$C^*(X)$ with cardinality $2^\omega$ and with $\lvert f_A - f_B \rvert \geq 1$ for $A \neq B$ subsets of $\{a_i\}$. Now suppose that $(g_z, Z_z; y \in Y)$ is a chain of metric compactifications of $X$ with sup $(g_z, Z_z) = \beta X$. Then let $P_z = \{ f \circ g_z; f \in C^*(Z_z) \}$. Then $P_z$ is the closed regular subring of $C^*(X)$ associated with $g_z, Z_z$. Since $(g_z, Z_z)$ is a chain, so is $(P_z)$ with $\bigcup_{z \in Z} P_z = C^*(X)$. By transfinite induction define a function $k: \{ a < \omega_1 \} \to (P_z)$ such that for $a < b < \omega_1, k(a) \subseteq k(b)$ and $k(a) \neq k(b)$. Then it can be shown that $\bigcup_{a \in a} k(a) = C^*(X)$. Consider the map $F: \{ A \subset \{ a_i \} \} \to \{ a < \omega_1 \}$ defined by $F(A) = \min\{ a; f_A \in k(a) \}$. Then $F$ is at most countable to one and onto a cofinal subset of $\{ a < \omega_1 \}$. This implies that $\omega_1 = 2^\omega$.

**Theorem 2.** If $X$ is discrete and $|X| = \omega_1$, then $\omega_{\omega+1} = 2^\omega$ if and only if $\beta X$ is the supremum of a chain of compactifications of $X$ each of which has weight $\omega_1$.

The proof of Theorem 2 is similar to that of Theorem 1 and so is omitted.

**References**


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**Some properties of the induced map**

by K. R. Gentry (Greensboro)

1. **Introduction.** We use the notation and terminology of Eilenberg and Steenrod [1] for inverse limit sequences (the index set $I$ is the positive integers) of topological spaces and continuous functions. If $(X, f)$ is an inverse limit sequence, then we have the bonding maps $f_n^I: X_n \to X_n$ (n \leq m) and the continuous projections $f_n: X_n \to X_n$. By a map $\psi$ from the inverse limit sequence $(X, f)$ to the inverse limit sequence $(X, g)$ we mean a sequence of continuous functions $\psi_n: X_n \to Y_n$ such that $\psi_n f_n^I = f_n^I \psi_{n+1}$ for all $n \in I$. The inverse limit of $\psi$ is the function $\psi_\infty$ from $X_\infty$ into $Y_\infty$ such that if $x \in X_\infty$, $x = (x_0, x_1, x_2, \ldots)$, then $\psi_\infty(x) = (\psi_0(x_0), \psi_1(x_1), \psi_2(x_2), \ldots)$.

Eilenberg and Steenrod have shown that (1) $\psi_\infty$ is a continuous function from $X_\infty$ into $Y_\infty$ (Theorem 3.13), and (2) if each $\psi_n$ is a 1-1 function of $X_n$ onto $Y_n$, then $\psi_\infty$ is a 1-1 function of $X_\infty$ onto $Y_\infty$ (Theorem 3.15). In this note we investigate further the relationship between properties of the $\psi_n$ and properties of $\psi_\infty$.

2. **Periodicity.** A continuous function $f$ from $X$ into itself is said to be periodic provided there exists a positive integer $n$ such that $f^n(x) = x$ for all $x \in X$. The least such integer $n$ is called the period of $f$. Assuming $X$ to be metric, $f$ is said to be almost periodic provided that for any $\epsilon > 0$ there exists a positive integer $n$ such that $d(f^n(x), x) < \epsilon$ for all $x \in X$.

Let $\mathcal{V}$ be a map of $(X, f)$ into itself, where each $X_n$ is metric. The following example shows that $\varphi_n$ may fail to be periodic even though each $\varphi_n$ is periodic. For each $n \in I$, let (1) $X_0 = \{ x \in E^n; |x| = 1 \}$, (2) $f_n^{I^+} = X_0$, $f_n^{I^{-}} = X_0$ be defined by $f_n^{I^+}(x) = x$, and (3) $\varphi_n: X_0 \to X_0$ be defined by $\varphi_n = \exp(2\pi i/2^n)$. Then $\mathcal{V} = (\varphi_n)$ is a map of $(X, f)$ into itself such that each $\varphi_n$ is periodic but $\varphi_\infty$ is not. However, noting $\varphi_n$ is almost periodic does suggest the following:

**Theorem 1.** If each $\varphi_n$ is periodic, then $\varphi_\infty$ is almost periodic.
Proof. Let the metric for $X_m$ be denoted by $d_m$ and choose the metric $d$ for $X_m$ such that if $x, y \in X_m$, $x = (a_1, x_1, y_1, 1, \ldots), y = (y_1, y_2, y_3, \ldots)$, then

$$d(x, y) = \sum_{n=1}^{\infty} \frac{d_m(a_n, x_n)}{2^n(1 + d_m(a_n, y_n))}.$$  

Let $\varepsilon > 0$ be given. There exists a positive integer $m$ such that

$$\sum_{n=m+1}^{\infty} \frac{1}{2^n} < \varepsilon.$$  

Since for any $x_n, y_n \in X_n$,

$$\frac{d_m(x_n, y_n)}{2^n(1 + d_m(x_n, y_n))} \leq \frac{1}{2^n},$$

we have that for any $x, y \in X_m$,

$$\sum_{n=m+1}^{\infty} \frac{d_m(x_n, y_n)}{2^n(1 + d_m(x_n, y_n))} < \varepsilon.$$  

Let $k_1$ denote the period of $y_m$ and let $k = k_1 k_2 \ldots k_m$. If $x \in X_m$, then

$$d(x, y(x)) = \sum_{n=1}^{m} \frac{d_m(x_n, y_n)}{2^n(1 + d_m(x_n, y_n))} + \sum_{n=m+1}^{\infty} \frac{d_m(x_n, y_n)}{2^n(1 + d_m(x_n, y_n))} < \varepsilon = \varepsilon.$$  

That the first sum is zero follows from the way $k$ was chosen.

It is not known to me whether each $y_m$ being almost periodic implies $x_m$ is almost periodic. However, assuming the $X_m$ are compact, we do get that the property of being almost periodic is preserved.

**Theorem 2.** If each $X_m$ is compact and each $y_m$ is almost periodic, then $x_m$ is almost periodic.

**Proof.** Let $\varepsilon > 0$ be given. There exists a positive integer $m$ such that for any $x_n, y_n \in X_m$,

$$\sum_{n=m+1}^{\infty} \frac{d_m(x_n, y_n)}{2^n(1 + d_m(x_n, y_n))} < \varepsilon.$$  

Since each $X_m$ is compact, the bonding maps are uniformly continuous. Using this fact, we get the following positive real numbers: a $\delta_{m-1}$ such that if $d_m(x_m, y_m) < \delta_{m-1}$, then $d_m(-f_{m-1}(x_m), -f_{m-1}(y_m)) < \varepsilon$; a $\delta_{m-1}$ such that if $d_m(x_m, y_m) < \delta_{m-1}$, then $d_m(-f_{m-1}(x_m), -f_{m-1}(y_m)) < \varepsilon$; and $\delta_{m-1}$ such that if $d_m(x_m, y_m) < \delta_{m-1}$, then $d_m(-f_{m-1}(x_m), -f_{m-1}(y_m)) < \varepsilon$.
Theorem 3. Suppose that (1) each \( y_i \) is interior and (2) for each \( n \in I, \)

\[
y_i^n(y_{i+1}) = y_{i+1}
\]

implies \((f_{i+1}^{n+1})^{-1}(a_i) \cap \psi_i^{c_i}(y_{i+1}) \neq \emptyset\). Then \( y_i \) is interior.

Proof. Let \( U_i \) be an open subset of \( X_i \). We can assume there exists a positive integer \( n \) and open subset \( U_n \) of \( X_n \) such that \( U_n = f_{i-n}^{n}(U_i) \). It follows that \( \psi_i f_{i-n}^{n}(U_n) = \psi_i f_{i-n}^{n}(U_i) \). For let \( y = (y_1, y_2, y_3, \ldots) \in \psi_i f_{i-n}^{n}(U_i) \). Then \( y = \psi_i(x_1, x_2, x_3, \ldots) \in \psi_i f_{i-n}^{n}(U_i) \) and so \( x = (x_1, x_2, x_3, \ldots) \in \psi_i f_{i-n}^{n}(U_i) \).

Now let \( y = (y_1, y_2, y_3, \ldots) \in \psi_i f_{i-n}^{n}(U_i) \). Then \( y \in \psi_i f_{i-n}^{n}(U_i) \) and so there is an \( x \in U_n \) such that \( \psi_i(x) = y \). Consider \( \varphi \) as \( (f_{i-n}^{n}(x_1), f_{i-n}^{n}(x_2), \ldots) \in \psi_i f_{i-n}^{n}(U_i) \) and so \( \psi_i(x) \in \psi_i f_{i-n}^{n}(U_i) \).

4. Monotone, light and compact maps. A continuous function \( f : X \to Y \) is said to be monotone provided that, for each point \( y \in Y \), the inverse image \( f^{-1}(y) \) is connected.

Lemma 1. The inverse limit of compact connected spaces is connected.

Theorem 5. Let \( \Psi \) be a map of \( (X, f) \) into \( (Y, g) \). Suppose each \( X_n \) is compact Hausdorff and each \( y_i \) is monotone. Then \( y_i \) is monotone.

Proof. Let \( y = (y_1, y_2, y_3, \ldots) \in X_\infty \). For each \( n \in I, \psi_{i-n}^{c_i}(y) \) is a compact, connected subset of \( X_n \). This follows since the \( y_i \) are monotone. If \( h_{i-n}^{c_i} \) is \( f_{i-n}^{c_i} \) restricted to \( \psi_{i-n}^{c_i}(y_{i+1}) \), then \( h_{i-n}^{c_i} \) is a continuous function from \( \psi_{i-n}^{c_i}(y_{i+1}) \) into \( \psi_{i-n}^{c_i}(y) \). Let \( C_m \) be the inverse limit of the inverse sequence \( \psi_{i-n}^{c_i}(y), h_{i-n}^{c_i} \). Since each \( \psi_{i-n}^{c_i}(y) \) is compact and connected, lemma 1 implies that \( C_m \) is connected. The proof of the theorem then follows since \( C_m \) is \( \psi_{i-n}^{c_i}(y) \).

A continuous function \( f : X \to Y \) is said to be light provided that, for each \( y \in Y \), the inverse image \( f^{-1}(y) \) is totally disconnected.

Lemma 2. The inverse limit of totally disconnected spaces is totally disconnected.