On the conjecture $2^X \approx I^\omega$

by

Neil R. Gray (Washington)

1. It is a conjecture of long standing [7] that if X is a Peano continuum, then its hyperspace 2^X is homeomorphic with the Hilbert cube I^ω. It is known that these spaces share several topological properties — for example, both are absolute retracts [7]. The purpose of this paper is to show that 2^X and I^ω share another topological property: in each of these spaces every point is *unstable*. This will be accomplished in two steps (Theorem 1 and Theorem 2) in order to make the methods used more understandable.

THEOREM 1. *If X is a polyhedron, then every point of 2^X is unstable.*

(By a *polyhedron* we mean the geometric realization of a finite, connected, simplicial complex.)

THEOREM 2. *If X is a Peano continuum, then every point of 2^X is unstable.*

DEFINITION. If (X, d) is a metric space, then 2^X is the space of non-empty closed subsets of X metrized by the Hausdorff metric, ϱ. We define the Hausdorff metric as follows: for each $x \in X$ and $E \in 2^X$, $\varepsilon > 0$ we define $\operatorname{dist}(x, E) = \inf\{d(x, y)|\ y \in E\}$ and $V_\varepsilon(E) = \{x \in X|\ \operatorname{dist}(x, E) < \varepsilon\}$. Then, if E and E' are points of 2^X we define $\varrho(E, E') = \inf\{\varepsilon > 0|\ E \subset V_\varepsilon(E')$ and $E' \subset V_\varepsilon(E)\}$.

DEFINITION. A point p in a space X is called *unstable* if for each open neighborhood U of p there is a homotopy h_t: $X \to X$ such that $h_0 = 1$, $p \notin h_1(X)$, and for all t, $h_t|_{X \setminus U} = 1$ and $h_t(U) \subset U$. (Here 1 denotes the identity mapping on X.) Thus a point p in a space X is unstable provided that every neighborhood U of p admits a deformation which is stationary on the boundary of U and which does not move any point onto p. For example, if X is a closed interval, then its unstable points are its endpoints. A point which is not unstable is called *stable*. It is clear that the property of being stable or unstable is a topological property.

That each point of I^ω is unstable can be seen as follows: the point whose every coordinate is zero is easily seen to be unstable (in much the same way as with the endpoint of an interval). Then, since we are dealing with a topological property and since I^ω is known to be homogeneous (see [4]), every point is unstable.

We will need the following results on unstable points and Peano continua.

THEOREM A [2]. *If (X, d) is a compact ANR and $p \in X$, then p is unstable if and only if for each $\gamma > 0$ there is a continuous γ-map of X which misses p.*

(A γ-map of X is a map $f\colon X \to X$ such that $d(x, f(x)) < \gamma$ for each $x \in X$. We say f misses p if $p \notin f(X)$.)

THEOREM B [1]. *If X is a Peano continuum, then there exists a convex metric for X which is compactible with the topology on X.*

(A metric d on X is convex if for each pair of points a, b in X there is a set $ab \subset X$ which is isometric with the closed line interval $[0; d(a, b)]$ under an isometry h such that $h(0) = a$ and $h(d(a, b)) = b$. There may be more than one such set.)

THEOREM C [7]. *If X is a Peano continuum, then 2^X is an AR.*

Combining theorems A and C with the fact that 2^X is compact, we see that our work is somewhat simplified. To show that a point $A \in 2^X$ is unstable we shall construct the γ-map of Theorem A rather than the homotopy of the definition.

In all that follows X is a Peano continuum and d is a convex metric on X. Lower case Roman letters $a, b, \ldots, y, z$ denote points of X while upper case letters $A, B, \ldots, Y, Z$ denote points of the metric space 2^X with metric ϱ. For example, $B(x, \gamma) = \{y \in X \mid d(x, y) < \gamma\}$ and $B(A, \gamma) = \{E \in 2^X \mid \varrho(A, E) < \gamma\}$.

2. Some lemmas.

LEMMA 1. *If g is a continuous γ-map of X, then the function f defined for each $E \in 2^X$ by $f(E) = g(E)$ is a continuous γ-map of 2^X.*

LEMMA 2. *If $g\colon X \to 2^X$ is continuous and $\varrho(\{x\}, g(x)) < \gamma$ for all $x \in X$, then the function f defined on 2^X by $f(E) = \bigcup\{g(x) \mid x \in E\}$ is a continuous γ-map of 2^X.*

LEMMA 3. *If $0 < \delta < \gamma$, then the function f defined on 2^X by $f(E) = \overline{V_\delta(E)}$ is a continuous γ-map of 2^X. Also, $\varrho(E, f(E)) \leqslant \delta$ for each E in 2^X.*

The proofs of these lemmas (see [3]) are rather routine exercises in the use of the Hausdorff metric. (Only Lemma 3 requires the convexity of d.) The next lemma is rather complicated — it exists to handle an especially awkward part of the theorem.

LEMMA 4. *Suppose that $A \in 2^X$, $q \in A$, $B(q, a) \subset A$, and $B(q, a)$ is isometric with a line interval of length $2a$. Then A is unstable in 2^X.*

Proof. We may as well assume that $B(q, a) = (-a; a)$, with $q = 0$. (We can use the given isometry to do this.) We suppose that $\gamma > 0$ is given, and we seek a γ-map which misses A. Our method is to map each element E of 2^X to its image in 2^X by altering E in a suitable fashion. Since our only information about A is that it contains the open interval $(-a, a)$ it seems natural to map A to $A\setminus(-\delta; \delta)$ for some $\delta < a$. The operation of "removing intervals" (of varying sizes) will not be continuous (essentially for the reason that one can find a sequence $\{E_n\}$ in 2^X such that $E_n \to A$ but $0 \notin E_n$ for each n). We shall remedy this situation by adjoining the point 0 to all sets which are sufficiently close to A. In order to make this "adjoining map" (which we shall call g) continuous we shall have to pave the way with a mapping h which has the property of making each $h(E)$ "nice" near 0. We shall define three maps: h, g, and f so that $2^X \xrightarrow{h} h(2^X) \xrightarrow{g} gh(2^X) \xrightarrow{f} fgh(2^X)$, where h and g have the stated properties, and f will be the map that "removes intervals". When convenient we shall use the usual properties of the line and write $x < y$, $d(x, y) = |y - x|$, etc., for points in $(-a; a)$.

DEFINITION OF h. Choose any $\mu > 0$ so that $\mu < \min\{\frac{1}{4}a, \frac{1}{4}\gamma\}$. Define h on 2^X by $h(E) = \overline{V_\mu(E)}$. Then since $h(E)$ contains $B(x, \mu)$ for each x in E, $h(E) \cap [0; 2\mu]$ will consist of at most two intervals. More precisely, for each $E \in 2^X$ such that $h(E) \cap [0; 2\mu] \neq \emptyset$, $h(E) \cap [0; 2\mu]$ is of the form

(I) $[0; e_1] \cup [e_2; 2\mu]$ with $0 \leqslant e_1 < e_2 \leqslant 2\mu$ or

(II) $[0; e_1]$ with $0 \leqslant e_1 < 2\mu$ or

(III) $[e_2; 2\mu]$ with $0 < e_2 \leqslant 2\mu$ or

(IV) $[0; e_2]$ with $e_2 = 2\mu$.

These statements are immediate from the assumption that d is a convex metric and that $B(q, a) = (-a; a)$.

DEFINITION OF g. We shall define g on $h(2^X)$. Our only requirements for g (besides continuity) is that g should adjoin 0 to sets $h(E)$ which are sufficiently close to A, and should not move any point very far.

Remark 1. Consider an open interval $(a; b) \subset [-\mu; 2\mu]$. If $E \in 2^X$ and $E \cap (a; b) = \emptyset$, then $\varrho(E, A) \geqslant \frac{1}{2}(b - a)$. This is obvious from the equation

$$(a; b) = B\left(\frac{a+b}{2}, \frac{b-a}{2}\right).$$

Remark 2. Suppose $E \in 2^X$ and $x \in E$. Suppose $E' \in 2^X$ is obtained from E by adjoining some points of $B(x, \delta)$ and omitting some points of $E \cap B(x, \delta)$ in such a way that $E' \cap B(x, \delta) \neq \emptyset$. Then $\varrho(E, E') < 2\delta$. This is clear because $V_{2\delta}(E) \supset E \cup B(x, \delta) \supset E'$ and $V_{2\delta}(E') \supset [E\setminus B(x, \delta)] \cup B(x, \delta) \supset E$.

We define g by showing how to alter each $h(E)$ in $h(2^X)$ to obtain $gh(E)$. We shall do it in such a way that $0 \in gh(E)$ whenever $\varrho(gh(E), A) \leqslant \frac{1}{3}\mu$. First, if $h(E) \cap [0; 2\mu] = \emptyset$, then $gh(E) = h(E)$. If $h(E) \cap [0; 2\mu] \neq \emptyset$ and $2\mu \notin h(E)$, then $h(E)$ is of type I or II and 0 is already in $h(E)$, so we again set $gh(E) = h(E)$. The only case remaining is that where $2\mu \in h(E)$. Then $h(E)$ is of type III or IV, and we focus our attention on e_2. First define $e = \max\{\mu, e_2\}$, and note that $e \in h(E)$. Then define $e' = e - \operatorname{dist}(e, 2\mu) = e - (2\mu - e) = 2e - 2\mu$. (Note that $e' = 0$ if $e_2 \leqslant \mu$, and $e' = 2\mu$ if $e_2 = 2\mu$.) Then e' is the point we adjoin to $h(E)$. Thus we define

$$gh(E) = \begin{cases} h(E) & \text{if} \quad 2\mu \notin h(E)\,, \\ h(E) \cup \{e'\} & \text{if} \quad 2\mu \in h(E)\,. \end{cases}$$

By examining each of the possible cases I through IV, we see that g is continuous on $h(2^X)$, and that $\varrho(h(E), gh(E)) \leqslant \mu$ for each set $h(E)$. Finally, we want to show that $0 \in gh(E)$ if $\varrho(gh(E), A) \leqslant \mu/3$.

Suppose $\varrho(gh(E), A) \leqslant \mu/3$ and $0 \notin gh(E)$. We shall arrive at a contradiction. By Remark 1, $gh(E) \cap [0; 2\mu] \neq \emptyset$, so $h(E)$ is of type III. Since we are assuming $\varrho(gh(E), A) \leqslant \mu/3$ we cannot have $(0; \frac{2}{3}\mu) \cap gh(E) = \emptyset$. (Other wisefor some $\varepsilon > 0$, $(-\varepsilon; \frac{2}{3}\mu) \cap gh(E) \neq \emptyset$ and; $(gh(E), A) \geqslant \mu + \varepsilon/2$.) Thus either $h(E) \cap (0; \frac{2}{3}\mu) \neq \emptyset$ or $e' \in (0; \frac{2}{3}\mu)$. But both these statements are incompatible with our hypotheses: If $e' \in (0; \frac{2}{3}\mu)$, then e_2 is the midpoint of the interval $[e'; 2\mu]$. Since $gh(E) \cap (e'; e_2) = \emptyset$ and $\frac{1}{2}(e_2 - e') = \frac{1}{4}(2\mu - e') > \frac{1}{4}(2\mu - \frac{2}{3}\mu) = \mu/3$, we have $\varrho(gh(E), A) > \mu/3$. If $h(E) \cap (0; \frac{2}{3}\mu) \neq \emptyset$, then, since $h(E)$ is of type III, we have $e_2 \leqslant \frac{2}{3}\mu < \mu$. Thus $e' = 0$ and $0 \in gh(E)$.

Definition of f. The function f is going to remove the open set $(-\mu/3; \mu/3)$ from A. To make f continuous we need some method for continuously varying the interval to be removed. To do this we define a real-valued function δ on 2^X by $\delta E = \max\{0, \mu/3 - \varrho(gh(E), A)\}$. Then δ is continuous because g and h are continuous. Now we define f on $gh(2^X)$:

$$fgh(E) = \begin{cases} gh(E) & \text{if} \quad \varrho(gh(E), A) \geqslant \mu/3\,, \\ [gh(E) \setminus (-\delta E; \delta E)] \cup \{-\delta E; \delta E\} & \text{if} \quad \varrho(gh(E), A) < \mu/3\,. \end{cases}$$

From Remark 2 and the fact that $0 \in gh(E)$ if $\varrho(gh(E), A) \leqslant \mu/3$, f is a $\frac{4}{3}\mu$-map (from one subset of 2^X into another). It is clear from the definition that f misses A. We now show that f is continuous. It is clear that f is continuous at any point $gh(E)$ such that $\varrho(gh(E), A) > \mu/3$ because f is the identity on an open (in $gh(2^X)$) neighbourhood of $gh(E)$.

Now suppose $\varrho(gh(E), A) = \mu/3$. Then since $\delta E = 0$, $\delta E'$ will be small for points $gh(E')$ near $gh(E)$. For such points we have either $fgh(E') = gh(E')$ or $fgh(E')$ is obtained from $gh(E')$ by altering $gh(E')$ in a small neighborhood of $0 \in gh(E')$. It then follows from Remark 2 that f is continuous at $gh(E)$.

Finally we consider the case $\varrho(gh(E), A) < \mu/3$. Since $\delta E > 0$ we can choose $\eta > 0$ so that $\delta E' > 0$ for any point $gh(E')$ such that $\varrho(gh(E), gh(E')) < \eta$. Let $B = gh(2^X) \cap B(gh(E), \eta)$. We can write $f = f_2 f_1$, where for each $gh(E') \in B$, $f_1(gh(E')) = gh(E') \cup \{-\delta E', \delta E'\}$ and $f_2(f_1 gh(E')) = f_1 gh(E') \setminus (-\delta E'; \delta E')$. Since B is a neighborhood (in $gh(2^X)$) of $gh(E)$, it will suffice to prove that f_1 is continuous at $gh(E)$ and f_2 is continuous at $f_1 gh(E)$. Since δ is continuous and $\delta E' > 0$ for each $gh(E') \in B$, it is clear that f_1 is continuous at $gh(E)$. We now show continuity of f_2 at $f_1 gh(E)$. Let $\varepsilon > 0$ be given. Choose $\beta \in (0; \varepsilon)$ so that $|\delta E - \delta E'| < \varepsilon$ when $\varrho(f_1 gh(E), f_1 gh(E')) < \beta$. Let $x \in f_2 f_1 gh(E)$. Since $x \in f_1 gh(E)$ there is a point $x' \in f_1 gh(E')$ with $d(x, x') < \beta$. If $x' \notin (-\delta E'; \delta E')$, then $x' \in f_2 f_1 gh(E')$, so $\operatorname{dist}(x, f_2 f_1 gh(E')) \leqslant d(x, x') < \beta$. If $x' \in (-\delta E'; \delta E')$, then, since $x \notin (-\delta E; \delta E)$, at least one of $|x - \delta E'|$ and $|x - (-\delta E')|$ is less than the larger of $|x - x'|$ and $|\delta E - \delta E'|$. Then $\operatorname{dist}(x, f_2 f_1 gh(E')) \leqslant \min\{|x - \delta E'|, |x - (-\delta E')|\} < \max\{|x - x'|, |\delta E - \delta E'|\} < \varepsilon$. Thus $V_\varepsilon(f_2 f_1 gh(E')) \supset f_2 f_1 gh(E)$. In a similar fashion we see $V_\varepsilon(f_2 f_1 gh(E)) \supset f_2 f_1 gh(E')$, and so conclude that $\varrho(f_2 f_1 gh(E), f_2 f_1 gh(E')) < \varepsilon$. Thus f_2 is continuous at $f_1 gh(E)$.

We have therefore defined a continuous function $fgh\colon 2^X \to 2^X$. This function misses A because f does. Also, it is a γ-map: if $E \in 2^X$ we have $\varrho(E, fgh(E)) \leqslant \varrho(E, h(E)) + \varrho(h(E), gh(E)) + \varrho(gh(E), fgh(E)) < \mu + \mu + \frac{4}{3}\mu < 4\mu < \gamma$. This completes the proof of Lemma 4.

3. Proof of Theorem 1 and Theorem 2.

Theorem 1. *If X is a polyhedron, then every point of 2^X is unstable.*

Proof. Let $A \in 2^X$. If $\operatorname{int} A = \emptyset$, then we suppose d to be a convex metric on X. Then, given $\varepsilon > 0$, we let $f\colon 2^X \to 2^X$ be defined by $f(E) = \overline{V_{\varepsilon/2}(E)}$. Then f is a continuous ε-map and f misses A because each $f(E)$ has non-empty interior. If $\operatorname{int} A \neq \emptyset$, then there is a point $q \in \operatorname{int} A$ with an open neighborhood which is a Euclidean n-ball lying in $\operatorname{int} A$. We consider two cases.

Case 1: $n = 1$. Again we suppose d to be a convex metric on X. Then some $B(q, a)$ is (isometric with) a line interval, so Lemma 4 ensures that A is unstable.

Case 2: $n \geqslant 2$. For this we do not impose a convex metric on X. Let $\varepsilon > 0$ be given. Let the open n-ball neighborhood of q be B_1, centered at q, of radius $\delta < \varepsilon/2$, and with boundary an $(n-1)$-sphere S_1. Let B_2 be an open n-ball centered at q with radius less than δ. Choose any point $p \in S_1$. Define $g_1\colon S_1 \cup \overline{B_2} \to 2^{S_1}$ by

$$g_1(x) = \begin{cases} \{x\} & \text{if} \quad x \in S_1, \\ \{p\} & \text{if} \quad x \in \overline{B}_2. \end{cases}$$

Now S_1 is a Peano continuum, so 2^{S_1} is an AR. Furthermore, $S_1 \cup \overline{B}_2$ is a closed subset, so there is a continuous extension g_2 of g_1 to all of $\overline{B}_1$. Now we define $g\colon X \to 2^X$ by

$$g(x) = \begin{cases} \{x\} & \text{if} \quad x \notin B_1, \\ g_2(x) & \text{if} \quad x \in \overline{B}_1. \end{cases}$$

Then g is continuous on X because it is continuous on both $\overline{B}_1$ and $X \backslash B_1$. As was remarked earlier, Lemma 2 does not require that the metric on X be convex. Thus the map $f(E) = U\{g(x)|\ x \in E\}$ is a continuous map of 2^X into itself. Also, since $g(x) = \{x\}$ if $x \in B_1$ and $g(B_1) \subset \overline{B}_1$, we have $\varrho(\{x\}, g(x)) \leqslant \operatorname{diameter} \overline{B}_1 < \varepsilon$. Thus f is an ε-map. Finally, f misses A because no $f(E)$ contains q.

The proof for Theorem 2 will be similar to that for Theorem 1. The crux of the matter in Case 2 of Theorem 1 is the existence of a neighborhood whose boundary is a Peano continuum. In the more general situation we are able to get by with a neighborhood whose boundary is *contained* in a Peano continuum. For this we need another lemma. (Definitions of terms here can be found in [5] and [6].)

LEMMA 5. *If q is not a local separating point of X and $V \neq X$ is any open neighborhood of q, there exists a neighborhood N of q such that*

(i) *N is closed and $N \subset V$,*

(ii) *$N = W \cup bW$, where $q \in W$, W is open, bW is a Peano continuum containing the boundary of N, and $bW \cap W = \emptyset$.*

Proof. We assume that d is a convex metric on X. There is a neighborhood P of q such that $B(q, \alpha) \subset P \subset V$ for some $\alpha > 0$, and P is a Peano continuum. Then, because q is not a local separating point of X, q is not a cut point of P. Let $F(q, \alpha)$ denote the boundary of $B(q, \alpha)$. Then $F(q, \alpha)$ is compact so by a result of Wilder ([6], p. 82), there is a $\delta > 0$ such that every pair of points x, y of $F(q, \alpha)$ is connected by an arc $A(x, y)$ lying in $P \backslash B(q, \alpha)$. Fix $x_0 \in F(q, \alpha)$ and let

$$F = U\{A(x_0, y)|\ y \in F(q, \alpha) \text{ and } A(x_0, y) \subset P \backslash B(q, \alpha)\}.$$

Then F is connected and $F(q, \alpha) \subset F$. Let β be a positive number such that $\beta < \frac{1}{2}\min\{\delta, \operatorname{dist}(P, X \backslash V)\}$. Define $bW = \overline{T_\beta(F)}$. (See [5], p. 21 for this construction.) Note that:

1. $q \notin bW$ because $\beta < \frac{1}{2}\delta$, and $\delta \leqslant \operatorname{dist}(q, F)$ because $F \subset P \backslash B(q, \delta)$.

2. bW is a Peano continuum because $T_\beta(F)$ has property S ([5], p. 21).

3. $bW \subset V$ because $\beta < \frac{1}{2}\operatorname{dist}(P, X \backslash V)$ and $F \subset P$ together ensure that $\overline{T_\beta(F)} \subset V$.

Finally, we define $W = B(q, \alpha) \backslash bW$ and define $N = W \cup bW$. Clearly W is open and $W \cap bW = \emptyset$. Note that by 1, $q \in W$. Note also that $N \subset V$ by 3 and the definition of N. Thus the only claims in (i) and (ii) that remain to be verified are:

(a) *N is closed,*

(b) *the boundary of $N \subset bW$.*

Proof of (a). Because of the relations $B(q, \alpha) \cup F(q, \alpha) = \overline{B(q, \alpha)}$ and $F(q, \alpha) \subset bW$ we have $N = B(q, \alpha) \cup bW = B(q, \alpha) \cup F(q, \alpha) \cup bW = \overline{B(q, \alpha)} \cup bW$. Thus N is the union of two closed sets.

Proof of (b). Suppose that x bolongs to the boundary of N. Because N is closed, $x \in N$. Then $x \notin W$ because W is an open subset of N. Thus $x \in N \backslash W = bW$.

THEOREM 2. *If X is a Peano continuum, then every point of 2^X is unstable.*

Proof. Let $A \in 2^X$. If $\operatorname{int} A = \emptyset$ we proceed exactly as in Theorem 1. If $\operatorname{int} A \neq \emptyset$ we consider two cases.

Case 1. Each point of A is a local separating point of X. Let $\gamma > 0$ be given. We can find points a and b of A so that a line segment ab lies in $\operatorname{int} A$. Let us consider ab to be parameterized; take a and b to be real numbers and take ab to be the closed interval $[a; b]$. If x and y are in ab we use $(x; y)$ to denote the open interval $\{z|\ z \in ab \text{ and } x < z < y\}$.

The points of order 2 in X are dense in ab because all but a countable number of the local separating points of X have order 2 ([5], p. 61), and all of the points of ab are local separating points of X. Thus we can choose $q \in ab$, $a < q < b$, so that q is of order 2. If, for some $\alpha > 0$, we have $B(q, \alpha) = (q-\alpha; q+\alpha)$, then we have an open subset of A which is a line interval. Then by Lemma 4, A is unstable in 2^X. Otherwise, every $B(q, \alpha)$ contains points not on $(q-\alpha; q+\alpha)$. Then choose α, $0 < \alpha < \gamma/2$, and an open set W, $q \in W \subset \overline{W} \subset B(q, \alpha)$ so that the boundary of W consists of two points of ab, say of point a', b', where $a < a' < q < b' < b$. This choice is possible because every small neighborhood of q must contain in its boundary two points of ab, yet we can choose an arbitrarily small neighborhood of q so as to contain only two points on its boundary.

Now $ab \cap \overline{W}$ is a closed interval (it is the subinterval of ab with endpoints a' and b'), so it is a retract of $\overline{W}$. We denote it by $a'b'$. Note that $W \backslash a'b' \neq \emptyset$. Let $g_1\colon \overline{W} \to a'b'$ be a retraction map. Now define $g\colon X \to X$ as follows:

$$g(x) = \begin{cases} g_1(x) & \text{if} \quad x \in \overline{W}, \\ x & \text{if} \quad x \notin W. \end{cases}$$

Then g is well-defined because $\overline{W} \cap \overline{(X \backslash W)} = \operatorname{bdry} W = \{a', b'\}$, and $g_1(a') = a'$, $g_1(b') = b'$. Clearly, then, g is continuous on X. Also, since

$g[\overline{W}] \subset \overline{W}$ and diam $\overline{W} <$ diam$B(q, a) < \gamma$, g is a γ-map. Then Lemma 1 assures us that the function f, defined for each $E \in 2^X$ by $f(E) = g(E)$ is a continuous γ-map of 2^X. Finally, f misses A because no image under f contains $W \backslash a'b'$, which is a non-empty subset of A.

Case 2. Some point q of A is not a local separating point of X. Let $\gamma > 0$ be given so that $B(q, \gamma/2) \neq X$. Let $N = W \cup bW$ be the closed neighborhood of q of Lemma 5, where $N \subset B(q, \gamma/2)$. Choose any closed neighborhood M of q such that $M \subset W$. Then $M \cup bW$ is a closed subset of N. Choose any $p \in bW$. Define g_1: $M \cup bW \to 2^{bW}$ as follows:

$$g_1(x) = \begin{cases} \{x\} & \text{if} \quad x \in bW, \\ \{p\} & \text{if} \quad x \in M. \end{cases}$$

Then, because bW and M are closed and disjoint, g_1 is continuous. Because bW is a Peano continuum, 2^{bW} is an AR. Then we have a continuous extension g_2 of g_1 to all of N, g_2: $N \to 2^{bW}$. Note that $q \notin g_2(x)$ for every $x \in N$, because $q \notin bW$. Now define g: $X \to 2^X$ as follows:

$$g(x) = \begin{cases} \{x\} & \text{if} \quad x \in \overline{X \backslash N}, \\ g_2(x) & \text{if} \quad x \in N. \end{cases}$$

Now g is well-defined because the boundary $N \cap \overline{X \backslash N}$ of N is a subset of bW, and $g_2(x) = g_1(x) = \{x\}$ on bW. Also, g is continuous on X and $\varrho(\{x\}, g(x)) < \gamma$ for all $x \in X$. Hence the conditions of Lemma 2 are satisfied by g, and the function f: $2^X \to 2^X$ defined by $f(E) = U\{g(x) | x \in E\}$ is a continuous γ-map. Finally, f misses A because $q \notin g(x)$ for each $x \in X$.

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WESTERN WASHINGTON STATE COLLEGE

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Compactification and the continuum hypothesis

by

James Keesling (Florida)

Introduction. The purpose of this note is to show the equivalence of the continuum hypothesis with a statement about the metrizable compactifications of a non-compact separable metric space. Also we give a necessary and sufficient condition that $2^{\omega_\alpha} = \omega_{\alpha+1}$ for any ordinal α.

Notation. We use the notation of [2] and [3]. In [3] there is a description of the well-known correspondence between the compactifications of a completely regular space X and the closed subrings of $C^*(X)$ which contain the constants and generate the topology of X. In this paper we call a subring of $C^*(X)$ *regular* if it contains the constants and generates the topology of X.

We denote a cardinal by ω and consider the cardinals as a subclass of the ordinals in the usual way. If α is an ordinal, then ω_α is the α-th infinite cardinal.

Main results. The results of this paper are Theorem 1 and Theorem 2.

THEOREM 1. *The continuum hypothesis holds if and only if for some (resp. all) non-compact separable metric space X, the Stone-Čech compactification, βX, of X is the supremum of a chain of metric compactifications of X.*

Proof. Suppose that the continuum hypothesis holds, i.e., $2^{\omega_0} = \omega_1$. Then $|C^*(X)| = 2^{\omega_0}$ and $C^*(X)$ can be subscripted $C^*(X) = \{g_\alpha\colon \alpha < \omega_1\}$ by the countable ordinals. Let F be a closed regular separable subring of $C^*(X)$. Then $e_F \beta F$ corresponds to a metric compactification of X. Now let $\gamma < \omega_1$ and let F_γ be the smallest closed regular subring of $C^*(X)$ containing F and $\{g_\alpha\colon \alpha \leqslant \gamma\}$. Then F_γ is separable and $\{e_{F_\gamma} \beta F_\gamma\colon \gamma < \omega_1\}$ is a chain of metric compactifications of X. Since $\bigcup_{\gamma<\omega_1} F_\gamma = C^*(X)$ we must have $\sup\{e_{F_\gamma} \beta F_\gamma\} = \beta X$.

Now let us show the converse. Let $\{x_i\}$ be a countable closed discrete subset of X which is non-compact separable metric. Let $A \subset \{x_i\}$ and associate with A the function f_A where $f_A | \{x_i\} = \chi_A$, the characteristic function of A on $\{x_i\}$. Then $\{f_A\colon A \subset \{x_i\}\}$ is a closed discrete subset of