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Remarks on Anderson's paper "On topological infinite deficiency"

by

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Suppose that the topological space X is the product of $\aleph_0$ copies of an interval J which is either closed or open. A closed subset A of X is said to be of *infinite deficiency* (briefly: *deficient*) in X if there exists a homeomorphism h of X onto itself such that, for infinitely many i, the natural projections $\pi_i h(A)$ are (at most) one-point sets in the interior of J.

The sets of infinite deficiency have been systematically investigated by R. D. Anderson in [1], [3]. The importance of these sets lies in their topological negligibility property (see condition (g) of Theorem 1 in this paper) and the property of extending homeomorphisms (here: Theorem 5); both properties have been established in their final form by Anderson, but the pioneer work in this respect was done by Klee ([9], [10]). For other results concerning negligibility see also [5], [6], [7], [8]. The theory of deficient sets can easily be transferred to the case of separable infinite-dimensional Fréchet spaces.

The present paper is a contribution to the theory of infinite deficiency. In Section 1 we establish some topological characterizations of sets of infinite deficiency. One of them (condition (ii) in Theorem 2), applied to F_σ sets rather than to closed sets, gives a characterization of σ-deficient sets, i.e. of countable unions of deficient sets. This class of sets, being a natural generalization of deficient sets, is discussed in Section 2 (1). Finally, in Section 3 we establish a theorem on extending homeomorphisms to the pair: Hilbert cube Q and its pseudointerior s, which is an analogue of the above-mentioned theorem of Anderson, dealing with a single space X which is either Q or s.

Our results are derived from two theorems of Anderson, which are stated explicitly as Theorem 1 in Section 1 and Theorem 5 in Section 3.

(1) Added in proof. σ-deficient sets (sets of type Z_σ) and their relations to problems of negligibility have been studied by R. D. Anderson and his colaborators, see, e.g. [5].

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0. Preliminaries. By N we denote the set of non-negative integers. Greek letters: α, β, γ denote non-void subsets of N. We define $\alpha^{\perp} = N \setminus \alpha$. For every topological space Z and every $\alpha \subset N$ we denote by Z^{α} the product $\mathop{\mathbf{P}}_{n\in\alpha} Z_n$ with $Z_n = Z$, endowed with the usual product topology. By π_α we denote the natural projection $\pi_\alpha\colon Z^N \to Z^\alpha$. $Q = I^N$ $(I = [-1, 1])$ is the Hilbert cube, and $s = (-1, 1)^N$ its *pseudointerior*. We consider Q and s with the standard metric

$$d(x, y) = \sum_{n\in N} 2^{-n}|x_n - y_n| .$$

Let (Z, ϱ) be a metric space and Y a topological space. By $\bar{\varrho}$ we shall denote the metric $\bar{\varrho}(f, g) = \sup_{y\in Y} \varrho\big(f(y), g(y)\big)$ defined on the set of continuous functions from Y into Z. By Z^Y we shall denote the same set endowed with the compact-open topology. For any subset $K \subset Z$ we shall denote by $G(Z, K)$ the set of all the homeomorphisms of the pair (Z, K) onto itself (called *autohomeomorphisms* of the pair (Z, K)); $G(Z) \overset{\mathrm{df}}{=} G(Z, Z)$ (the set of all autohomeomorphisms of the space Z). By $e \in G(Z)$ we denote any identity map. Unless otherwise stated the spaces $G(Q, s)$ and $G(Q)$ will be considered with the metric $\bar{d}$.

We shall use the following lemma.

LEMMA 1. *For any complete metric space (Z, ϱ), the space $\big(G(Z), \Psi\big)$ with metric Ψ, given by:*

$$\Psi(f, g) = \bar{\varrho}(f, g) + \bar{\varrho}(f^{-1}, g^{-1})$$

is also a complete metric space. If the space (Z, ϱ) is compact, the metrics Ψ and $\bar{\varrho}$ induces the same topology on $G(Z)$.

An easy proof is left to the reader.

Let X be either Q or s. A closed subset K of X is called *straight* if either K is empty or if there exists an infinite set α such that $\pi_\alpha(K)$ is a subset of s consisting of a single point. We say that $K \subset X$ is of *infinite deficiency* if there exists an $f \in G(X)$ such that $f(K)$ is straight.

Sets of infinite deficiency will be also called briefly *deficient*; countable unions of deficient sets will be called *σ-deficient*.

We say that closed subset K of an infinite-dimensional separable Fréchet space F is of *infinite deficiency* if there exists an $f \in G(F)$ such that $f(K)$ is a subset of a closed linear subspace of infinite linear deficiency.

A subset K of a topological space Y has *property Z* in Y if K is closed and, for any non-void, open and homotopically trivial set U, $U \setminus K$ is also non-void and homotopically trivial.

1. Characterization of sets of infinite deficiency. Anderson's results for the sets of infinite deficiency can be summarized as follows.

THEOREM 1. *Let K be a closed subset of the Hilbert cube Q and let $M = K \cap s$. Then the following conditions are equivalent:*

(a) *K has infinite deficiency in Q;*

(b) *there exists an $f \in G(Q, s)$ such that $f(K)$ is straight in Q;*

(c) *K has property Z in Q;*

(d) *there exists an $f \in G(Q)$ such that $f(s \setminus K) = s$;*

(e) *there exists an $f \in G(Q)$ such that $f(s \cup K) = s$;*

(f) *M has infinite deficiency in the space s;*

(g) *for every subset $U \subset s$ which is open relative to s there is an $f \in G(s)$ such that $f(U) = U \setminus M$ and $f(x) = x$ for $x \in s \setminus U$;*

(h) *for every subset $U \subset s$ which is open relative to s the sets $U \setminus M$ and U are homeomorphic;*

(j) *M has property Z in s;*

(k) *for every homeomorphism f of s onto a Fréchet space F the image (M) has infinite deficiency in F.*

The above results can be found in [3] (not always explicitly) except (g), which is a corollary of Theorem 9.2 of [4]. The last four conditions can also be regarded as characterizations of deficient subsets of the space s. This follows from the fact that every closed subset of the space s can be represented in a form $K \cap s$ where K is the closure relative to Q of the given set. Let us note also that taking, in the condition (e), $f = e$, we find that every compact subset of s has infinite deficiency both with respect to s and with respect to Q. Similarly, by (d), every compact subset of the pseudoboundary $Q \setminus s$ has infinite deficiency in Q.

Now, using the above theorem, we shall prove an additional characterization of sets of infinite deficiency.

THEOREM 2. *Suppose that X is either Q or s and K is a closed subset of X. Then the following conditions are equivalent:*

(a) *K has infinite deficiency in X.*

(i) *There exists a homotopy $h\colon X \times [0, 1] \to X$ such that $h_0 = e$ and $h_t(X) \cap K = \emptyset$ for $t \in (0, 1]$.*

(ii) *For every $n \in N$, the set $\{f \in X^{I^n}\colon f(I^n) \cap K = \emptyset\}$ is dense in X^{I^n}.*

(iii) *For every metric space (Z, ϱ) including X as its retract, there is a retraction $r\colon Z \xrightarrow{\text{onto}} X$ such that $r^{-1}(K) = K$.*

Proof. (a) $\Rightarrow$ (i).

1. $X = Q$. By Theorem 1 (d), there exists an $f \in G(Q)$ such that $f(s \setminus K) = s$. The homotopy $h(x, t) = f^{-1}\big((1-t)\cdot f(x)\big)$ satisfies (i); moreover we have $h(s \times [0, 1]) \subset s$.

2. $X = s$. By Theorem 1, $\overline{K}$ is of infinite deficiency in Q. Consequently, there exists a suitable homotopy $h\colon Q\times[0,1]\to Q$ such that $h_t(Q)\cap\overline{K} = \emptyset$ for $t\in(0,1]$, $h_0 = e$ and $h(s\times[0,1])\subset s$. The homotopy $h|_{s\times[0,1]}$ satisfies (i).

(i) $\Rightarrow$ (ii) obvious.

(ii) $\Rightarrow$ (a). By the equivalence of (i) and (f) of Theorem 1 it is sufficient to demonstrate that K has property Z in X. Thus, let U be open, non-void and homotopically trivial, and let $f\colon \partial I^n\to U\setminus K$. By assumption, f has an extension $F\colon I^n\to U$. Let us note that the number

$$\varepsilon = \min\left[d\big(F(I^n), s\setminus U\big), d\big(f(\partial I^n), K\big)\right]$$

is positive; consequently, there exists a mapping $H\in s^{I^n}$ such that $\bar{d}(F, H) < \varepsilon$ and $H(I^n)\cap K = \emptyset$. Then $h = H|\partial I^n\colon \partial I^n\to U\setminus K$ is homotopically trivial and homotopic to f (the homotopy is given by e.g. $G(x,t) = tf(x) + (1-t)h(x)$). This implies that also $f\colon \partial I^n\to U\setminus K$ is homotopically trivial.

(i) $\Rightarrow$ (iii). Let f be a retraction Z onto X. We put $r(z) = h_{u(z)}\big(f(z)\big)$, $z\in Z$, where $u(z) = \min\big(1, \varrho(z, X)\big)$.

(iii) $\Rightarrow$ (i). We put $Z = X\times[0,1]$; let r be a retraction which satisfies condition (iii). We define $h_t(x)$ as $r(x,t)$.

Conditions (i) and (iii) have been introduced by W. Kuperberg in connection with the study of stable points. The equivalence (i) $\Leftrightarrow$ (iii) has also been established by W. Kuperberg. Clearly, Theorem 2 holds true for any separable Fréchet space F. The following sufficient conditions for being of infinite deficiency are consequences of Theorem 2.

Corollary 1. *Let K be a closed subsed of X ($X = Q$ or s) such that for every finite a, $\pi_{a\perp}(K)\neq X$. Then K is deficient in X.*

Proof. We shall show that condition (ii) of Theorem 2 is satisfied. Let $f\in X^{I^n}$ and $\varepsilon > 0$. Let us choose a finite set a such that $\sum_{n\notin a} 2^{-n} < \varepsilon$. We define $g\in I^{I^n}$ by

$$(\pi_a g)(x) = (\pi_a f)(x);\ (\pi_{a\perp} g)(x) = a, \quad x\in I^n,$$

where $a\in X\setminus\pi_{a\perp}(K)$. Then $g(I^n)\cap K = \emptyset$ and $\bar{d}(f, g) < \varepsilon$.

Corollary 2. *Suppose that F is a separable, infinite-dimensional Fréchet space, ϱ an invariant metric on F and $\tau_n\colon F\to F$ a sequence of projections such that for every $x\in F$, $\varrho\big(x, \tau_n(x)\big)\searrow 0$. Let $E_n = \tau_n(F)$. If K is a closed subset of F such that $K\cap\bigcup_{n\in N} E_n = \emptyset$, then K is of infinite deficiency in F.*

Proof. Let $f\in F^{I^n}$ and $\varepsilon > 0$. Since the sequence of functions $\varphi_l(x) = \varrho\big(f(x), \tau_l f(x)\big)$ is decreasing, by the Dini theorem there exists a $k\in N$ such that $\bar{\varrho}(f, \tau_l f) < \varepsilon$ for $l\geq k$. The map $g = \tau_k f$ satisfies

$$g(I^n)\cap K = \emptyset \quad\text{and}\quad \bar{\varrho}(f, g) < \varepsilon.$$

Remark. The class of subsets of Q satisfying the assumption of Corollary 1 coincides with the class of *weakly thin sets* in the sense of Anderson [1], and the statement of Corollary 1 concerning Q is a consequence of [1], Section 3.

2. Characterization of σ-deficient sets. Since X admits a complete metric, the Baire category theorem together with condition (ii) in Theorem 2 gives the following characterization of σ-deficient sets:

Proposition 1. *Let X be either Q or s and let K be an F_σ subset of X. Then K is σ-deficient in X if and only if for any $n\in N$ the set $\{f\in X^{I^n}\colon f(I^n)\cap K = \emptyset\}$ is dense in X^{I^n}.*

Corollary 3. *Let K be a σ-deficient subset of X. Then K is deficient in X if and only if K is closed in X.*

Proof. It is a consequence of Proposition 1 and Theorem 2. This result has been established by R. D. Anderson.

In order to obtain further characterizations of σ-deficient sets, we shall need the following lemma.

Lemma 2. *Suppose that $\varepsilon > 0$ and M_1, M_2, M_3 are deficient subsets of Q such that $M_1\cap M_2 = \emptyset$. Then, there exists an $f\in G(Q, s)$ such that $f(M_1)\cap M_3 = \emptyset$ and $f|_{M_2} = e$, $\bar{d}(f, e) < \varepsilon$.*

Proof. According to Corollary 2, the set $L = M_1\cup M_2\cup M_3$ is of infinite deficiency. Hence, there is a $g\in G(Q, s)$ such that $g(L)$ is straight, say,

$$(\pi_a g)(L) = a, \quad a\in s, \quad \bar{\bar{a}} = \aleph_0.$$

Let δ be a positive number such that $d(x, y) < \delta$ implies $d\big(g^{-1}(x), g^{-1}(y)\big) < \varepsilon$. Clearly, there exists an isotopy h_t, $t\in[0,1]$ of Q with the following properties: $h_t\in G(Q, s)$, $\bar{d}(h_t, e) < \varepsilon$ for $t\in[0,1]$, $h_0 = e$ and $h_t(a)\neq a$ for $t\in(0,1]$. Let g_1 be given by the formulas:

$$\varphi_{a\perp}\big(g_1(x)\big) = \pi_{a\perp}(x), \quad (\pi_a g_1)(x) = h_{u(x)}(x)$$

where $u(x) = \min\Big(1, d\big(\pi_{a\perp}(x), (\pi_{a\perp} g)(M_2)\big)\Big)$.

We put $f = g^{-1} g_1 g$.

Theorem 3. *Let K be a subset of Q of type F_σ. Then the following conditions are equivalent.*

(iv) *K is a σ-deficient set.*

(v) *For any σ-deficient set L, there exists an $f\in G(Q)$ such that $f(K)\cap L = \emptyset$.*

(vi) *There exists an* $f \in G(Q)$ *such that* $f(K) \subset s$.

(vii) *There exists an* $f \in G(Q)$ *such that* $f(K) \subset Q \backslash s$.

(viii) *For every* σ*-compact subset* L *of the pseudointerior* s, *there exists an* $f \in G(Q, s)$ *such that* $f(K) \cap L = \emptyset$.

Proof. (iv) implies (v). Suppose that $K = \bigcup_{n \in N} K_n$, $L = \bigcup_{n \in N} L_n$, where K_n and L_n are deficient. By Lemma 2, for any pair $i, j \in N$, the set $\{f \in G(Q): f(K_i) \cap L_j = \emptyset\}$ is open and dense in $G(Q)$. Since $G(Q)$ is complete-metrizable (Lemma 1), the classical Baire theorem shows that the set $\{f \in G(Q): f(K) \cap L = \emptyset\}$ is non-empty.

(v) implies (vi). This follows from the fact that $Q \backslash s$ is σ-deficient (the countable union of end-slices, each of which is deficient).

(vi) implies (vii). Let a_i, $i = 0, 1, \ldots$, be infinite pair-wise disjoint subsets of N such that $\bigcup_{i \in N} a_i = N$. Let $f_1 \in G(Q)$ be such that $f_1(K) \subset s$. Since K is of type F_σ, we conclude that there are compact sets $K_n \subset s$, $n \in N$, such that

$$f_1(K) = \bigcup_{i \in N} K_i .$$

For each $i \in N$ the set $\pi_{a_i}(K_i)$ is a compact subset of s. Hence, by Theorem 1, (d) $\Leftrightarrow$ (e), there are $g_i \in G(Q)$ such that

$$g_i\big(\pi_{a_i}(K_i)\big) \subset Q \backslash s , \quad \text{for each } i \in N .$$

The cartesian product of the maps g_i, i.e. the map $g \in G(Q)$ defined by the condition $\pi_{a_i} g = g_i$, takes $f_1(K)$ to the pseudo-boundary. Hence $f = g f_1$ satisfies statement (vii).

(vii) implies (iv). Let f be as in (vii). Then $f(K)$ is an F_σ subset of the pseudoboundary, and therefore is σ-deficient. But this implies that K itself is σ-deficient.

(iv) implies (viii). For each $x, y \in (-1, 1)^a$, let us write

$$\varrho_a(x, y) = \sum_{n \in a} 2^{-n} \min\left(1, \left|\tan\frac{\pi}{2}x_n - \tan\frac{\pi}{2}y_n\right|\right),$$

and let $\varrho = \varrho_N$. Then it is easy to see that each ϱ_a is a complete metric for the space $(-1, 1)^a$ compatible with the product topology of this space. Hence, using Lemma 1, we conclude that the set $G(Q, s)$ turns into a complete metric space under the metric

$$\psi(f, g) = \bar{d}(f, g) + \bar{d}(f^{-1}, g^{-1}) + \bar{\varrho}(f_{|s}, g_{|s}) + \bar{\varrho}(f_{|s}^{-1}, g_{|s}^{-1}) .$$

Using the classical Baire theorem, we reduce the proof of our implication to that of the following

Lemma 3. *If* M *is a deficient set in* Q *and* L *is a compact set of* s, *then the set* $A = \{f \in G(Q, s): f(M) \cap L = \emptyset\}$ *is a dense open subset of* $G(Q, s)$ *in the topology induced by the metric* ψ.

Proof. It is obvious that A is open. To prove that it is also dense, assume that we are given an $\varepsilon > 0$ and an $h \in G(Q, s)$. Pick a finite set $a \subset N$ with $\sum_{n \notin a} 2^{-n} < \varepsilon/8$, i.e.

(1) $$\varrho_{a^\perp} < \varepsilon/8 .$$

Denote by $\mathfrak{R}$ the collection of all sets $R \subset Q$ which are of the form

(2) $$R = \mathbf{P}_{i \in \beta}[a_i, b_i] \times I^{\beta^\perp}, \quad \text{where} \quad \beta \supset a, \ \bar{\bar{\beta}} < \aleph_0, \ -1 < a_i < b_i < 1 .$$

We claim that there are $D, D_1 \in \mathfrak{R}$ such that

(3) $$h^{-1}(L) \subset \operatorname{int} D_1 , \quad h(D_1) \subset D .$$

In fact, let K be the intersection of all the sets $R \in \mathfrak{R}$ such that $h^{-1}(L) \subset \operatorname{int} R$. Then $h(K)$ is a compact subset of s, and therefore there is a set $D \in \mathfrak{R}$ with $h(K) \subset \operatorname{int} D$, i.e. $K \subset h^{-1}(\operatorname{int} D)$. By the standard compactness argument we can pick a finite collection of sets $R_1, \ldots, R_j$ in $\mathfrak{R}$ such that $h^{-1}(L) \subset \operatorname{int} R_i$ for $i \leqslant j$ and $R_1 \cap \ldots \cap R_j \subset h^{-1}(\operatorname{int} D)$. Then the set D together with $D_1 = R_1 \cap \ldots \cap R_j$ satisfies conditions (3).

Let us now continue the proof of the lemma. By (2) we have $D = T \times I^{\beta^\perp}$, where D is a finite-dimensional closed cube contained in the cube $\operatorname{int} I^\beta$, $\bar{\bar{\beta}} < \aleph_0$. The function ϱ_β is uniformly continuous on $T \times T$. Hence there is a $\delta_1 > 0$ such that, for any $x, y \in D$, the condition $\sum_{n \in \beta} 2^{-n}|x_n - y_n| < \delta_1$ implies

$$\varrho(x, y) = \varrho_\beta(\pi_\beta x, \pi_\beta y) + \varrho_{\beta^\perp}(\pi_{\beta^\perp} x, \pi_{\beta^\perp} y) < \varepsilon/8 + \varepsilon/8 = \varepsilon/4 ,$$

and therefore

(4) $$d(x, y) < \delta_1 \quad \text{implies} \quad \varrho(x, y) < \varepsilon/4 \quad \text{for} \quad x, y \in D .$$

Similarly, there is a $\delta_2 > 0$ such that

(5) $$d(x, y) < \delta_2 \quad \text{implies} \quad \varrho(x, y) < \varepsilon/4 \quad \text{for} \quad x, y \in D_1 ,$$

and, moreover, such that

(6) $$d(x, y) < \delta_2 \quad \text{implies} \quad d\big(h(x), h(y)\big) < \delta_1 \quad \text{for all } x, y \in Q .$$

By Theorem 2, condition (i), the set ∂D_1 has property Z with respect to D_1 regarded as a Hilbert cube. Hence, by Lemma 2 (applied to $M_1 = h^{-1}(L)$, $M_2 = \partial D_1$ and $M_3 = M \cap D_1$), there is a $g \in G(D_1, \operatorname{int} D_1 \cap s)$ such that

(7) $$d(g, e) < \delta_2 , \quad g\big(h^{-1}(L)\big) \cap (M \cap D_1) = \emptyset \quad \text{and} \quad g|_{\partial D_1} = e .$$

Extending g as identity beyond D_1, we may assume without loss of generality that $g \in G(Q, s)$ and g is supported on D_1.

We define $f = hg^{-1}$. By (7), $f(M) \cap L = \emptyset$. Hence to complete the proof of the lemma, we have to show that $\psi(f, h) < \varepsilon$. Observe that the condition $f(x) \neq h(x)$ implies $x \in D_1$; thus by (3) we find that $f(x) \neq h(x)$ implies $h(x) \in D$ and $f(x) \in D$. Hence, using the estimation $d(g, e) < \delta_2$ in (7) and conditions (6), (5), (4), we conclude that each of the numbers $\bar{d}(f, h)$, $\bar{d}(f^{-1}, h^{-1})$, $\bar{\varrho}(f_{|s}, h_{|s})$, $\bar{\varrho}(f_{|s}^{-1}, h_{|s}^{-1})$ is less than $\varepsilon/4$. Whence $\psi(f, h) < \varepsilon$.

Proof of the implication (viii) $\Rightarrow$ (iv). Suppose that K is an F_σ set in Q satisfying (viii). Let, for each $n \in N$,

$$(*) \qquad E_n = \{x \in s\colon\ \pi_i(x) = 0 \text{ for all } i > n\}.$$

Each E_n is a countable union of compact sets in s. Hence, by (viii), there is an $f \in G(Q, s)$ such that $f(K) \cap \bigcup_{n \in N} E_n = \emptyset$. Representing $f(K)$ as $f(K) = \bigcup_{i \in N} K_i$, a countable union of compact sets, we find that each set $K_i \cap s$ satisfies the assumption of Corollary 1 ($\pi_{\alpha\perp}(K_i \cap s) \not\ni 0$ for each finite $\alpha \subset N$). Thus $K_i \cap s$ is deficient in s for each $i \in N$. Hence, by Theorem 1, the sets K_i are deficient in Q, and therefore both $f(K)$ and K are σ-deficient.

COROLLARY 3. *There exists an autohomeomorphism $f \in G(Q)$ which takes the pseudoboundary of Q into the pseudointerior* (cf. [3], Theorem 11.1).

Proof. $Q \backslash s$ is clearly of type F_σ. Hence the statement follows from the implication (vii) $\Rightarrow$ (vi).

THEOREM 4. *Let K be a F_σ-subset of the space s. Then the following conditions are equivalent.*

(ix) *K is σ-deficient in s.*

(x) *For every subset L of s which is a countable union of compact sets there exists an $f \in G(s)$ such that $f(K) \cap L = \emptyset$.*

Proof. (ix) $\Rightarrow$ (x). Let $K = \bigcup_{i \in N} K_i$ where K_i are deficient sets. The sets $\bar{K}_i$ are of infinite deficiency in Q (Theorem 1); thus the condition (viii) of Theorem 3 gives the existence of $h \in G(Q, s)$ such that $h(\bigcup_{i \in N} \bar{K}_i) \cap L = \emptyset$. We put $f = h|_s$.

(x) $\Rightarrow$ (ix). Let $K = \bigcup_{i \in N} L_i$, where L_i are closed in s, and let $f \in G(s)$ be such that $f(K) \cap \bigcup_{i \in N} E_n = \emptyset$, where E_n are given by $(*)$. Then $f(L_i)$ satisfies the assumption of Corollary 1 and consequently is deficient in s. We conclude that L_i is deficient in s, and K is a σ-deficient set.

4. Extensions of homeomorphisms between deficient sets. The following theorem is proved in [3]:

THEOREM 5. *Let K_1 and K_2 be deficient subsets of X (X is Q or S), and let $f\colon K \xrightarrow{\text{onto}} K_2$ be a homeomorphism. Then, there exists an $f \in G(X)$ such that $F|_{K_1} = f$. In the case where K_1 and K_2 are compact subsets of s, the autohomeomorphism F can be chosen from the set $G(Q, s)$.*

The second part of the theorem can be extended as follows.

THEOREM 6. *Let K_1 and K_2 be deficient subset of Q and let f be a homeomorphism between the pairs $(K_1, K_1 \cap s)$ and $(K_2, K_2 \cap s)$. Then there exists an $F \in G(Q, s)$ such that $F|_{K_1} = f$.*

Proof 1. We consider the case $K_1 = K_2$. By Theorem 1 (e), there exists a $g \in G(Q)$ such that $g(s \cup K_1) = s$. According to Theorem 5, the homeomorphism $h = gfg^{-1}\colon g(K_1) \to g(K_1)$ can be extended to a $H \in G(Q, s)$. Clearly, $F = g^{-1}Hg$ gives the desired extension.

2. We pass to the general case. Applying Lemma 2 with $M_1 = K_1$, $M_3 = K_2$, $M_2 = \emptyset$ we conclude that there exists an $h \in G(Q, s)$ such that $h(K_1) \cap K_2 = \emptyset$. Let $K_3 = h(K_1)$ and $K = K_2 \cup K_3$. According to 1°, the homeomorphism $f_1\colon K \to K$ given by

$$f_1(x) = \begin{cases} fh^{-1}(x), & x \in K_3, \\ hf^{-1}(x), & x \in K_2 \end{cases}$$

can be extended to an $F_1 \in G(Q, s)$. $F = F_1 h \in G(Q, s)$ is the desired autohomeomorphism.

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