $\{x_i\}$. Now suppose that $\{g_\gamma Z_\gamma\colon \gamma \in \Gamma\}$ is a chain of metric compactifications of X with $\sup\{g_\gamma Z_\gamma\} = \beta X$. Then let $F_\gamma = \{f \circ g_\gamma\colon f \in C^*(Z_\gamma)\}$. Then F_γ is the closed regular subring of $C^*(X)$ associated with $g_\gamma Z_\gamma$. Since $\{g_\gamma Z_\gamma\}$ is a chain, so is $\{F_\gamma\}$ with $\bigcup_{\gamma \in \Gamma} F_\gamma = C^*(X)$. By transfinite induction define a function $h\colon \{\alpha < \omega_1\} \to \{F_\gamma\}$ such that for $\alpha < \beta < \omega_1$, $h(\alpha) \subset h(\beta)$ and $h(\alpha) \neq h(\beta)$. Then it can be shown that $\bigcup_{\alpha < \omega_1} h(\alpha) = C^*(X)$. Consider the map $F\colon \{A\colon A \subset \{x_i\}\} \to \{\alpha\colon \alpha < \omega_1\}$ defined by $F(A) = \min\{\alpha\colon f_A \in h(\alpha)\}$. Then F is at most countable to one and onto a cofinal subset of $\{\alpha\colon \alpha < \omega_1\}$. This implies that $\omega_1 = 2^{\omega_0}$.

THEOREM 2. *If X is discrete and $|X| = \omega_\alpha$, then $\omega_{\alpha+1} = 2^{\omega_\alpha}$ if and only if βX is the supremum of a chain of compactifications of X each of which has weight ω_α.*

The proof of Theorem 2 is similar to that of Theorem 1 and so is omitted.

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Some properties of the induced map

by

K. R. Gentry. (Greensboro)

1. Introduction. We use the notation and terminology of Eilenberg and Steenrod [1] for inverse limit sequences (the index set I is the positive integers) of topological spaces and continuous functions. If (X, f) is an inverse limit sequence, then we have the bonding maps $f_n^m\colon X_m \to X_n$ $(n \leq m)$ and the continuous projections $f_n\colon X_\infty \to X_n$. By a map Ψ from the inverse limit sequence (X, f) to the inverse limit sequence (Y, g) we mean a sequence of continuous functions $\psi_n\colon X_n \to Y_n$ such that $\psi_n f_n^{n+1} = f_n^{n+1}\psi_{n+1}$ for all $n \in I$. The inverse limit of Ψ is the function ψ_∞ from X_∞ into Y_∞ such that if $x \in X_\infty$, $x = (x_1, x_2, x_3, \ldots)$, then $\psi_\infty(x) = \big(\psi_1(x_1), \psi_2(x_2), \psi_3(x_3), \ldots\big)$.

Eilenberg and Steenrod have shown that (1) ψ_∞ is a continuous function from X_∞ into Y_∞ (Theorem 3.13), and (2) if each ψ_n is a 1-1 function of X_n onto Y_n, then ψ_∞ is a 1-1 function of X_∞ onto Y_∞ (Theorem 3.15). In this note we investigate further the relationship between properties of the ψ_n and properties of ψ_∞.

2. Periodicity. A continuous function f from X into itself is said to be *periodic* provided there exists a positive integer n such that $f^n(x) = x$ for all $x \in X$. The least such integer n is called the *period* of f. Assuming X to be metric, f is said to be *almost periodic* provided that for any $\varepsilon > 0$ there exists a positive integer n such that $d\big(x, f^n(x)\big) < \varepsilon$ for all $x \in X$.

Let Ψ be a map of (X, f) into itself, where each X_n is metric. The following example shows that ψ_∞ may fail to be periodic even though each ψ_n is periodic. For each $n \in I$, let (1) $X_n = \{z \in R^2 \,|\, |z| = 1\}$, (2) $f_n^{n+1}\colon X_{n+1} \to X_n$ be defined by $f_n^{n+1}(z) = z^2$, and (3) $\psi_n\colon X_n \to X_n$ be defined by $\psi_n(z) = \exp(2\pi i/2^n)\cdot z$. Then $\Psi = \{\psi_n\}$ is a map of (X, f) into itself such that each ψ_n is periodic but ψ_∞ is not. However, noting ψ_∞ is almost periodic does suggest the following:

THEOREM 1. *If each ψ_n is periodic, then ψ_∞ is almost periodic.*

Proof. Let the metric for X_n be denoted by d_n and choose the metric d for X_∞ such that if $x, y \in X_\infty$, $x = (x_1, x_2, x_3, \ldots)$, $y = (y_1, y_2, y_3, \ldots)$, then

$$d(x, y) = \sum_{n=1}^{\infty} \frac{d_n(x_n, y_n)}{2^n[1 + d_n(x_n, y_n)]}.$$

Let $\varepsilon > 0$ be given. There exists a positive integer m such that

$$\sum_{n=m+1}^{\infty} \frac{1}{2^n} < \varepsilon.$$

Since for any $x_n, y_n \in X_n$,

$$\frac{d_n(x_n, y_n)}{2^n[1 + d_n(x_n, y_n)]} < \frac{1}{2^n}$$

we have that for any $x, y \in X_\infty$,

$$\sum_{n=m+1}^{\infty} \frac{d_n(x_n, y_n)}{2^n[1 + d_n(x_n, y_n)]} < \varepsilon.$$

Let k_n denote the period of ψ_n and let $k = k_1 k_2 \ldots k_m$. If $x \in X_\infty$, then

$$d(\psi_\infty^k(x), x)$$

$$= \sum_{n=1}^{m} \frac{d_n(\psi_n^k(x_n), x_n)}{2^n[1 + d_n(\psi_n^k(x_n), x_n)]} + \sum_{n=m+1}^{\infty} \frac{d_n(\psi_n^k(x_n), x_n)}{2^n[1 + d_n(\psi_n^k(x_n), x_n)]} < 0 + \varepsilon = \varepsilon.$$

That the first sum is zero follows from the way k was chosen.

It is not known to me whether each ψ_n being almost periodic implies ψ_∞ is almost periodic. However, assuming the X_n are compact, we do get that the property of being almost periodic is preserved.

THEOREM 2. *If each X_n is compact and each ψ_n is almost periodic, then ψ_∞ is almost periodic.*

Proof. Let $\varepsilon > 0$ be given. There exists a positive integer m such that for any $x, y \in X_\infty$,

$$\sum_{n=m+1}^{\infty} \frac{d_n(x_n, y_n)}{2^n[1 + d_n(x_n, y_n)]} < \frac{\varepsilon}{2}.$$

Since each X_n is compact, the bonding maps are uniformly continuous. Using this fact, we get the following positive real numbers: a δ_{m-1} such that if $d_m(x_m, y_m) < \delta_{m-1}$, then $d_{m-1}(f_{m-1}^m(x_m), f_{m-1}^m(y_m)) < \varepsilon/2m$; a δ_{m-2} such that if $d_m(x_m, y_m) < \delta_{m-2}$, then $d_{m-2}(f_{m-2}^m(x_m), f_{m-2}^m(y_m)) < \varepsilon/2m$, ... a δ_1 such that if $d_m(x_m, y_m) < \delta_1$, then $d_1(f_1^m(x_m), f_1^m(y_m)) < \varepsilon/2m$. Now since ψ_m is almost periodic there exists a positive integer k such that $d_m(x_m, \psi_m^k(x_m)) < \min(\varepsilon/2m, \delta_1, \delta_2, \ldots, \delta_{m-1})$ for all $x_m \in X_m$.

Let $x \in X_\infty$. We will show that $d(x, \psi_\infty^k(x)) < \varepsilon$. By the way k was chosen so that $d_m(x_m, \psi_m^k(x_m)) < \varepsilon/2m$. Also $d_{m-1}(x_{m-1}, \psi_{m-1}^k(x_{m-1})) = d_{m-1}(f_{m-1}^m(x_m), \psi_{m-1}^k f_{m-1}^m(x_m)) = d_{m-1}(f_{m-1}^m(x_m), f_{m-1}^m \psi_m^k(x_m)) < \varepsilon/2m$. The last equality follows from the fact that $\psi_j^n f_j^{j+1} = f_j^{j+1} \psi_{j+1}^n$ for all positive integers j and n, and the last inequality follows since $d_m(x_m, \psi_m^k(x_m)) < \delta_{m-1}$. Similarly it can be shown that $d_n(x_n, \psi_n^k(x_n)) < \varepsilon/2m$ for all n, $1 \leqslant n \leqslant m-2$. So

$$d(x, \psi_\infty^k(x)) = \sum_{n=1}^{m} \frac{d_n(x_n, \psi_n^k(x_n))}{2^n[1 + d_n(x_n, \psi_n^k(x_n))]} + \sum_{n=m+1}^{\infty} \frac{d_n(x_n, \psi_n^k(x_n))}{2^n[1 + d_n(x_n, \psi_n^k(x_n))]}$$

$$< m\left(\frac{\varepsilon}{2m}\right) + \frac{\varepsilon}{2} = \varepsilon.$$

3. Interior and closed maps. A function $f\colon X \to Y$ is said to be *interior* (*closed*) if it is continuous and the image of every open (closed) subset of X is open (closed) in Y. Let Ψ be a map of (X, f) into (Y, g). The following example shows that each ψ_n being interior does not imply that ψ_∞ is interior. For each $n \in I$, let (1) X_n be the positive integers with discrete topology, (2) $f_n^{n+1}\colon X_{n+1} \to X_n$ be the identity map, (3) Y_n be 2^n distinct points $\{y(n, 1), y(n, 2), \ldots, y(n, 2^n)\}$ with discrete topology,

(4) $$g_n^{n+1}\colon Y_{n+1} \to Y_n$$

be defined by

$$g_n^{n+1}[y(n+1, j)] = \begin{cases} y(n, j) & 1 \leqslant j \leqslant 2^n, \\ y(n, j - 2^n) & 2^n + 1 \leqslant j \leqslant 2^{n+1} \end{cases}$$

and

(5) $$\psi_n\colon X_n \to Y_n$$

be defined by $\psi_n(1) = y(n, 1)$, $\psi_n(2) = y(n, 2), \ldots, \psi_n(2^n) = y(n, 2^n)$, $\psi_n(2^n+1) = y(n, 1)$, $\psi_n(2^n+2) = y(n, 2), \ldots, \psi_n(2^n+2^n) = y(n, 2^n), \ldots$ $\ldots, \psi_n(2^n+2^n+1) = y(n, 1), \ldots$ Each ψ_n is interior and onto but ψ_∞ is not interior. For $(\{1\} \times X_2 \times X_3 \times \ldots) \cap X_\infty = (1, 1, 1, \ldots)$ is open in X_∞ but $\psi_\infty[(1, 1, 1, \ldots)]$ is not open in Y_∞. To see $\psi_\infty[(1, 1, 1, \ldots)]$ is not open in Y_∞ we note that Y_∞ is homeomorphic to the Cantor set.

Under certain restrictive but useful conditions we do get that the property of being interior is preserved.

THEOREM 3. *Suppose that* (1) *each* ψ_n *is interior and* (2) *for each* $n \in I$,

$$\psi_n(x_n) = y_n\,, \qquad g_n^{n+1}(y_{n+1}) = y_n$$

implies $(f_n^{n+1})^{-1}(x_n) \cap \psi_{n+1}^{-1}(y_{n+1}) \neq \emptyset$. *Then* ψ_∞ *is interior.*

Proof. Let U_∞ be an open subset of X_∞. We can assume there exists a positive integer n and open subset U_n of X_n such that $U_\infty = f_n^{-1}(U_n)$. It follows that $\psi_\infty f_n^{-1}(U_n) = g_n^{-1}\psi_n(U_n)$. For let $y = (y_1, y_2, y_3, \ldots) \in \psi_\infty f_n^{-1}(U_n)$. Then $y = \psi_\infty(x_1, x_2, x_3, \ldots)$ where $x_n \in U_n$. Thus $\psi_n(x_n) \in \psi_n(U_n)$ and so $y = \psi_\infty(x_1, x_2, x_3, \ldots) \in g_n^{-1}\psi_n(U_n)$. Now let $y = (y_1, y_2, y_3, \ldots) \in g_n^{-1}\psi_n(U_n)$. Then $y_n \in \psi_n(U_n)$ and so there is an $x_n \in U_n$ such that $\psi_n(x_n) = y_n$. Consider $z = \big(f_1^n(x_n), f_2^n(x_n), \ldots, f_{n-1}^n(x_n), x_n, z_{n+1}, z_{n+2}, \ldots\big)$ where $z_{n+1} \in (f_n^{n+1})^{-1}(x_n) \cap \psi_{n+1}^{-1}(y_{n+1})$, $z_{n+2} \in (f_{n+1}^{n+2})^{-1}(z_{n+1}) \cap \psi_{n+2}^{-1}(y_{n+2}), \ldots$; $z \in f_n^{-1}(U_n)$ and so $\psi_\infty(z) \in \psi_\infty f_n^{-1}(U_n)$. Since $\psi_\infty(z) = y$ we have that $y \in \psi_\infty f_n^{-1}(U_n)$.

ψ_n being interior and g_n being continuous implies $g_n^{-1}\psi_n(U_n)$ is open. Thus $\psi_\infty(U_\infty)$ is open since we have shown $\psi_\infty(U_\infty) = g_n^{-1}\psi_n(U_n)$.

Applying a theorem of Himmelberg we are able to get conditions which imply that closed is preserved.

THEOREM 4. *Suppose* (1) *each* X_n *is Hausdorff,* (2) *each* ψ_n *is onto and closed and* (3) $\psi_n^{-1}(y_n)$ *is compact for all* $y_n \in Y_n$, $n \in I$. *Then* ψ_∞ *is closed.*

Proof. By Theorem 1 of [2], $\theta\colon \prod X_n \to \prod Y_n$ defined by $\theta(x_1, x_2, x_3, \ldots) = \big(\psi_1(x_1), \psi_2(x_2), \psi_3(x_3), \ldots\big)$ is closed. Since each X_n is Hausdorff X_∞ is a closed subset of $\prod X_n$. Thus ψ_∞ which is θ restricted to X_∞ is a closed map of X_∞ into Y_∞.

4. Monotone, light and compact maps. A continuous function $f\colon X \to Y$ is said to be *monotone* provided that, for each point $y \in Y$, the inverse image $f^{-1}(y)$ is connected.

LEMMA 1. *The inverse limit of compact connected spaces is connected.*

THEOREM 5. *Let* Ψ *be a map of* (X, f) *into* (Y, g). *Suppose each* X_n *is compact Hausdorff and each* ψ_n *is monotone. Then* ψ_∞ *is monotone.*

Proof. Let $y = (y_1, y_2, y_3, \ldots) \in Y_\infty$. For each $n \in I$, $\psi_n^{-1}(y_n)$ is a compact, connected subset of X_n. This follows since the ψ_n are monotone. If h_n^{n+1} is f_n^{n+1} restricted to $\psi_{n+1}^{-1}(y_{n+1})$, then h_n^{n+1} is a continuous function from $\psi_{n+1}^{-1}(y_{n+1})$ into $\psi_n^{-1}(y_n)$. Let C_∞ be the inverse limit of the inverse sequence $\big(\psi_n^{-1}(y_n), h_n^{n+1}\big)$. Since each $\psi_n^{-1}(y_n)$ is compact and connected lemma 1 implies that C_∞ is connected. The proof of the theorem then follows since $C_\infty = \psi_\infty^{-1}(y)$.

A continuous function $f\colon X \to Y$ is said to be *light* provided that, for each $y \in Y$, the inverse image $f^{-1}(y)$ is totally disconnected.

LEMMA 2. *The inverse limit of totally disconnected spaces is totally disconnected.*

THEOREM 6. *Let* Ψ *be a map of* (X, f) *into* (Y, g). *If each* ψ_n *is light, then* ψ_∞^{-1} *is light.*

Proof. Let $y = (y_1, y_2, y_3, \ldots) \in Y_\infty$. Since the ψ_n are light, each $\psi_n^{-1}(y_n)$ is totally disconnected. If h_n^{n+1} is f_n^{n+1} restricted to $\psi_{n+1}^{-1}(y_{n+1})$, h_n^{n+1} is continuous function from $\psi_{n+1}^{-1}(y_{n+1})$ into $\psi_n^{-1}(y_n)$. Let D_∞ be the inverse limit of the inverse sequence $\big(\psi_n^{-1}(y_n), h_n^{n+1}\big)$. By Lemma 2, D_∞ is totally disconnected. The proof of the theorem follows since $D_\infty = \psi_\infty^{-1}(y)$.

A continuous function $f\colon X \to Y$ is said to be *compact* if the inverse image of every compact set is compact.

THEOREM 7. *Let* Ψ *be a map of* (X, f) *into* (Y, g). *If* Y_∞ *is Hausdorff and each* ψ_n *is compact, then* ψ_∞ *is compact.*

Proof. Let C be a compact subset of Y_∞. Then $\psi_\infty^{-1}(C) \subset \big[\psi_1^{-1}\big(g_1(C)\big) \times \psi_2^{-1}\big(g_2(C)\big) \times \ldots\big] \cap X_\infty$. Each $g_n(C)$ is compact and, since the ψ_n are compact, each $\psi_n^{-1}\big(g_n(C)\big)$ is compact. So $\big[\psi_1^{-1}\big(g_1(C)\big) \times \psi_2^{-1}\big(g_2(C)\big) \times \ldots\big] \cap X_\infty$ is compact. C being compact and a subset of a Hausdorff space implies C is closed. Thus $\psi_\infty^{-1}(C)$ is a closed subset of *a* compact set and hence compact.

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