$g[\overline{W}]\subset\overline{W}$ and diam $\overline{W}<\operatorname{diam}B(q,a)<\gamma$, g is a γ-map. Then Lemma 1 assures us that the function f, defined for each $E\in 2^X$ by $f(E)=g(E)$ is a continuous γ-map of 2^X. Finally, f misses A because no image under f contains $W\setminus a'b'$, which is a non-empty subset of A.

Case 2. Some point q of A is not a local separating point of X. Let $\gamma>0$ be given so that $B(q,\gamma/2)\neq X$. Let $N=W\cup bW$ be the closed neighborhood of q of Lemma 5, where $N\subset B(q,\gamma/2)$. Choose any closed neighborhood M of q such that $M\subset W$. Then $M\cup bW$ is a closed subset of N. Choose any $p\in bW$. Define g_1: $M\cup bW\to 2^{bW}$ as follows:

$$g_1(x)=\begin{cases}\{x\} & \text{if} \quad x\in bW,\\ \{p\} & \text{if} \quad x\in M.\end{cases}$$

Then, because bW and M are closed and disjoint, g_1 is continuous. Because bW is a Peano continuum, 2^{bW} is an AR. Then we have a continuous extension g_2 of g_1 to all of N, g_2: $N\to 2^{bW}$. Note that $q\notin g_2(x)$ for every $x\in N$, because $q\notin bW$. Now define g: $X\to 2^X$ as follows:

$$g(x)=\begin{cases}\{x\} & \text{if} \quad x\in\overline{X\setminus N},\\ g_2(x) & \text{if} \quad x\in N.\end{cases}$$

Now g is well-defined because the boundary $N\cap\overline{X\setminus N}$ of N is a subset of bW, and $g_2(x)=g_1(x)=\{x\}$ on bW. Also, g is continuous on X and $\varrho(\{x\},g(x))<\gamma$ for all $x\in X$. Hence the conditions of Lemma 2 are satisfied by g, and the function f: $2^X\to 2^X$ defined by $f(E)=\bigcup\{g(x)|\ x\in E\}$ is a continuous γ-map. Finally, f misses A because $q\notin g(x)$ for each $x\in X$.

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Compactification and the continuum hypothesis

by

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Introduction. The purpose of this note is to show the equivalence of the continuum hypothesis with a statement about the metrizable compactifications of a non-compact separable metric space. Also we give a necessary and sufficient condition that $2^{\omega_\alpha}=\omega_{\alpha+1}$ for any ordinal α.

Notation. We use the notation of [2] and [3]. In [3] there is a description of the well-known correspondence between the compactifications of a completely regular space X and the closed subrings of $C^*(X)$ which contain the constants and generate the topology of X. In this paper we call a subring of $C^*(X)$ *regular* if it contains the constants and generates the topology of X.

We denote a cardinal by ω and consider the cardinals as a subclass of the ordinals in the usual way. If α is an ordinal, then ω_α is the α-th infinite cardinal.

Main results. The results of this paper are Theorem 1 and Theorem 2.

THEOREM 1. *The continuum hypothesis holds if and only if for some (resp. all) non-compact separable metric space X, the Stone-Čech compactification, βX, of X is the supremum of a chain of metric compactifications of X.*

Proof. Suppose that the continuum hypothesis holds, i.e., $2^{\omega_0}=\omega_1$. Then $|C^*(X)|=2^{\omega_0}$ and $C^*(X)$ can be subscripted $C^*(X)=\{g_\alpha:\ \alpha<\omega_1\}$ by the countable ordinals. Let F be a closed regular separable subring of $C^*(X)$. Then $e_F\beta F$ corresponds to a metric compactification of X. Now let $\gamma<\omega_1$ and let F_γ be the smallest closed regular subring of $C^*(X)$ containing F and $\{g_\alpha:\ \alpha\leqslant\gamma\}$. Then F_γ is separable and $\{e_{F_\gamma}\beta F_\gamma:\ \gamma<\omega_1\}$ is a chain of metric compactifications of X. Since $\bigcup_{\gamma<\omega_1}F_\gamma=C^*(X)$ we must have $\sup\{e_{F_\gamma}\beta F_\gamma\}=\beta X$.

Now let us show the converse. Let $\{x_i\}$ be a countable closed discrete subset of X which is non-compact separable metric. Let $A\subset\{x_i\}$ and associate with A the function f_A where $f_A|\ \{x_i\}=\chi_A$, the characteristic function of A on $\{x_i\}$. Then $\{f_A:\ A\subset\{x_i\}\}$ is a closed discrete subset of

$C^*(X)$ with cardinality 2^{ω_0} and with $\|f_A - f_B\| \geqslant 1$ for $A \neq B$ subsets of $\{x_i\}$. Now suppose that $\{g_\gamma Z_\gamma: \gamma \in \Gamma\}$ is a chain of metric compactifications of X with $\sup\{g_\gamma Z_\gamma\} = \beta X$. Then let $F_\gamma = \{f \circ g_\gamma: f \in C^*(Z_\gamma)\}$. Then F_γ is the closed regular subring of $C^*(X)$ associated with $g_\gamma Z_\gamma$. Since $\{g_\gamma Z_\gamma\}$ is a chain, so is $\{F_\gamma\}$ with $\bigcup_{\gamma \in \Gamma} F_\gamma = C^*(X)$. By transfinite induction define a function $h: \{\alpha < \omega_1\} \to \{F_\gamma\}$ such that for $\alpha < \beta < \omega_1$, $h(\alpha) \subset h(\beta)$ and $h(\alpha) \neq h(\beta)$. Then it can be shown that $\bigcup_{\alpha < \omega_1} h(\alpha) = C^*(X)$. Consider the map $F: \{A: A \subset \{x_i\}\} \to \{\alpha: \alpha < \omega_1\}$ defined by $F(A) = \min\{\alpha: f_A \in h(\alpha)\}$. Then F is at most countable to one and onto a cofinal subset of $\{\alpha: \alpha < \omega_1\}$. This implies that $\omega_1 = 2^{\omega_0}$.

THEOREM 2. *If X is discrete and $|X| = \omega_\alpha$, then $\omega_{\alpha+1} = 2^{\omega_\alpha}$ if and only if βX is the supremum of a chain of compactifications of X each of which has weight ω_α.*

The proof of Theorem 2 is similar to that of Theorem 1 and so is omitted.

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Some properties of the induced map

by

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1. Introduction. We use the notation and terminology of Eilenberg and Steenrod [1] for inverse limit sequences (the index set I is the positive integers) of topological spaces and continuous functions. If (X, f) is an inverse limit sequence, then we have the bonding maps $f_n^m: X_m \to X_n$ $(n \leqslant m)$ and the continuous projections $f_n: X_\infty \to X_n$. By a map Ψ from the inverse limit sequence (X, f) to the inverse limit sequence (Y, g) we mean a sequence of continuous functions $\psi_n: X_n \to Y_n$ such that $\psi_n f_n^{n+1} = f_n^{n+1}\psi_{n+1}$ for all $n \in I$. The inverse limit of Ψ is the function ψ_∞ from X_∞ into Y_∞ such that if $x \in X_\infty$, $x = (x_1, x_2, x_3, \ldots)$, then $\psi_\infty(x) = \big(\psi_1(x_1), \psi_2(x_2), \psi_3(x_3), \ldots\big)$.

Eilenberg and Steenrod have shown that (1) ψ_∞ is a continuous function from X_∞ into Y_∞ (Theorem 3.13), and (2) if each ψ_n is a 1-1 function of X_n onto Y_n, then ψ_∞ is a 1-1 function of X_∞ onto Y_∞ (Theorem 3.15). In this note we investigate further the relationship between properties of the ψ_n and properties of ψ_∞.

2. Periodicity. A continuous function f from X into itself is said to be *periodic* provided there exists a positive integer n such that $f^n(x) = x$ for all $x \in X$. The least such integer n is called the *period* of f. Assuming X to be metric, f is said to be *almost periodic* provided that for any $\varepsilon > 0$ there exists a positive integer n such that $d\big(x, f^n(x)\big) < \varepsilon$ for all $x \in X$.

Let Ψ be a map of (X, f) into itself, where each X_n is metric. The following example shows that ψ_∞ may fail to be periodic even though each ψ_n is periodic. For each $n \in I$, let (1) $X_n = \{z \in R^2 | |z| = 1\}$, (2) $f_n^{n+1}: X_{n+1} \to X_n$ be defined by $f_n^{n+1}(z) = z^2$, and (3) $\psi_n: X_n \to X_n$ be defined by $\psi_n(z) = \exp(2\pi i/2^n)\cdot z$. Then $\Psi = \{\psi_n\}$ is a map of (X, f) into itself such that each ψ_n is periodic but ψ_∞ is not. However, noting ψ_∞ is almost periodic does suggest the following:

THEOREM 1. *If each ψ_n is periodic, then ψ_∞ is almost periodic.*