On discontinuous additive functions

by

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One of the classical problems of analysis is this:

Let \( T \) be a set on the real line \( \mathbb{R} \) or, more generally, in the \( n \)-dimensional Euclidean space \( \mathbb{R}^n \), and let \( f \) be a real-valued function which is defined in \( \mathbb{R}^n \) and additive, i.e. satisfies Cauchy's functional equation:

\[
 f(x + y) = f(x) + f(y)
\]

for \( x, y \in \mathbb{R}^n \). Suppose that \( f \) is upper-bounded on \( T \). What conditions upon the set \( T \) imply the continuity of \( f \)?

The same problem can be stated for functions which are defined in some convex domain \( \mathcal{D} \subset \mathbb{R}^n \) and satisfy Jensen's inequality (2) instead of (1):

\[
 f \left( \frac{x + y}{2} \right) \leq \frac{f(x) + f(y)}{2}
\]

for \( x, y \in \mathcal{D} \). Such functions will be referred to as \( Q \)-convex. This expression is justified by the observation that they satisfy also the inequality

\[
 f \left( (1 - \alpha)x + \alpha y \right) \leq (1 - \alpha)f(x) + \alpha f(y)
\]

for \( x, y \in \mathcal{D}, \alpha \) a rational, \( 0 < \alpha < 1 \); and the latter is an immediate consequence of the generalized Jensen formula

\[
 f \left( \frac{x_1 + \ldots + x_q}{q} \right) \leq \frac{1}{q} \left( f(x_1) + \ldots + f(x_q) \right)
\]

for \( x_1, \ldots, x_q \in \mathcal{D}, q = 1, 2, 3, \ldots \), which may be found in any textbook on convex functions, e.g., [2]. In order to obtain (5) for \( \alpha = \frac{p}{q}, p = 0, 1, \ldots, q; q! \) it suffices to set in (4) \( x_1 = \ldots = x_{q-p} = x_1, x_{q-p+1} = \ldots = x_q = y \).

R. Ger and M. Kuczma introduce in [1] the following set classes:

A set \( TC \subset \mathbb{R}^n \) belongs to the class \( A \) iff every \( Q \)-convex function \( f: A \rightarrow \mathbb{R} \), \( T \subset C \subset \mathbb{R}^n \), upper-bounded on \( T \), is continuous in \( A \).

A set \( TC \subset \mathbb{R}^n \) belongs to the class \( B \) iff every additive function \( f: \mathbb{R}^n \rightarrow \mathbb{R} \), upper-bounded on \( T \), is continuous.

References


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A set \( T \subset \mathbb{R}^n \) belongs to the class \( \mathcal{C} \) iff every additive function \( f: \mathbb{R}^n \to \mathbb{R} \), (upper- and lower-) bounded on \( T \), is continuous.

Obviously, we have the inclusions

\[
\mathcal{A} \subset \mathcal{B} \subset \mathcal{C}.
\]

The problem indicated may be restated as follows: describe the members of the classes \( \mathcal{A} \) and \( \mathcal{B} \).

As is well known (A. Ostrowski [6] and S. Marcus [5]), the sets with positive inner Lebesgue measure belong to \( \mathcal{A} \). For a set \( T \subset \mathbb{R}^n \), let \( J(T) \) be the smallest midpoint convex set containing \( T \). For a set \( A \subset \mathbb{R}^n \), it is called midpoint convex provided \( \frac{1}{2} (a + A) \subset A \). Now, the condition

\[
\sigma_0 (J(T)) > 0
\]

is sufficient for a set \( T \) to belong to the class \( \mathcal{A} \) (M. Kuczma [3], R. Ger and M. Kuczma [1]); here \( \sigma_0 \) denotes the usual inner Lebesgue measure. A conjecture of S. Marcus (cf. [4]) says that condition (6) is also necessary for the relation \( T \subset \mathcal{A} \). However, this turns out to be incorrect (see the example after Theorem 3 below).

A natural question arises whether inclusions (5) are proper. It is easy to see that \( \mathcal{B} \neq \mathcal{C} \). In fact, let \( f: \mathbb{R}^n \to \mathbb{R} \) be any discontinuous additive function and let \( K = \{ x \in \mathbb{R}^n : f(x) < 0 \} \). Clearly, \( K \subset \mathcal{B} \). On the other hand, together with any point \( x \), the set \( K \) contains the sequence \( (k_n) \), where \( k_n = (1, 2, 3, \ldots) \), and thus any nonzero additive function is unbounded on \( K \). It follows that \( K \subset \mathcal{C} \). (Another example of a set belonging to \( \mathcal{B} \) may be found in [1].)

The authors of [1] conjecture that the inclusion \( \mathcal{A} \subset \mathcal{B} \) is actually an equality. It is so indeed and this fact is established in the present paper (Theorem 4). This result reduces certain problems concerning the boundedness and continuity of \( \mathcal{Q} \)-convex functions to the corresponding ones for additive functions; and these are sometimes more convenient to handle.

The method employed in the proof is that of an analysis of the geometric structure of vector spaces over the field \( \mathbb{Q} \) of rational numbers. This seems reasonable in view of the fact that an additive function \( \mathbb{R}^n \to \mathbb{R} \) is just a linear map between those sets, regarded as vector spaces over \( \mathbb{Q} \). In the sequel we shall make use of this fact without an explicit statement.

One of the tools most frequently used in the arguments concerning convexity and separation in real vector spaces (and consequently in several branches of analysis) is the famous theorem of Hahn and Banach; one of its alternative formulations reads as follows:

Let \( E \) be a vector space over the field \( \mathbb{R} \) of the reals and let \( C \) be a convex subset of \( E \) with the following property: for every \( x \in E \) a number \( c_0 > 0 \) exists such that \( t x \in C \) for \( 0 < t < c_0 \). Then every linear functional \( f: \mathbb{R} \to \mathbb{R} \) defined on a subspace \( X \) of \( E \), upper-bounded on \( X \times C \), admits an extension to a functional \( F: \mathbb{R} \to \mathbb{R} \), upper-bounded on \( C \).

An analogue of this theorem for vector spaces over \( \mathbb{Q} \) would be of much use for us. Two possible generalizations may be stated for, instead of considering only functionals, i.e. linear maps into \( \mathbb{Q} \), we may take into account functionals with values in the order-completion \( \mathbb{R} \) of \( \mathbb{Q} \). Note that linearity with respect to rational scalars is just additivity. We thus have two questions:

**Question 1.** Let \( E \) be a vector space over \( \mathbb{Q} \) and let \( C \) be a subset of \( E \) with the following properties:

(i) \( 1 - a^\circ C + a\mathbb{C} \subset C \) for \( a \in \mathbb{Q}, 0 < a < 1 \);

(ii) for every \( x \in E \) a number \( c_0 > 0 \) exists such that \( a x \in C \) for \( a \in \mathbb{Q}, 0 < a < c_0 \).

Further, let \( f: \mathbb{R} \to \mathbb{R} \) be an additive function defined on a subspace \( X \) of \( E \), upper-bounded on \( X \times C \). Can it be extended to an additive function \( F: \mathbb{R} \to \mathbb{R} \), upper-bounded on \( C \)?

**Question 2.** Let \( E, C, X \) be as in Question 1 and let \( f: \mathbb{R} \to \mathbb{R} \) be an additive function, upper-bounded on \( X \times C \). Can it be extended to an additive function \( F: \mathbb{R} \to \mathbb{R} \), upper-bounded on \( C \)?

We cannot expect a positive answer to Question 1. For if \( E = E, C = \{ -1, 1 \}, X = \mathbb{R} \), \( f(\xi) = \xi \) for \( \xi \in C \), then the conditions of Question 1 are fulfilled, and yet \( f \) clearly admits no extension to an additive function \( E \to \mathbb{R} \), upper-bounded on \( C \). However, whereas the answer to Question 1 is "no", the answer to Question 2 is "yes" (Theorem 1), and this turns out to be sufficient for our purposes.

The following terminology and notations will be used in the sequel: A subset \( C \) of a vector space \( E \) over \( \mathbb{Q} \) will be called \( \mathcal{Q} \)-convex iff

\[
(1 - a^\circ C \cap a\mathbb{C} \subset C)
\]

holds for \( a \in \mathbb{Q}, 0 < a < 1 \).

Let \( x_0 \in \mathcal{A} \cap E \). We say that the set \( A \) is \( \mathcal{Q} \)-radial at the point \( x_0 \) iff for every \( a \in E \) a number \( c_0 > 0 \) exists such that \( x_0 + a x \in A \) whenever \( a \in \mathbb{Q}, 0 < a < c_0 \).

Intuitively speaking, a set \( A \) is \( \mathcal{Q} \)-radial at \( x_0 \) if it contains a "radical segment" in each direction from the point \( x_0 \).

In the theory of convexity in real vector spaces, notions concerning to those defined above are well known. The prescript "\( \mathcal{Q} \)-" is employed here to emphasize the fact that rational scalars only are involved.

The set-theoretic operations are denoted as usual \( \cup, \cap, \setminus, \bigcup, \bigcap \), whereas the symbols \( \vdash \) (or absence of a dot) and \( \Sigma \), applied to sets,
stand for the algebraic operations on them. \( f_{\lambda A} \) is the restriction of a function \( f \) (defined in a larger domain) to the set \( A \).

Finally we make the following convention:

Greek letters always denote rational numbers.

\[ x \]

\[ y \]

**Lemma.** Let \( E \) be a vector space over \( \mathbb{Q} \), \( X \) and \( Y \) subspaces of \( E \) such that \( X \subseteq Y \) and \( Y \) has codimension 1 in \( Y \); let \( C \) be a \( Q \)-convex subset of \( E \), \( Q \)-radial at the point 0; finally, let \( f: X \to \mathbb{R} \) be an additive function such that \( f_{{\mathbb{R}} \cap C} \leq 1 \). Then there exists an additive function \( g: Y \to \mathbb{R} \) with the property \( g|_X = f \), \( g|_{\mathbb{R} \cap C} \leq 1 \).

**Proof.** According to the supposition we may write

\[ Y = X + QY, \]

where \( y \) is a point in \( Y \).

Consider the sets \( U, V \subseteq X \times Q \), defined by

\[ U = \{(x, \xi) \in X \times Q : \xi > 0, \quad \frac{x - y}{\xi} \in C \}, \]

\[ V = \{(x, \xi) \in X \times Q : \xi > 0, \quad \frac{x + y}{\xi} \in C \}. \]

The set \( C \) is \( Q \)-radial at 0, thus, given an \( x \in X \), the points \( (x, \pm y)|_{\xi} \) are in \( C \) for sufficiently large \( \xi \). It follows that the sets \( U \) and \( V \) are non-empty. Write

\[ u = \text{l.u.b. } \{ f(x) - \xi : (x, \xi) \in U \}, \]

\[ v = \text{g.l.b. } \{ f(x) : (x, \xi) \in V \}, \]

\[ -\infty < u \leq +\infty, \quad -\infty < v < +\infty. \]

We are going to show that

\[ u \leq v \]

and thus \( u \) and \( v \) are proper real numbers.

Suppose, on the contrary, that \( u < v \). Then there exists a pair \( (a, a) \in U \) such that \( f(a) - a > v \), and there exists a pair \( (a, \beta) \in V \) such that \( f(a) - a > \beta - f(a) \) or, equivalently,

\[ f(a + z) > a + \beta. \]

According to the definition of \( U \) and \( V \), we have \( x, z \in X \), \( \alpha, \beta > 0 \),

\[ \frac{x - y}{\alpha} \in C \quad \text{and} \quad \frac{x + y}{\beta} \in C. \]

Hence, in view of (7)

\[ \frac{x + z}{a + \beta} > a + \beta, \quad \frac{x - y}{\alpha} < \frac{x + y}{\beta} \in C, \]

a contradiction to (10) since \( f|_{\mathbb{R} \cap C} \leq 1 \) and so \( f(x + z) \leq a + \beta \). Consequently (9) holds; let \( c \in [u, v] \). Define the desired function \( g \) by

\[ g(x) = f(x), \quad g(y) = c \]

and extend it by \( Q \)-linearity onto the whole of \( Y \) (see (8)). It remains to verify that \( g|_{\mathbb{R} \cap C} \leq 1 \).

Let \( x + ay \in Y \setminus C \); we may assume \( a \neq 0 \). If \( a > 0 \), then

\[ \frac{x}{a} \in V, \quad \frac{1 - f(a)}{a} > v \implies v > \epsilon \]

and

\[ g(x + ay) = f(x) + ay \leq 1. \]

If \( a < 0 \) then

\[ \frac{x}{a} \in U, \quad \frac{1 - f(a)}{a} \leq u \leq c \]

and (11) also holds.

The following theorem is the key to our further considerations and gives a positive answer to Question 2 above. Its proof, as well as that of the preceding lemma, is essentially the same as in the real case.

**Theorem.** Let \( E \) be a vector space over \( \mathbb{Q} \), \( X \) a subspace of \( E \), and \( C \) a \( \mathbb{Q} \)-convex subset of \( E \), \( Q \)-radial at the point 0. If \( f: X \to \mathbb{R} \) is an additive function such that \( f_{{\mathbb{R}} \cap C} \leq 1 \), then there exists an additive function \( F: E \to \mathbb{R} \) with the properties \( F|_X = f \), \( F|_{\mathbb{R} \cap C} \leq 1 \).

**Proof.** Consider the family \( \mathfrak{M} \) of all pairs \( (X', f') \), where \( X' \) is a subspace of \( E \), \( f' : X' \to \mathbb{R} \) an additive function such that

\[ X \subseteq X' \subseteq E, \quad f'|_X = f, \quad f'|_{\mathbb{R} \cap C} \leq 1. \]

\( \mathfrak{M} \) is non-empty since \( (X, f) \in \mathfrak{M} \). It is ordered by the relation \( \preceq \), defined by

\[ (X', f') \preceq (X'', f'') \quad \text{if} \quad X' \subseteq X'' \quad \text{and} \quad f'|_{X'} = f''|_{X'}. \]

If \( \mathfrak{M} \) is any subfamily of \( \mathfrak{M} \), linearly ordered by \( \preceq \), then the pair \( \bigcup \mathfrak{M}, \tilde{f} \), where \( \tilde{f} \) being defined in the obvious way, is again in \( \mathfrak{M} \). It follows, in virtue of Zorn's lemma, that \( \mathfrak{M} \) contains a pair \( (X_{\text{max}}, f_{\text{max}}) \), maximal with respect to \( \preceq \). It suffices to show that \( X_{\text{max}} \subseteq E \) and \( f_{\text{max}} \) will then be the desired extension \( F \).

Suppose that there exists a \( g \in EN \) with \( g \neq 0 \). Then the space \( Y = X_{\text{max}} + QY \) satisfies the conditions of the preceding lemma with \( X_{\text{max}} \) for \( X \) and the pair \( (Y, g) \), where \( g \) is the function from the assertion of the lemma, would be a proper extension of the pair \( (X_{\text{max}}, f_{\text{max}}) \) thus contradicting the maximality of the latter.

We shall now be concerned with the construction of certain discontinuous additive functions.
Theorem 2. Let \( C \) be a \( Q \)-concave subset of the real line \( \mathbb{R} \), \( Q \)-radial at some point. Then either \( C \) is an interval or there exists a discontinuous additive function \( \Phi : \mathbb{R} \to \mathbb{R} \), upper-bounded on \( C \).

Proof. Assume at the moment that \( C \) is \( Q \)-radial at 0.

Suppose that \( C \) is not an interval. Then there exist \( x \) and \( y \) such that

\[
(12) \quad x \in C, \quad y \in C ,
\]

\[
(13) \quad 0 < x < y \quad \text{or} \quad 0 > x > y .
\]

Consider the set \( S \subset Q \times Q \), defined by

\[
S = \{ (\xi, \eta) : (1 + \xi)x + \eta y \in C, \eta > 0 \}.
\]

\( S \) is non-void since

\[
(14) \quad (-1, 1) \in S.
\]

Write

\[
s = \text{l.u.b.} \left\{ \frac{\xi}{\eta} : (\xi, \eta) \in S \right\};
\]

we shall see that \( s < \infty \). \( C \) is \( Q \)-radial at 0; so \( -\alpha y \in C \) for sufficiently small \( \alpha \), say, for \( 0 < \alpha < \alpha_0 \) where \( \alpha \) is a positive constant. Suppose \( (\xi, \eta) \in S \).

\[
(15) \quad \frac{\eta}{\xi} y \in C , \quad (1 + \xi)x + \eta y \in C
\]

(by the definition of \( S \)), whence, in view of (7),

\[
s = \left( \frac{1}{1 + \xi} \right) (1 + \xi)x + \eta y \in C ,
\]

contrary to (12). Consequently, the quotient \( \frac{\eta}{\xi} \) is upper-bounded for \( (\xi, \eta) \in S \) and \( s \) is a real number; furthermore, \( s > -1 \) on account of (14).

Put

\[
f(x) = 1, \quad f(y) = -s < 1
\]

an extend \( f \) by \( Q \)-linearity to an additive function \( f : \mathbb{R} \to \mathbb{R} \), where \( \mathbb{X} = Qx + Qy \); this is possible since \( x \) and \( y \) are \( Q \)-linearly independent in view of (12) and (13) and the \( Q \)-convexity of \( C \).

We are going to show that

\[
(16) \quad f(x + \alpha y) \leq 1.
\]

Suppose, on the contrary, that there exists a point \( ax + \beta y \in C \) with

\[
(17) \quad f(ax + \beta y) > 1 \quad \text{or, equivalently,}
\]

\[
(18) \quad \beta s < \alpha - 1.
\]

Consider three possible situations:

1. \( \beta > 0 \). Then \( (\alpha - 1) / \beta > s \), which is impossible since \( (\alpha - 1, \beta) \in S \).
2. \( \beta = 0 \). Then \( \alpha > 1 \), \( ax \in C \) whence, by the \( Q \)-convexity of \( C \), \( x \in C \), contrary to (15).
3. \( \beta < 0 \). Then \( s > (1 - \alpha) \theta \) and there exists, by the definition of \( \theta \), a pair \( (\xi, \eta) \in S \) such that \( \xi \eta > (1 - \alpha) \theta \), \( s > 0 \), whence

\[
(19) \quad f(ax + \beta y) = f((1 + \xi)x + \eta y) < f(1 + \xi)x + \eta y.
\]

Write

\[
\zeta = \frac{\eta}{\eta - \beta} (1 + \xi); \quad \eta \frac{\eta - \beta}{\eta - \beta};
\]

then, by (17), \( \zeta > 1 \). The pair \( (\xi, \eta) \) is in \( S \); so \( (1 + \xi)x + \eta y \in C \). Hence, in view of (17), the points

\[
(20) \quad \xi = \frac{1}{\zeta} (ax + \beta y), \quad \eta = \frac{1}{\zeta} (1 + \xi)x + \eta y
\]

and finally

\[
(21) \quad x = \frac{\eta}{\eta - \beta} u + \frac{\beta}{\eta - \beta} \eta
\]

are in \( C \). However, this is impossible, since \( \zeta = s \), as is easy to verify.

Consequently (16) follows and \( f \) may be extended to an additive function \( \Phi : \mathbb{R} \to \mathbb{R} \) such that \( F|_C < \infty \), on account of Theorem 1. \( F \) is discontinuous in view of (13), (15) and the fact that every continuous additive function \( R \to \mathbb{R} \) is a constant multiple of the identity.

The theorem is thus proved in the case where the set \( C \) is \( Q \)-radial at 0. If it is \( Q \)-radial at some other point \( a_0 \), then the set \( C - a_0 \) is \( Q \)-radial at 0 and the preceding case applies to the finding of a discontinuous additive function \( F : \mathbb{R} \to \mathbb{R} \) such that \( F|_{C - a_0} < \infty \), provided \( C \) is not an interval. It follows that \( F|_C < 1 + F(a_0) \).

Theorem 3. Let \( C \) be a \( Q \)-convex subset of the \( n \)-dimensional Euclidean space \( \mathbb{R}^n \), \( Q \)-radial at some point. Then either \( C \) contains a ball or there exists a discontinuous additive function \( \Phi : \mathbb{R}^n \to \mathbb{R} \), upper-bounded on \( C \).

Proof. We may assume that \( C \) is \( Q \)-radial at the origin, replacing \( C \), if necessary, by an appropriate translate of it (see the last argument in the proof of Theorem 2). Assume further that there exists no discontinuous additive function on \( C \). We are going to show that the origin is then an interior point of \( C \).

The proof will be by induction on \( n \). The case of \( n = 1 \) is the consequence of the preceding theorem since \( 0 \) is certainly an interior point of any interval, \( Q \)-radial at 0.

Assume that the assertion is true for some \( n \) and suppose that \( CC \subset \mathbb{R}^{n+1} \). Let \( \eta_1, \ldots, \eta_n \) be the usual orthonormal basis for \( \mathbb{R}^{n+1} \); thus \( \mathbb{R}^{n+1} = \mathbb{R}_a + \mathbb{R}_b \).
We have
(22) \[ R = \bigcup_{i \in I} X_i, \]
where
(23) \[ X_i = \sum_{j \in I} Qh_j. \]

K is defined as the set of those \(x\) for which the coefficient \(a_{\alpha}\) in the expansion (21) is positive. In fact, \(K\) has the desired properties (i), (ii), (iii):

(i) Straightforward verification.

(ii) Suppose that \(w(\mathbb{K}) > 0\); then the set \(K + K\) would contain an interval of positive length, in virtue of the theorem of H. Steinhaus, [7]. However, this is impossible since \(K + K = K\) and \(K\) is disjoint with the set \(Qh_1 - h_1\), dense in \(E\).

(iii) Let \(f: R \to R\) be an additive function such that \(f(x) < M\). For \(x \in X_i\), we have \(x + h_i \in K\); so \(f(x) < M - f(h_i)\). Thus \(f|_{X_i}\) is upper-bounded and, consequently, \(\lim_{x \to a} f(x) = 0\) since \(X_1\) is a \(Q\)-linear subspace of \(R\). Now, (iii) follows in view of (22).

Recall the definition of the classes \(A\) and \(B\).

The above example contradicts the conjecture of S. Marcus, mentioned in the introduction, that all sets which are in \(A\) enjoy the property (6); for \(w(\mathbb{K}) = 0\) by (i) and (ii), yet \(K \in B\) by (iii) and thus also \(K \in A\), on account of the following

**Theorem 4.** \(A = B\).

In other words: Let \(T\) be a set in \(E^n\); suppose that there exists a \(Q\)-continuous function \(f: A \to E\), defined and discontinuous in a convex domain \(A\), such that \(f\) is upper-bounded on \(T\). Then there exists a discontinuous additive function \(B \to E\), upper-bounded on \(T\).

**Proof.** Suppose \(f|_T \leq M\) and write \(C = \{x \in \mathbb{R}: f(x) < M\}\). The set \(C\) is \(Q\)-convex in view of (3). It is not difficult to see that \(C\) is \(Q\)-radial at each of its points. In fact, let \(x \in C\) and \(x \neq 0\); there exists a \(\gamma > 0\) such that \(x + y \gamma \in A\). Now, we have by (3),

\[ f(x_0 + ax) = f \left( \left(1 - \frac{a}{\gamma} \right) x_0 + \frac{a}{\gamma} (x_0 + y \gamma) \right) \leq \left(1 - \frac{a}{\gamma} \right) f(x_0) + \frac{a}{\gamma} f(x_0 + y \gamma) \]

for \(0 < a < \gamma\); so

\[ \limsup_{x \to a} f(x_0 + ax) < f(x_0) < M \]

and

\[ f(x_0 + ax) < M \]

for small \(a\); but this means just that \(C\) is \(Q\)-radial at \(x_0\).

Certainly \(C\) contains no ball, for a \(Q\)-convex function, upper-bounded on a ball, is necessarily continuous. Finally, observe that \(T \subset C\).

The assertion follows in virtue of Theorem 3.
Remarks on Anderson’s paper

“On topological infinite deficiency”

by

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Suppose that the topological space $X$ is the product of $n$ copies of an interval $J$ which is either closed or open. A closed subset $A$ of $X$ is said to be of infinite deficiency (briefly: deficient) in $X$ if there exists a homeomorphism $h$ of $X$ onto itself such that, for infinitely many $i$, the natural projections $\pi_i(h(A))$ are (at most) one-point sets in the interior of $J$.

The sets of infinite deficiency have been systematically investigated by R. D. Anderson in [1], [3]. The importance of these sets lies in their topological negligibility property (see condition (g) of Theorem 1 in this paper) and the property of extending homeomorphisms (here: Theorem 5); both properties have been established in their final form by Anderson, but the pioneer work in this respect was done by Klee ([9], [10]). For other results concerning negligibility see also [5], [6], [7], [8]. The theory of deficient sets can easily be transferred to the case of separable infinite-dimensional Fréchet spaces.

The present paper is a contribution to the theory of infinite deficiency. In Section 1 we establish some topological characterizations of sets of infinite deficiency. One of them (condition (ii) in Theorem 2), applied to $\mathbb{F}_n$ sets rather than to closed sets, gives a characterization of $\sigma$-deficient sets, i.e. of countable unions of deficient sets. This class of sets, being a natural generalization of deficient sets, is discussed in Section 2 (1). Finally, in Section 3 we establish a theorem on extending homeomorphisms to the pair: Hilbert cube $Q$ and its pseudointerior $s$, which is an analogue of the above-mentioned theorem of Anderson, dealing with a single space $X$ which is either $Q$ or $s$.

Our results are derived from two theorems of Anderson, which are stated explicitly as Theorem 1 in Section 1 and Theorem 5 in Section 3.

(1) Added in proof. $\sigma$-deficient sets (sets of type $\mathbb{Z}_1$) and their relations to problems of negligibility have been studied by R. D. Anderson and his collaborators, see, e.g., [5].