versally measurable set, for each $\varepsilon > 0$, there exist Borel subsets U and V of G such that $U \subset E \subset V$ and $\mu(V) - \mu(U) < \varepsilon$. Then $f(U)$ and $f(V)$ are Borel sets satisfying $f(U) \subset f(E) \subset f(V)$ and $\lambda(f(V)) - \lambda(f(U)) < \varepsilon$. Hence $f(E)$ is a universally measurable set.

Lemma 2, the analogous result for universal null sets, follows from the proof of Lemma 1.

LEMMA 2. *If G is an element of $\mathfrak{B}$, E is a universal null subset of G, and g is g one to one Borel measurable function on G, then $g(E)$ is a universal null set.*

In the course of establishing Theorem 1, it will be convenient to denote by J the set of irrational numbers.

THEOREM 1. *If E is a universally measurable subset of R and f is a bimeasurable function on R, then $f(E)$ is a universally measurable set.*

Proof. Let $B = R - (S_1 \cup S_2 \cup S_3)$ where S_1 is the set of rational numbers, $S_2 = \{f^{-1}(x);\ x \in S_1\}$, and $S_3 = \{x;\ f^{-1}(f(x))$ is uncountable$\}$. Then B is a Borel subset of J, $f(B)$ is a Borel subset of J, and the restriction $g = f|B$ of f to B is semi-regular: for each x in R, $g^{-1}(x)$ is a countable set. Hence it follows from page 243 of [2] that there is a sequence $\{B_i\}$ of pairwise disjoint Borel subsets of B such that $\bigcup B_i = B$ and the restrictions $g_i = g|B_i$ are one to one Borel measurable functions. Lemma 1 tells us that each $g_i(E \cap B_i)$ is a universally measurable set. Thus, since $f(R - B)$ is a countable set,

$$f(E) = f(E \cap B) \cup f(E - B) = \{\bigcup g_i(E \cap B_i)\} \cup f(E - B)$$

is a universally measurable set.

Applying Lemma 2 instead of Lemma 1 yields Theorem 2.

THEOREM 2. *If E is a universal null subset of R and f is a continuous bimeasurable function on R, then $f(E)$ is a universal null set.*

Banach's characterization of CBV functions implies that the CBV function h constructed in [1] satisfies $m(\{x;\ h^{-1}(x)$ is uncountable$\}) = 0$, where m denotes the Lebesgue measure. An examination of the construction for h shows that there is a perfect set P such that $h^{-1}(x)$ is uncountable if $x \in P$.

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On discontinuous additive functions

by

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One of the classical problems of analysis is this:

Let T be a set on the real line R or, more generally, in the n-dimensional Euclidean space R^n, and let f be a real-valued function which is defined in R^n and *additive*, i.e. satisfies Cauchy's functional equation:

$$f(x+y) = f(x) + f(y) \tag{1}$$

for $x, y \in R^n$. Suppose that f is upper-bounded on T. What conditions upon the set T imply the continuity of f?

The same problem may be stated for functions which are defined in some convex domain $\Delta \subset R^n$ and satisfy Jensen's inequality (2) instead of (1):

$$f\left(\frac{x+y}{2}\right) \leqslant \frac{f(x)+f(y)}{2} \tag{2}$$

for $x, y \in \Delta$. Such functions will be referred to as *Q-convex*. This expression is justified by the observation that they satisfy also the inequality

$$f\big((1-a)x + ay\big) \leqslant (1-a)f(x) + af(y) \tag{3}$$

for $x, y \in \Delta$, a rational, $0 \leqslant a \leqslant 1$; and the latter is an immediate consequence of the generalized Jensen formula

$$f\left(\frac{x_1 + \ldots + x_q}{q}\right) \leqslant \frac{1}{q}\big(f(x_1) + \ldots + f(x_q)\big) \tag{4}$$

for $x_1, \ldots, x_q \in \Delta$, $q = 1, 2, 3, \ldots$, which may be found in any textbook on convex functions, e.g. [2]. In order to obtain (3) for $a = p/q$, $p = 0, 1, \ldots, q$, it suffices to set in (4) $x_1 = \ldots = x_{q-p} = x$, $x_{q-p+1} = \ldots \ldots = x_q = y$.

R. Ger and M. Kuczma introduce in [1] the following set classes:

A set $T \subset R^n$ belongs to the class $\mathcal{A}$ iff every Q-convex function $f\colon \Delta \to R$, $T \subset \Delta \subset R^n$, upper-bounded on T, is continuous in Δ.

A set $T \subset R^n$ belongs to the class $\mathfrak{B}$ iff every additive function $f\colon R^n \to R$, upper-bounded on T, is continuous.

A set $T\subset R^n$ belongs to the class $\mathfrak{C}$ iff every additive function $f\colon R^n\to R$, (upper- and lower-) bounded on T, is continuous.

Obviously, we have the inclusions

$$\mathfrak{A}\subset\mathfrak{B}\subset\mathfrak{C}. \tag{5}$$

The problems indicated may now be restated as follows: describe the members of the classes $\mathfrak{A}$ and $\mathfrak{B}$.

As is well known (A. Ostrowski [6] and S. Marcus [5]), the sets with positive inner Lebesgue measure belong to $\mathfrak{A}$. For a set $T\subset R^n$, let $J(T)$ be the smallest midpoint convex set containing T (a set $A\subset R^n$ is called *midpoint convex* provided $\frac{1}{2}(A+A)\subset A$). Now, the condition

$$m_i\big(J(T)\big)>0 \tag{6}$$

is sufficient for a set T to belong to the class $\mathfrak{A}$ (M. Kuczma [3], R. Ger and M. Kuczma [1]); here m_i denotes as usual the inner Lebesgue measure. A conjecture of S. Marcus (cf. [4]) says that condition (6) is also necessary for the relation $T\in\mathfrak{A}$. However, this turns out to be inexact (see the example after Theorem 3 below).

A natural question arises whether inclusions (5) are proper. It is easy to see that $\mathfrak{B}\neq\mathfrak{C}$. In fact, let $f\colon R^n\to R$ be any discontinuous additive function and let $K=\{x\in R^n\colon f(x)\leqslant 0\}$. Clearly, $K\notin\mathfrak{B}$. On the other hand, together with any point x, the set K contains the sequence (kx), $k=1,2,3,\ldots$, and thus any nonzero additive function is unbounded on K. It follows that $K\in\mathfrak{C}$. (Another example of a set belonging to $\mathfrak{C}\setminus\mathfrak{B}$ may be found in [1].)

The authors of [1] conjecture that the inclusion $\mathfrak{A}\subset\mathfrak{B}$ is actually an equality. It is so indeed and this fact is established in the present paper (Theorem 4). This result reduces certain problems concerning the boundedness and continuity of Q-convex functions to the corresponding ones for additive functions; and these are sometimes more convenient to handle.

The method employed in the proof is that of an analysis of the geometric structure of vector spaces over the field Q of rational numbers. This seems reasonable in view of the fact that an additive function $R^n\to R$ is just a linear map between those sets, regarded as vector spaces over Q. In the sequel we shall make use of this fact without an explicit statement.

One of the tools most frequently used in the arguments concerning convexity and separation in *real* vector spaces (and consequently in several branches of analysis) is the famous theorem of Hahn and Banach; one of its alternative formulations reads as follows:

Let E be a vector space over the field R of the reals and let C be a convex subset of E with the following property: for every $x\in E$ a number $c_x>0$ exists such that $tx\in C$ for $0\leqslant t<c_x$. Then every linear functional $f\colon X\to R$ defined on a subspace X of E, upper-bounded on $X\cap C$, admits an extension to a functional $F\colon E\to R$, upper-bounded on C.

An analogue of this theorem for vector spaces over Q would be of much use for us. Two possible generalizations may be stated, for instead of considering only functionals, i.e. linear maps into Q, we may take into account functions with values in the order-completion R of Q. Note that linearity with respect to rational scalars is just additivity. We thus have two questions:

QUESTION 1. Let E be a vector space over Q and let C be a subset of E with the following properties:

(i) $(1-a)C+aC\subset C$ for $a\in Q$, $0\leqslant a\leqslant 1$;

(ii) for every $x\in E$ a number $c_x>0$ exists such that $ax\in C$ for $a\in Q$, $0\leqslant a<c_x$.

Further, let $f\colon X\to Q$ be an additive function defined on a subspace X of E, upper-bounded on $X\cap C$. Can it be extended to an additive function $F\colon E\to Q$, upper-bounded on C?

QUESTION 2. Let E, C, X be as in Question 1 and let $f\colon X\to R$ be an additive function, upper-bounded on $X\cap C$. Can it be extended to an additive function $F\colon E\to R$, upper-bounded on C?

We cannot expect a positive answer to Question 1. For if $E=R$, $C=[-1,1]$, $X=Q$, $f(\xi)=\xi$ for $\xi\in Q$, then the conditions of Question 1 are fulfilled, and yet f clearly admits no extension to an additive function $R\to Q$, upper-bounded on C. However, whereas the answer to Question 1 is "no", the answer to Question 2 is "yes" (Theorem 1), and this turns out to be sufficient for our purposes.

The following terminology and notations will be used in the sequel:

A subset C of a vector space E over Q will be called *Q-convex* iff

$$(1-a)C+aC\subset C \tag{7}$$

holds for $a\in Q$, $0\leqslant a\leqslant 1$.

Let $x_0\in A\subset E$. We say that the set A is *Q-radial at the point x_0* iff for every $x\in E$ a number $c_x>0$ exists such that $x_0+ax\in A$ whenever $a\in Q$, $0\leqslant a<c_x$.

Intuitively speaking, a set A is Q-radial at x_0 if it contains a "rational segment" in each direction from the point x_0.

In the theory of convexity in real vector spaces, notions corresponding to those defined above are well known. The prescript "Q-" is employed here to emphasize the fact that rational scalars only are involved.

The set-theoretic operations are denoted as usual $\cup$, $\cap$, $\setminus$, $\bigcup$, $\times$, whereas the symbols $+$, $\cdot$ (or absence of a dot) and $\sum$, applied to sets, stand for the algebraic operations on them. $f|_A$ is the restriction of a function f (defined in a larger domain) to the set A.

Finally we make the following convention:

Greek letters always denote rational numbers.

* * *

LEMMA. *Let E be a vector space over Q, X and Y subspaces of E such that $X \subset Y$ and X has codimension 1 in Y; let C be a Q-convex subset of E, Q-radial at the point 0; finally, let $f\colon X \to R$ be an additive function such that $f|_{X\cap C} \leqslant 1$. Then there exists an additive function $g\colon Y \to R$ with the properties $g|_X = f$, $g|_{Y\cap C} \leqslant 1$.*

Proof. According to the supposition we may write

$$Y = X + Qy\,, \tag{8}$$

where y is a point in $Y \setminus X$.

Consider the sets $U, V \subset X \times Q$, defined by

$$U = \left\{(x, \xi)\colon\ x \in X\,,\ \xi > 0\,,\ \frac{x-y}{\xi} \in C\right\},$$

$$V = \left\{(x, \xi)\colon\ x \in X\,,\ \xi > 0\,,\ \frac{x+y}{\xi} \in C\right\}.$$

The set C is Q-radial at 0, thus, given an $x \in X$, the points $(x \pm y)/\xi$ are in C for sufficiently large ξ. It follows that the sets U and V are non-void. Write

$$u = \text{l.u.b.}\ \{f(x) - \xi\colon\ (x, \xi) \in U\}\,,$$
$$v = \text{g.l.b.}\ \{\xi - f(x)\colon\ (x, \xi) \in V\}\,,$$

$-\infty < u \leqslant +\infty$, $-\infty \leqslant v < +\infty$. We are going to show that

$$u \leqslant v \tag{9}$$

and thus u and v are proper real numbers.

Suppose, on the contrary, that $u > v$. Then there exists a pair $(x, \alpha) \in U$ such that $f(x) - \alpha > v$; and there exists a pair $(z, \beta) \in V$ such that $f(x) - \alpha > \beta - f(z)$ or, equivalently,

$$f(x+z) > \alpha + \beta\,. \tag{10}$$

According to the definition of U and V, we have $x, z \in X$, $\alpha, \beta > 0$,

$$\frac{x-y}{\alpha} \in C \quad\text{and}\quad \frac{z+y}{\beta} \in C\,.$$

Hence, in view of (7)

$$\frac{x+z}{\alpha+\beta} = \frac{\alpha}{\alpha+\beta}\cdot\frac{x-y}{\alpha} + \frac{\beta}{\alpha+\beta}\cdot\frac{z+y}{\beta} \in C\,,$$

a contradiction to (10) since $f|_{X\cap C} \leqslant 1$ and so $f(x+z) \leqslant \alpha+\beta$.

Consequently (9) holds; let $c \in [u, v]$. Define the desired function g by

$$g|_X = f\,, \qquad g(y) = c$$

and extend it by Q-linearity onto the whole of Y (see (8)). It remains to verify that $g|_{Y\cap C} \leqslant 1$.

Let $x + ay \in Y \cap C$; we may assume $a \neq 0$. If $a > 0$, then

$$\left(\frac{x}{a}, \frac{1}{a}\right) \in V\,, \qquad\text{so}\qquad \frac{1-f(x)}{a} \geqslant v \geqslant c$$

and

$$g(x+ay) = f(x) + ac \leqslant 1\,. \tag{11}$$

If $a < 0$ then

$$\left(-\frac{x}{a}, -\frac{1}{a}\right) \in U\,, \qquad\text{so}\qquad \frac{1-f(x)}{a} \leqslant u \leqslant c$$

and (11) also holds.

The following theorem is the key to our further considerations and gives a positive answer to Question 2 above. Its proof, as well as that of the preceding lemma, is essentially the same as in the real case.

THEOREM 1. *Let E be a vector space over Q, X a subspace of E, and C a Q-convex subset of E, Q-radial at the point 0. If $f\colon X \to R$ is an additive function such that $f|_{X\cap C} \leqslant 1$, then there exists an additive function $F\colon E \to R$ with the properties $F|_X = f$, $F|_C \leqslant 1$.*

Proof. Consider the family $\mathfrak{M}$ of all pairs (X', f'), where X' is a subspace of E, $f'\colon X' \to R$ an additive function such that

$$X \subset X' \subset E\,, \qquad f'|_X = f\,, \qquad f'|_{X'\cap C} \leqslant 1\,.$$

$\mathfrak{M}$ is non-void since $(X, f) \in \mathfrak{M}$. It is ordered by the relation $\precsim$, defined by

$$(X', f') \precsim (X'', f'') \quad\text{iff}\quad X' \subset X'' \text{ and } f''|_{X'} = f'\,.$$

If $\mathfrak{M}_0$ is any subfamily of $\mathfrak{M}$, linearly ordered by $\precsim$, then the pair $\big(\bigcup_{(X',f')\in\mathfrak{M}_0} X', \tilde{f}\big)$, $\tilde{f}$ being defined in the obvious way, is again in $\mathfrak{M}$. It follows, in virtue of Zorn's lemma, that $\mathfrak{M}$ contains a pair $(X_{\max}, f_{\max})$, maximal with respect to $\precsim$. It suffices to show that $X_{\max} = E$; $f_{\max}$ will then be the desired extension F.

Suppose that there exists a $y \in E \setminus X_{\max}$. Then the space $Y = X_{\max} + Qy$ satisfies the conditions of the preceding lemma with $X_{\max}$ for X; and the pair (Y, g), where g is the function from the assertion of the lemma, would be a proper extension of the pair $(X_{\max}, f_{\max})$, thus contradicting the maximality of the latter.

We shall now be concerned with the construction of certain discontinuous additive functions.

THEOREM 2. *Let C be a Q-convex subset of the real line R, Q-radial at some point. Then either C is an interval or there exists a discontinuous additive function $F\colon R\to R$, upper-bounded on C.*

Proof. Assume at the moment that C is Q-radial at 0.

Suppose that C is not an interval. Then there exist x and y such that

$$x\notin C\,,\quad y\in C\,, \tag{12}$$

$$0<x<y \quad \text{or} \quad 0>x>y\,. \tag{13}$$

Consider the set $S\subset Q\times Q$, defined by

$$S=\{(\xi,\eta)\colon (1+\xi)x+\eta y\in C,\ \eta>0\}\,.$$

S is non-void since

$$(-1,1)\in S\,. \tag{14}$$

Write

$$s=\text{l.u.b.}\left\{\frac{\xi}{\eta}\colon (\xi,\eta)\in S\right\};$$

we shall see that $s<+\infty$. C is Q-radial at 0; so $-ay\in C$ for sufficiently small a, say, for $0\leqslant a<c$, where c is a positive constant. Suppose $(\xi,\eta)\in S$; then $\xi/\eta\leqslant 1/c$ since otherwise we would have $\xi/\eta>1/c$, $\xi>0$, $0<\eta/\xi<c$, and

$$-\frac{\eta}{\xi}y\in C\,,\qquad (1+\xi)x+\eta y\in C$$

(by the definition of S), whence, in view of (7),

$$x=\frac{1}{1+\xi}\big((1+\xi)x+\eta y\big)+\frac{\xi}{1+\xi}\left(-\frac{\eta}{\xi}y\right)\in C,$$

contrary to (12). Consequently, the quotient ξ/η is upper-bounded for $(\xi,\eta)\in S$ and s is a real number; furthermore, $s\geqslant -1$ on account of (14).

Put

$$f(x)=1\,,\qquad f(y)=-s\leqslant 1 \tag{15}$$

an extend f by Q-linearity to an additive function $f\colon X\to R$, where $X=Qx+Qy$; this is possible since x and y are Q-linearly independent in view of (12) and (13) and the Q-convexity of C.

We are going to show that

$$f|_{X\cap C}\leqslant 1\,. \tag{16}$$

Suppose, on the contrary, that there exists a point $ax+\beta y\in C$ with $f(ax+\beta y)>1$ or, equivalently,

$$\beta s<a-1\,.$$

Consider three possible situations:

1. $\beta>0$. Then $(a-1)/\beta>s$, which is impossible since $(a-1,\beta)\in S$.

2. $\beta=0$. Then $a>1$, $ax\in C$ whence, by the Q-convexity of C, $x\in C$, contrary to (12).

3. $\beta<0$. Then $s>(1-a)/(-\beta)$ and there exists, by the definition of s, a pair $(\xi,\eta)\in S$ such that $\xi/\eta>(1-a)/(-\beta)$, $\eta>0$, whence

$$a\eta>\eta+\beta\xi\,. \tag{17}$$

Write

$$\zeta=\frac{a\eta-\beta(1+\xi)}{\eta-\beta}\,;$$

then, by (17), $\zeta>1$. The pair (ξ,η) is in S; so $(1+\xi)x+\eta y\in C$. Hence, in view of (7), the points

$$u=\frac{1}{\zeta}(ax+\beta y)\,,\qquad v=\frac{1}{\zeta}\big((1+\xi)x+\eta y\big)$$

and finally

$$z=\frac{\eta}{\eta-\beta}u+\frac{-\beta}{\eta-\beta}v$$

are in C. However, this is impossible, since $z=x$, as is easy to verify.

Consequently (16) follows and f may be extended to an additive function $F\colon R\to R$ such that $F|_C\leqslant 1$, on account of Theorem 1. F is discontinuous in view of (13), (15) and the fact that every continuous additive function $R\to R$ is a constant multiple of the identity.

The theorem is thus proved in the case where the set C is Q-radial at 0. If is is Q-radial at some other point x_0, then the set $C-x_0$ is Q-radial at 0 and the preceding case applies to the finding of a discontinuous additive function $F\colon R\to R$ such that $F|_{C-x_0}\leqslant 1$, provided C is not an interval. It follows that $F|_C\leqslant 1+F(x_0)$.

THEOREM 3. *Let C be a Q-convex subset of the n-dimensional Euclidean space R^n, Q-radial at some point. Then either C contains a ball or there exists a discontinuous additive function $F\colon R^n\to R$, upper-bounded on C.*

Proof. We may assume that C is Q-radial at the origin, replacing C, if necessary, by an appropriate translate of it (see the last argument in the proof of Theorem 2). Assume further that there exists no discontinuous additive function, upper-bounded on C. We are going to show that the origin is then an interior point of C.

The proof will be by induction on n. The case of $n=1$ is the consequence of the preceding theorem since 0 is certainly an interior point of any interval, Q-radial at 0.

Assume that the assertion is true for some n and suppose that $C\subset R^{n+1}$. Let $e_0,\ldots,e_n$ be the usual orthonormal basis for R^{n+1}; thus $R^{n+1}=Re_0+$

$+R^n$, where R^n stands for $Re_1+\ldots+Re_n$. There exists neither a discontinuous additive function $f\colon Re_0\to R$ nor a discontinuous additive function $f\colon R^n\to R$, upper-bounded on $Re_0\cap C$ resp. $R^n\cap C$ since otherwise such a function might be extended to a discontinuous additive function $F\colon R^{n+1}\to R$, upper-bounded on C, according to Theorem 1 —and this is a contradiction to our assumption. Thus the origin is in the (1-dimensional resp. n-dimensional) interior of the two sets $Re_0\cap C$ and $R^n\cap C$, on account of Theorem 2 and the induction hypothesis. In orther words, there exists an $r>0$ such that

(18) $$\{te_0\colon |t|<r\}\cup\{a\in R^n\colon \|a\|<r\}\subset C;$$

$\|\cdot\|$ is here the Euclidean norm.

We shall prove that the ball $\{x\in R^{n+1}\colon \|x\|<r/\sqrt{2}\}$ is contained in C. Suppose that $x=a+x_0e_0$, where $a\in R^n$, $x_0\in R$, is a point in R^{n+1} such that

(19) $$\|x\|^2=\|a\|^2+|x_0|^2<r^2/2;$$

the proof will be complete if we show that $x\in C$.

If $\|a\|=0$ or $|x_0|=0$, then the assertion follows from (18) and (19). We may thus assume that both summands in (19) are positive. We have, by (19), $\|a\|+|x_0|<r$ and hence

$$0<|x_0|<r-\|a\|<r.$$

So there exists an $a\in Q$ such that

(20) $$0<\frac{|x_0|}{r}<a<1-\frac{\|a\|}{r}<1.$$

Write $s=1/(1-a)$, $t=x_0/a$. Then $sa\in C$, $te_0\in C$ in view of (18) and (20), whence

$$x=a+x_0e_0=(1-a)sa+ate_0\in C.$$

The following example indicates that the assumption that C is Q-radial at some point cannot be omitted in Theorems 2 and 3.

EXAMPLE. *There exists a set $K\subset R$ with the properties:*

(i) *K is Q-convex;*

(ii) $m_i(K)=0$;

(iii) *$f\equiv 0$ is the only additive function $R\to R$, upper-bounded on K.*

Proof. Let $\{h_i\}_{i<I}$ be a Hamel basis for the vector space R over Q, indexed by the ordinals smaller than the least ordinal I of power continuum. Every $x\in R$, $x\neq 0$, has a unique expansion

(21) $$x=a_1h_{i_1}+\ldots+a_nh_{i_n},\quad i_1<\ldots<i_n,\quad a_1\neq 0,\ldots,a_n\neq 0.$$

We have

(22) $$R=\bigcup_{i<I}X_i,$$

where

$$X_i=\sum_{j<i}Qh_j.$$

K is defined as the set of those x's for which the coefficient a_n in the expansion (21) is positive. In fact, K has the desired properties (i), (ii), (iii):

(i) Straightforward verification.

(ii) Suppose that $m_i(K)>0$; then the set $K+K$ would contain an interval of positive length, in virtue of the theorem of H. Steinhaus, [7]. However, this is impossible since $K+K=K$ and K is disjoint with the set Qh_1-h_2, dense in R.

(iii) Let $f\colon R\to R$ be an additive function such that $f|_K\leqslant M$. For $x\in X_i$ we have $x+h_i\in K$; so $f(x)\leqslant M-f(h_i)$. Thus $f|_{X_i}$ is upper-bounded and, consequently, $f|_{X_i}\equiv 0$ since X_i is a Q-linear subspace of R. Now, (iii) follows in view of (22).

Recall the definition of the classes $\mathcal{A}$ and $\mathcal{B}$.

The above example contradicts the conjeture of S. Marcus, mentioned in the introduction, that all sets which are in $\mathcal{A}$ enjoy the property (6); for $m_i(J(K))=0$ by (i) and (ii), yet $K\in\mathcal{B}$ by (iii) and thus also $K\in\mathcal{A}$, on account of the following

THEOREM 4. $\mathcal{A}=\mathcal{B}$.

In other words: *Let T be a set in R^n; suppose that there exists a Q-convex function $f\colon \Delta\to R$, defined and discontinuous in a convex domain Δ, $T\subset\Delta\subset R^n$ and that f is upper-bounded on T. Then there exists a discontinuous additive function $R^n\to R$, upper-bounded on T.*

Proof. Suppose $f|_T<M$ and write $C=\{x\in\Delta\colon f(x)<M\}$. The set C is Q-convex in view of (3). It is not difficult to see that C is Q-radial at each of its points. In fact, let $x_0\in C$ and $x\in R^n$; Δ is open, so there exists a $\gamma>0$ such that $x_0+\gamma x\in\Delta$. Now we have, by (3),

$$f(x_0+ax)=f\left(\left(1-\frac{a}{\gamma}\right)x_0+\frac{a}{\gamma}(x_0+\gamma x)\right)\leqslant\left(1-\frac{a}{\gamma}\right)f(x_0)+\frac{a}{\gamma}f(x_0+\gamma x)$$

for $0\leqslant a\leqslant\gamma$; so

$$\limsup_{a\to 0}f(x_0+ax)\leqslant f(x_0)<M$$

and

$$f(x_0+ax)<M$$

for small a; but this means just that C is Q-radial at x_0.

Certainly C contains no ball, for a Q-convex function, upper-bounded on a ball, is necessarily continuous. Finally, observe that $T\subset C$.

The assertion follows in virtue of Theorem 3.

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Remarks on Anderson's paper "On topological infinite deficiency"

by

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Suppose that the topological space X is the product of $\aleph_0$ copies of an interval J which is either closed or open. A closed subset A of X is said to be of *infinite deficiency* (briefly: *deficient*) in X if there exists a homeomorphism h of X onto itself such that, for infinitely many i, the natural projections $\pi_i h(A)$ are (at most) one-point sets in the interior of J.

The sets of infinite deficiency have been systematically investigated by R. D. Anderson in [1], [3]. The importance of these sets lies in their topological negligibility property (see condition (g) of Theorem 1 in this paper) and the property of extending homeomorphisms (here: Theorem 5); both properties have been established in their final form by Anderson, but the pioneer work in this respect was done by Klee ([9], [10]). For other results concerning negligibility see also [5], [6], [7], [8]. The theory of deficient sets can easily be transferred to the case of separable infinite-dimensional Fréchet spaces.

The present paper is a contribution to the theory of infinite deficiency. In Section 1 we establish some topological characterizations of sets of infinite deficiency. One of them (condition (ii) in Theorem 2), applied to F_σ sets rather than to closed sets, gives a characterization of σ-deficient sets, i.e. of countable unions of deficient sets. This class of sets, being a natural generalization of deficient sets, is discussed in Section 2 [1]. Finally, in Section 3 we establish a theorem on extending homeomorphisms to the pair: Hilbert cube Q and its pseudointerior s, which is an analogue of the above-mentioned theorem of Anderson, dealing with a single space X which is either Q or s.

Our results are derived from two theorems of Anderson, which are stated explicitly as Theorem 1 in Section 1 and Theorem 5 in Section 3.

[1] Added in proof. σ-deficient sets (sets of type Z_σ) and their relations to problems of negligibility have been studied by R. D. Anderson and his colaborators, see, e.g. [5].