let (X_d, d') denote the (metric) quotient space obtained by identifying points which are zero distance apart. D has the obvious partial order induced by the quasi-order $\leqslant$. The canonical map $f_{\alpha\beta}$ is the one induced by Id: $(X, d_\beta) \to (X, d_\alpha)$, where of course $d_\alpha \leqslant d_\beta$. Then the natural maps $(X, D) \to \prod_{d \in D} (X, d) \to \prod_{d \in D} (X_d, d')$ give the obvious M-Lipschitz isomorphism onto $\operatorname{lim\,proj}(X_d, d')$.

For (X, D) an M-Lipschitz structure, define $\mathrm{Lip}(X, D)$ to be $[f\colon X \to R \mid f$ is a bounded, M-Lipschitz function$]$. (We use the customary metric for R.) For any pseudometric d on X, set $\mathrm{Lip}(X, d) = \{f\colon X \to R \mid |f(x) - f(y)| \leqslant kd(x, y)$ for some $k > 0\}$. $\mathrm{Lip}(X, d)$ is a Banach space [8]. In [2] it is shown that $d \leqslant e$ iff Id: $\mathrm{Lip}(X, d) \to \mathrm{Lip}(X, e)$ is a continuous imbedding. Hence $\{\mathrm{Lip}(X_d, d') \mid d \in D\}$ is an inductive family of Banach spaces. We topologize $\mathrm{Lip}(X, D)$ with the inductive limit topology.

As was mentioned before, from the fact that D is countably generated, we cannot conclude that D is generated by a single pseudometric. However, we do have

6.6. THEOREM. *Let D be countably generated. Then* $\mathrm{Lip}(X, D)$ *is a Frechet space iff D is generated by a single pseudometric.*

Proof. If D is generated by d, then $\mathrm{Lip}(X, D) \approx \mathrm{Lip}(X, d)$ by the identity map.

Now suppose $\mathrm{Lip}(X, D)$ is a Frechet space and D is generated by $\{d_i\}$, $i = 1, 2, \ldots$ We assume without loss of generality that $d_i \leqslant d_{i+1}$ for each i. The inductive limit topology is the strongest locally convex topology making all of the injection maps $\mathrm{Lip}(X, d_i) \to \mathrm{Lip}(X, D)$ continuous. By a theorem of Grothendieck [5], $\mathrm{Lip}(X, D)$ is isomorphic to $\mathrm{Lip}(X, d_n)$ for some n via the injection map. Then since $\mathrm{Lip}(X, d_n) \to \mathrm{Lip}(X, d_{n+k})$ must be an isomorphism for each k, $d_n \approx d_{n+k}$. Thus D is generated by d_n.

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Topologies uniquely determined by their continuous self map

by

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The major content of this paper is the search for topologies which are unique among the topologies for a given set with respect to their continuous self maps. Several classes of such spaces shall be given. Among them are the locally Euclidean, T_2-spaces; the separable metric, locally connected continua; all spaces of CW-complexes; and the non-discrete, cofinite spaces.

For notation let a pair (X, U) denote a topological space if X is the set of points in the space, and U is the collection of all *closed* subsets of the space. This variation from standard is a convenience to this study. If (X, U) is a topological space, let $C(U)$ denote the collection of all functions from X into X which are continuous with respect to U. A space (X, U) is *special* if and only if the only topology V on X such that $C(U) = C(V)$ is the topology $V = U$. If Q is a class of spaces, then a space (X, U) in Q is *Q-special* if and only if the only topology V on X such that (X, V) is in Q, and $C(V) = C(U)$ is the topology $V = U$. A space (X, U) is *T_1-special* if and only if it is Q-special when Q denotes the class of all T_1 spaces.

Now the problem may be described as the search for special spaces. The method will involve the study of spaces which are both maximal and minimal in the lattice of T_1-spaces with respect to their continuous self maps. Then conditions shall be given under which a T_1-special space is special. In the process a class of spaces which are absolutely minimal T_1-spaces with respect to their self maps shall be studied. There is a close relationship between this study and the study of S-admissibility [3]. This relationship shall be clearified, and several theorems on the construction of S-admissible classes shall be given.

Additional notation. If (X, U) and (Y, V) are spaces, let $C(U, V)$ denote the set of continuous functions from (X, U) into (Y, V). Let Y^X denote the set of all functions from X into Y. Let 2^X denote the set of all subsets of X. If f and g are two functions such that the composite function f of g is defined, then let $f \circ g$ denote that composite. If X is a set, let 1_X denote the identity map on X.

1. Generated spaces.

DEFINITION 1.1. A T_1-space (X, U) is *generated* if and only if for every T_1 topology V on X such that $C(U) \subseteq C(V)$, $U \subseteq V$.

If $F \subseteq X^X$, then an *F-fiber* is the inverse of a point under a map in F. It is simple to establish that a space (X, U) is generated if and only if the set of all $C(U)$-fibers is a subbasis for U (recall that U is the collection of closed sets). In fact we have

THEOREM 1.1. *If $F \subseteq X^X$, then there is a coarsest T_1-topology U on X such that $F \subseteq C(U)$. Also this topology is generated.*

Proof. Let F' denote the smallest subset of X^X which is closed under composition and contains F, 1_X and all constant maps of X into X. Let U be the topology for which the F'-fibers are a subbasis. Since $1_X \in F'$, (X, U) is T_1. Since F' is closed under composition, the inverse of an F'-fiber under a map in F' is still an F'-fiber. Thus $F \subseteq F' \subseteq C(U)$. Since the F'-fibers are closed in any T_1 space on X for which the members of $C(U)$ are continuous, (X, U) is a generated space.

It should be mentioned that the class of S^*-spaces (spaces (X, U) such that the $C(U)$-fibers form a *basis* for U) differs little from the class of generated spaces [4]. We shall construct an example to show that they are not identical. Let C denote the Cantor set on the unit interval of the real axis in the complex plane. Let A denote the unit interval on the imaginary axis. We shall construct a topology on $C \cup A \cup \{-i\}$. Let $C \cup A$ inherit its topology from the plane. A basic open set for $-i$ shall be any set which contains $-i$ and all but a finite number of non-zero points in C. This is a generated space. Let K consist of $-i$, i and all members of C which are greater than $\frac{1}{2}$. Any continuous self map which takes K to a point must also take 0 to the same point. Thus this is not an S^*-space.

DEFINITION 1.2. If X is an infinite set, and a is an infinite cardinal number, then let $X/a = \{K \subseteq X\colon K = X$ or cardinal of K is less than $a\}$.

If a is the cardinal of the integers, then $(X, X/a)$ is the cofinite space. For every infinite cardinal a, $(X, X/a)$ is a T_1-space. If a is greater than the cardinal of X, then $(X, X/a)$ is the discrete space.

THEOREM 1.2. *If X is an infinite set, and a is an infinite cardinal number, then $C(X/a) = \{f \in X^X\colon f$ is constant or for every $x \in X$, the cardinal of $f^{-1}(x)$ is less than $a\}$.*

Proof. If $f \in C(X/a)$, then since $(X, X/a)$ is T_1, f is constant or each $f^{-1}(x)$ is closed and has cardinal less than a. Now suppose $g \in X^X$, and each $g^{-1}(x)$ has cardinal less than a. If K is a non-trivial member of X/a, then K has cardinal less than a. Thus $g^{-1}(K) = \bigcup g^{-1}(x)$ $(x \in K)$ has cardinal less than a. Hence $g \in C(X/a)$.

THEOREM 1.3. *If X is infinite, and a is an infinite cardinal, then $(X, X/a)$ is generated.*

In fact, by Theorem 1.2 every closed set in $(X, X/a)$ is a $C(X/a)$-fiber.

COROLLARY 1.3. *All cofinite spaces and all discrete spaces are generated.*

DEFINITION 1.3. If M is a collection of T_1-spaces, and (X, U) is a T_1-space, then (X, U) is *related to M* if and only if for every topology V on X such that for each $(Y, W) \in M$, $C(U, W) \subseteq C(V, W)$, $U \subseteq V$. A T_1-space (X, U) is *related to a T_1-space* (Y, V) if and only if (X, U) is related to the singleton collection $\{(Y, V)\}$.

This is just a particular situation involving projective limits in T_1-spaces. It is not difficult to verify the following two equivalences:

DEFINITION 1.3(A). If (X, U) is a T_1-space, and M is a collection of T_1-spaces, then a *U-M-block* is the inverse of a closed set in some space (Y, V) in M under a function in $C(U, V)$. Then (X, U) is *related to M* if and only if the collection of all U-M-blocks is a subbasis for U.

DEFINITION 1.3(B). If (X, U) is a T_1-space, and M is a collection of T_1-spaces, then (X, U) is *related to M* if and only if (X, U) can be embedded in some product of copies of members of M (not necessarily finite — no restriction on the number of copies of each member of M).

It is interesting to note the similarity between the next theorem and the fact that a compact, T_2-space can be embedded in a product of copies of members of a set M of T_2-spaces provided the continuous maps of the space into the members of M form a point separating collection. (A collection of maps on a common domain X is *point separating* if and only if for each pair of distinct points in X there is a map in the collection which takes this pair to a pair of distinct points.)

It will be helpful to note that if (X, U) is a space and $G \subseteq U$, then G is a subbasis for U if and only if for each $L \in U$ and $p \in X-L$, a finite subset of G covers L but not p.

THEOREM 1.4. *If (X, U) is generated, M is a set of T_1-spaces, and $F = \bigcup \{C(U, V)\colon (Y, V) \in M\}$ is point separating, then (X, U) is related to M.*

Proof. Let $\emptyset \neq L \in U$ and $p \in X-L$. Since (X, U) is generated, the set of $C(U)$-fibers is a subbasis for U. Thus there exist $C(U)$-fibers $K_1, K_2, \ldots, K_n$ such that $L \subseteq \bigcup K_i$, and $p \notin \bigcup K_i$. Now $K_1 = f^{-1}(x)$ for some $f \in C(U)$ and $x \in X$. Since $f(p) \neq x$, there exists $(Y, V) \in M$ and $g \in C(U, V)$ such that $g(f(p)) \neq g(x)$. Now $K_1 \subseteq (g \circ f)^{-1}(g(x))$, and $p \notin (g \circ f)^{-1}(g(x))$. Similarly each K_i is a subset of some F-fiber which does not contain p. Thus L is covered by a finite set of F-fibers which does not cover p. Since each F-fiber is a U-M-block, the set of U-M-blocks is a subbasis for U, and (X, U) is related to M.

THEOREM 1.5. *If (X, U) is related to M, and each space in M is generated, and for every $(Y, V) \in M$, $C(V, U)$ is point separating; then (X, U) is generated.*

Proof. Let $\emptyset \neq L \in U$, and $p \in X - L$. Since (X, U) is related to M, there exist U-M-blocks $K_1, K_2, \ldots, K_n$ such that $L \subseteq \bigcup K_i$, and $p \notin \bigcup K_i$. Thus $K_1 = f^{-1}(H)$ for some $(Y, V) \in M$, $f \in C(U, V)$ and $H \in V$. Since (Y, V) is generated, and $f(p) \notin H$, there exist $g_1, g_2, \ldots, g_m \in C(V)$ and $y_1, y_2, \ldots, y_m \in Y$ such that $H \subseteq \bigcup g_i^{-1}(y_i)$, and $f(p) \notin \bigcup g_i^{-1}(y_i)$. Thus $K_1 \subseteq \bigcup (g_i \circ f)^{-1}(y_i)$, and $p \notin \bigcup (g_i \circ f)^{-1}(y_i)$. Let $F = \bigcup \{C(U, W)$: $(Z, W) \in M\}$. Then each $g_i \circ f \in F$, and K_1 is a subset of a finite union of F-fibers which does not contain p. Similarly, each K_i is a subset of such a finite union. Thus there exist F-fibers $J_1, J_2, \ldots, J_u$ such that $L \subseteq \bigcup J_i$, and $p \notin \bigcup J_i$. Now for some $(Z, W) \in M$, $h \in C(U, W)$ and $z \in Z$, $J_1 = h^{-1}(z)$. Thus $h(p) \neq z$. Since $C(W, U)$ is point separating, there exists $t \in C(W, U)$ such that $t(h(p)) \neq t(z)$. Hence $J_1 \subseteq (t \circ h)^{-1}(t(z))$, and $p \notin (t \circ h)^{-1}(t(z))$. Similarly, each J_i is a subset of some $C(U)$-fiber which does not contain p. Thus L is covered by a finite number of $C(U)$-fibers which do not cover p. Hence the set of $C(U)$-fibers is a subbasis for U, and (X, U) is generated.

Due to Theorem 1.4, Theorem 1.5 may be stated as follows:

THEOREM 1.5(A). *If M is a set of generated spaces, (X, U) is homeomorphic to a subspace of a product of copies of members of M, and for every $(Y, V) \in M$, (Y, V) is homeomorphic to a subspace of a product of copies of (X, U), then (X, U) is generated.*

THEOREM 1.6. *If for every $a \in A$, X_a is a generated space, then $\prod X_a$ $(a \in A)$ is generated.*

Proof. Since each X_a can be embedded in $\prod X_a$, Theorem 1.5 is applicable.

THEOREM 1.7. *If (A, V) is a subspace of a generated space (X, U), and $C(U, V)$ is point seperating, then (A, V) is generated.*

Proof. See Theorem 1.5.

THEOREM 1.8. *If (X, U) is generated, (X, V) is T_1, and $V \subseteq U$, then (X, U) is related to (X, V).*

Proof. Since $V \subseteq U$, $1_X \in C(U, V)$. Thus $C(U, V)$ is point separating. Then by Theorem 1.4, (X, U) is related to (X, V).

THEOREM 1.9 (Magill). *If (X, U) is completely regular and T_2, and (X, U) contains a copy of the unit interval, then (X, U) is generated* [4].

THEOREM 1.10. *If (X, U) is T_2 and completely regular, (X, V) is generated, and $U \subseteq V$, then (X, V) is completely regular and T_2.*

Proof. By Theorem 1.8, (X, V) is related to (X, U). Hence (X, V) can be embedded in a product of copies of (X, U). Hence (X, V) is T_2 and completely regular.

THEOREM 1.11. *If (X, U) is generated and T_2, then (X, U) is T_3.*

Proof. Let $K \in U$ and $p \in X - K$. Since (X, U) is generated, there exists a finite union of the form $\bigcup f_i^{-1}(p_i)$ where each $f_i \in C(U)$, and each $p_i \in X$, and such that $K \subseteq \bigcup f_i^{-1}(p_i)$, and $p \notin \bigcup f_i^{-1}(p_i)$. For each i, $f_i(p) \neq p_i$, and there are disjoint open sets $u(i)$ and $v(i)$ such that $f_i(p) \in v(i)$, and $p_i \in u(i)$. Thus $\bigcup f_i^{-1}(u(i))$ covers K, and $\bigcap f_i^{-1}(v(i))$ covers p. Hence K and p are separated, and (X, U) is regular.

THEOREM 1.12. *Assume (X, U) is compact, T_2, and for every $p, q \in X$ with $p \neq q$, there exist $C(U)$-fibers $K_1, K_2, \ldots, K_n$ such that $\bigcup K_i = X$, and no K_i contains both p and q. Then (X, U) is generated.*

Proof. By Theorem 1.1 there is a coarsest T_1-topology V on X such that $C(U) \subseteq C(V)$, and (X, V) is generated. Thus $V \subseteq U$. To show that (X, V) is T_2 let $p, q \in X$ with $p \neq q$. There exist $C(U)$-fibers $K_1, K_2, \ldots, K_n$ such that $\bigcup K_i = X$, and no K_i contains both p and q. Since $C(U) \subseteq C(V)$, and (X, V) is T_1, each $K_i \in V$. Let $L_p = \bigcup \{K_i\colon p \in K_i\}$, and $L_q = \bigcup \{K_i\colon p \notin K_i\}$. Then $L_p, L_q \in V$, $L_p \cup L_q = X$, $p \in L_p$, $q \in L_q$, $q \notin L_p$ and $p \notin L_q$. Thus $X - L_q$ and $X - L_p$ separate p and q in (X, V). Thus (X, V) is T_2. But (X, U) is minimal T_2 because (X, U) is compact ([7], Theorem 16.21, p. 126). Thus $V \subseteq U$ implies that $V = U$. Hence (X, U) is generated.

In closing this section it is worthy of mention that K. D. Magill, Jr. has given an example of a non-degenerate continuum by H. Cook which is not generated ([4] and [1]).

2. Upper special spaces.

DEFINITION 2.1. A space (X, U) is *upper special* if and only if the only topology V on X such that $U \subseteq V$, and $C(U) = C(V)$ is the topology $V = U$.

THEOREM 2.1. *If (X, U) is generated and upper special, then (X, U) is T_1-special.*

Proof. Trivial.

DEFINITION 2.2. A space (X, U) is *full* if and only if (X, U) is a T_1-space with no isolated points, and the only topology V on X with no isolated points and such that $U \subseteq V$ and $C(U) \subseteq C(V)$ is the topology $V = U$.

THEOREM 2.2. *If (X, U) is full, then (X, U) is upper special.*

Proof. This is a result of the fact that a point p of a T_1-space is isolated if and only if some function which takes the complement of p to a point in the complement of p and leaves p fixed is continuous.

DEFINITION 2.3. If $x \in X$ and $f \in X^X$, then f *is* $x-x$ if and only if $f^{-1}(x) = x$.

DEFINITION 2.4. If $x \in X$, $A \subseteq X$, and $F \subseteq X^X$, then A is *x-large over F* if and only if $A \subseteq X - x$ and there exist $f_1, f_2, \ldots, f_n \in F$ such that for $1 \leqslant i \leqslant n$, f_i is $x-x$, and $X - x = \bigcup f_i^{-1}(A)$ $(1 \leqslant i \leqslant n)$.

DEFINITION 2.5. Let X be an infinite set, and $F \subseteq X^X$. Define condition F-1, condition F-2, and condition F-3 as follows:

CONDITION F-1. *The map $1_X \in F$, and for every $f, g \in F$, $f \circ g \in F$.*

CONDITION F-2. *If $\{K_1, K_2, \ldots, K_n\}$ is a finite set of F-fibers, then either $X = \bigcup K_i$ or $X - \bigcup K_i$ is infinite.*

CONDITION F-3. *If $x \in X$ and $A \subseteq X - x$, then either A is x-large over F or $[(X-x) - A]$ is x-large over F.*

DEFINITION 2.6. If $F \subseteq X^X$, then the *F-collection* is the set $\{K \subseteq X$: if $f_1, f_2, \ldots, f_n \in F$, then $\bigcup f_i^{-1}(K) = X$ or $X - \bigcup f_i^{-1}(K)$ is infinite$\}$.

THEOREM 2.3. *Suppose X is an infinite set, $F \subseteq X^X$, F satisfies* F-1, F-2 *and* F-3, *and U is the F-collection. Then (X, U) is a full topological space, $F \subseteq C(U)$, $C(U)$ satisfies $C(U)$-1, $C(U)$-2 and $C(U)$-3, and U is the $C(U)$-collection.*

Proof. (The proof is lengthy and will be presented as a sequence of lemmas.)

LEMMA 1. *The set $X \in U$, and each finite subset of X is in U.*

Proof. If K is a finite, non-empty subset of X, let $f_1, f_2, \ldots, f_n \in F$. Since K is finite, each $f_i^{-1}(K)$ is a finite union of F-fibers. Thus $\bigcup f_i^{-1}(K)$ is a finite union of F-fibers. By condition F-2, $K \in U$. That $X \in U$, and $\emptyset \in U$ is obvious.

LEMMA 2. *If $x \in X$, $A \subseteq B \subseteq X - x$ and A is x-large over F, then B is x-large over F.*

Proof. Since A is x-large over F, there exist $x-x$ functions $f_1, f_2, \ldots, f_n$ in F such that $X - x = \bigcup f_i^{-1}(A) \subseteq \bigcup f_i^{-1}(B)$. For each i, $f_i(x) = x \notin B$. Hence $x \notin \bigcup f_i^{-1}(B)$, and $X - x = \bigcup f_i^{-1}(B)$. Thus B is x-large over F.

LEMMA 3. *If $x \in X$ and $X - x = A_1 \cup A_2 \cup \ldots \cup A_m$, then there exists i such that $1 \leqslant i \leqslant m$, and A_i is x-large over F.*

Proof. For $m = 1$, the result follows from the fact that $1_X \in F$. If $m = 2$, and A_1 is not x-large over F, then since A_2 contains the complement of A_1, A_2 is x-large over F by condition F-3 and Lemma 2. Now suppose the lemma holds for $m = k \geqslant 2$. If $X - x = (A_1 \cup A_2) \cup A_3 \cup \cup \ldots \cup A_k \cup A_{k+1}$, then either some A_i is x-large over F or $A_1 \cup A_2$ is x-large over F. If $A_1 \cup A_2$ is x-large over F, then there exist $x-x$ functions $f_1, f_2, \ldots, f_n$ in F such that

$$X - x = \bigcup f_j^{-1}(A_1 \cup A_2) = \left(\bigcup f_j^{-1}(A_1)\right) \cup \left(\bigcup f_j^{-1}(A_2)\right) \quad (1 \leqslant j \leqslant n).$$

As in the case $m = 2$ we have sufficient conditions to assume without loss of generality that $\bigcup f_j^{-1}(A_1)$ is x-large over F. Thus there exist $x-x$ functions $g_1, g_2, \ldots, g_s$ such that

$$\begin{aligned} X - x &= \bigcup g_t^{-1}[\bigcup f_j^{-1}(A_1) \ (1 \leqslant j \leqslant n)] \quad (1 \leqslant t \leqslant s) \\ &= \bigcup (f_j \circ g_t)^{-1}(A_1) \quad (1 \leqslant j \leqslant n\,,\ 1 \leqslant t \leqslant s)\,. \end{aligned}$$

Since each $f_j \circ g_t$ is $x-x$, A_1 is x-large over F.

LEMMA 4. *If $K \notin U$, then there exist $x \in X$ and $f_1, f_2, \ldots, f_n \in F$ such that $X - x = \bigcup f_i^{-1}(K)$ $(1 \leqslant i \leqslant n)$.*

Proof. Since $K \notin U$, there exist $g_1, g_2, \ldots, g_m \in F$ such that the set $X - \bigcup g_i^{-1}(K)$ $(1 \leqslant i \leqslant m)$ is non-empty and finite. Now let the set $X - \bigcup g_i^{-1}(K) = \{x, x_1, x_2, \ldots, x_p\}$. Thus $X - x = \{x_1, x_2, \ldots, x_p\} \cup \left(\bigcup g_i^{-1}(K)\right)$. Now $\{x_1, x_2, \ldots, x_p\}$ is not x-large over F because $\{x_1, x_2, \ldots, x_p\}$ is finite and is a member of U by Lemma 1. Thus by Lemma 3, $\bigcup g_i^{-1}(K)$ is x-large over F. Hence there exist $x-x$ functions $h_1, h_2, \ldots, h_s \in F$ such that

$$\bigcup h_j^{-1}[\bigcup g_i^{-1}(K) \ (1 \leqslant i \leqslant m)] \ (1 \leqslant j \leqslant s) = X - x\,.$$

This implies that

$$\bigcup (g_i \circ h_j)^{-1}(K) \ (1 \leqslant i \leqslant m\,,\ 1 \leqslant j \leqslant s) = X - x\,.$$

This completes the proof.

LEMMA 5. *If $K_1, K_2 \in U$ then $K_1 \cup K_2 \in U$.*

Proof. Suppose not. By Lemma 4 there exist $x \in X$ and $f_1, f_2, \ldots, f_n \in F$ such that $X - x = \left(\bigcup f_i^{-1}(K_1)\right) \cup \left(\bigcup f_i^{-1}(K_2)\right)$. Using Lemma 3 we assume without loss of generality that $\bigcup f_i^{-1}(K_1)$ is x-large over F. This implies that $K_1 \notin U$. This is a contradiction.

LEMMA 6. *If $\{K_b\colon b \in B\} \subseteq U$, then $\bigcap K_b \, (b \in B) = K \in U$.*

Proof. Suppose $K \notin U$. By Lemma 4 there exist $x \in X$ and $f_1, f_2, \ldots, f_n \in F$ such that $X - x = \bigcup f_i^{-1}(K)$ $(1 \leqslant i \leqslant n)$. For $1 \leqslant i \leqslant n$ there exists $i' \in B$ such that $f_i(x) \notin K_{i'}$ because $f_i(x) \notin K$. Now $\bigcup f_i^{-1}(K) \subseteq \bigcup f_i^{-1}(K_{i'})$ because $K \subseteq K_{i'}$. Since $x \notin \bigcup f_i^{-1}(K_{i'})$, $X - x = \bigcup f_i^{-1}(K_{i'})$. By Lemma 3 there exists s such that $f_s^{-1}(K_{s'})$ is x-large over F. Thus there exist $x-x$ functions $g_1, g_2, \ldots, g_m$ in F such that $X - x = \bigcup g_j^{-1}\left(f_s^{-1}(K_{s'})\right)$ $(1 \leqslant j \leqslant m)$. This implies that $X - x = \bigcup (f_s \circ g_j)^{-1}(K_{s'})$ $(1 \leqslant j \leqslant m)$. Hence $K_{s'} \notin U$. This is a contradiction.

LEMMA 7. *The space (X, U) is a T_1-space, and $F \subseteq C(U)$.*

Proof. By Lemma 1, Lemma 5 and Lemma 6, (X, U) is a T_1-space. Suppose there exists $f \in F$ such that $f \notin C(U)$. Then there exists $K \in U$ such that $f^{-1}(K) \notin U$. By Lemma 4 there exist $x \in X$ and $f_1, f_2, \ldots, f_n \in F$

such that $X-x=\bigcup f_i^{-1}(f^{-1}(K))$ $(1\leqslant i\leqslant n)=\bigcup (f\circ f_i)^{-1}(K)$. Thus $K\notin U$. This is a contradiction.

LEMMA 8. *The space (X, U) has no isolated points.*

Proof. Suppose p is an isolated point of (X, U). Then $X-p\in U$. But $1_X^{-1}(X-p)=X-p$, and $X-(X-p)$ is finite. Thus $X-p\notin U$. This is a contradiction.

LEMMA 9. *If $\{K_1, K_2, \ldots, K_n\}$ is a finite set of $C(U)$-fibers, then either $\bigcup K_i=X$ or $X-\bigcup K_i$ is infinite.*

Proof. Since (X, U) is T_1, each K_i is closed and $\bigcup K_i$ is closed. Since (X, U) has no isolated points, $X-\bigcup K_i$ is infinite or empty.

LEMMA 10. *The collection $C(U)$ satisfies $C(U)$-1, $C(U)$-2 and $C(U)$-3.*

Proof. That $C(U)$-1 is satisfied is obvious, and $C(U)$-2 follows from Lemma 9. Now $C(U)$-3 holds because by Lemma 7, $F\subseteq C(U)$.

LEMMA 11. *The collection U is the $C(U)$-collection.*

Proof. Let U' denote the $C(U)$-collection. By Lemma 7, $F\subseteq C(U)$. Thus $U'\subseteq U$. Suppose there exists $K\in U-U'$. Then there exist $f_1, f_2, \ldots, f_n\in C(U)$ such that $X\neq\bigcup f_i^{-1}(K)$, and $X-\bigcup f_i^{-1}(K)$ is finite. Since $K\in U$, and each $f_i\in C(U)$, $\bigcup f_i^{-1}(K)\in U$. This contradicts Lemma 8. Thus $U'=U$.

LEMMA 12. *The space (X, U) is full.*

Proof. Suppose not. Then there exists a topology V on X such that $U\subseteq V$, $U\neq V$, $C(U)\subseteq C(V)$, and (X, V) has no isolated points. Let $v\in V-U$. Then there exist $x\in X$ and $f_1, f_2, \ldots, f_n\in C(U)$ such that $X-x=\bigcup f_i^{-1}(v)$ by Lemma 4. Since $v\in V$, and each $f_i\in C(V)$, $\bigcup f_i^{-1}(v)\in V$. Thus $X-x\in V$. Thus x is an isolated point of (X, V). This is a contradiction.

Now Theorem 2.3 follows from Lemma 7, Lemma 10, Lemma 11 and Lemma 12.

THEOREM 2.4. *Suppose (X, U) is a T_1-space which has no isolated points, and for every $x\in X$ and for every $A\subseteq X-x$ such that x is a limit point of A with respect to U, A is x-large over $C(U)$. Then $C(U)$ satisfies $C(U)$-1, $C(U)$-2 and $C(U)$-3, and U is the $C(U)$-collection. Thus (X, U) is full.*

Proof. It is clear that $C(U)$-1, $C(U)$-2 and $C(U)$-3 hold. Let U' denote the $C(U)$-collection. Suppose $u\in U$. If $f_1, f_2, \ldots, f_n\in C(U)$, then $\bigcup f_i^{-1}(u)\in U$. Since (X, U) has no isolated points, $\bigcup f_i^{-1}(u)=X$ or $X-\bigcup f_i^{-1}(u)$ is infinite. Thus $u\in U'$, and $U\subseteq U'$. Suppose $K\subseteq X$, and $K\notin U$. Then there exists $p\in X-K$ such that p is a limit point of K with respect to U. Thus K is p-large over $C(U)$. Hence $K\notin U'$. Thus $U=U'$, and U is the $C(U)$-collection.

3. More about upper special spaces.

DEFINITION 3.1. A space (A, V) is an *h-retract of* (X, U) if and only if (A, V) is a subspace of (X, U), $h\in C(U, V)$ and the restriction, $h|A$, of h to A is 1_A.

THEOREM 3.1. *Suppose (X, U) is T_1, and (A, V) is an h-retract of (X, U).*

(I) *If (A, V) is upper special, then for every topology W on X such that $U\subseteq W$ and $C(U)=C(W)$, (A, V) is a subspace of (X, W).*

(II) *If (A, V) is special, then for every topology W on X such that $C(U)=C(W)$, (A, V) is a subspace of (X, W).*

Proof. (I) Let (A, V') be the subspace of (X, W). Since $h\in C(U)=C(W)$, (A, V') is a retract of (X, W). Thus $C(V)$ is the collection of restrictions to A of functions in $C(U, V)$, and $C(V')$ is the collection of restrictions to A of functions in $C(W, V')$. Hence $C(U)=C(W)$ implies that $C(V)=C(V')$. Now $U\subseteq W$ implies that $V\subseteq V'$. Thus since (A, V) is upper special, $V=V'$.

(II) As in (I) we conclude that $C(V)=C(V')$. Thus $V=V'$ because (A, V) is special.

DEFINITION 3.2. (I) If W is a collection of spaces, and (X, U) is a space, then (X, U) is *anti-related to* W if and only if U is the finest topology on X such that for every $(Y, V)\in W$ each element of $C(V, U)$ is continuous from (Y, V) into X.

(II) If (X, U) and (Y, V) are spaces, then (X, U) is *anti-related to* (Y, V) if and only if (X, U) is anti-related to the singleton collection $\{(Y, V)\}$.

It is clear that if f is an identification map of (X, U) onto (Y, V), then (Y, V) is anti-related to (X, U). Also if (X, U) is coherent with respect to a collection W of subspaces of (X, U), then (X, U) is anti-related to W. (The space (X, U) is *coherent* with respect to W if for each $K\subseteq X$ such that $K\cap A$ is closed in A for every $A\in W$, $K\in U$ [6].) (Sometimes called the *weak* topology.)

THEOREM 3.2. *Suppose W and Z are collections of subspaces of (X, U), and (X, U) is coherent with respect to Z. If each member of Z is the image under an identification map of some member of W, then (X, U) is anti-related to W.*

Proof. Suppose not. Then there exists a topology V on X such that V is strictly finer than U, and for each $(A, T)\in W$, $C(T, U)\subseteq C(T, V)$. There is a $K\in V$ such that for some $(B, S)\in Z$, $K\cap B\notin S$. Thus if (B, S') is the subspace of (X, V), then $S'\neq S$. Since $U\subset V$, $S\subset S'$. For some $(A, T)\in W$, (B, S) is the image under an identification map f of (A, T).

Now $f \notin C(T, S')$. Hence $f \in C(T, U)$, and $f \notin C(T, V)$. This contradicts the fact that $C(T, U) \subseteq C(T, V)$.

THEOREM 3.3. *If (X, U) is anti-related to a collection W of upper special retracts of (X, U), then (X, U) is upper special.*

Proof. Suppose V is a topology on X such that $U \subseteq V$ and $C(U) = C(V)$. By Theorem 3.1 (I) each space in W is a subspace of (X, V). If (A, T) is an h-retract of (X, U) in W, and $f \in C(T, U)$, then $f \circ h \in C(U) = C(V)$. Hence $f = (f \circ h| A) \in C(T, V)$. Thus for each $(A, T) \in W$, $C(T, U) \subseteq C(T, V)$. Now since (X, U) is anti-related to W, $V = U$. Hence (X, U) is upper special.

We now seek sufficient conditions under which a finite product of copies of a space (X, U) is coherent with respect to the collection of all copies of (X, U) in the product space.

DEFINITION 3.3. A space (X, U) is *sequential* if and only if (X, U) is T_1, and for every $A \subseteq X$ and for every limit point p of A there exists a sequence of distinct points of A which converges to p. A space (X, U) is *homosequential* if and only if (X, U) is sequential, and if $x_1, x_2, \ldots$ is a sequence of distinct points converging to a point x, and f is a function from $\{x, x_1, x_2, \ldots\}$ into X such that $f(x_1), f(x_2), \ldots$ converges to $f(x)$, then there exists $g \in C(U)$ such that g agrees with f at x and on some subsequence of $x_1, x_2, \ldots$

It should be mentioned here that T_1 and first countable imply sequential, but the converse is not true. A cofinite space provides an example. It is also clear that convergence of a sequence does not imply unique convergence.

THEOREM 3.4. *If for $1 \leqslant i \leqslant n$, X_i is a copy of a homosequential space (X, U), and $\prod X_i$ $(1 \leqslant i \leqslant n)$ is sequential, then $\prod X_i$ is coherent with respect to the collection of all copies of (X, U) in $\prod X_i$.*

We need the following

LEMMA. *If $x, x_1, x_2, \ldots$ is a sequence of distinct points of X which converges to x, and f is a function from $\{x, x_1, x_2, \ldots\}$ into $\prod X_i$ $(1 \leqslant i \leqslant n)$ such that $f(x_1), f(x_2), \ldots$ converges to $f(x)$, then there exists a continuous map g from (X, U) into $\prod X_i$ such that g agrees with f at x and on some subsequence of $x_1, x_2, \ldots$*

Proof. The case $n = 1$ is a result of the fact that (X, U) is homosequential. Assume the result for $n \leqslant k$, and assume that $f(x_1), f(x_2), \ldots$ is a sequence of points of $\prod X_i$ $(1 \leqslant i \leqslant k+1)$ which converges to $f(x)$. Let P_1 and P_2 denote the projection maps of $\prod X_i$ $(1 \leqslant i \leqslant k+1)$ onto X_1 and onto $\prod X_i$ $(2 \leqslant i \leqslant k+1)$ respectively. By the inductive hypothesis there is a continuous map g of (X, U) into X_1 which agrees with $P_1 \circ f$ at x and on some subsequence Q_1 of $x_1, x_2, \ldots$ Similarly, there is a map h

of (X, U) into $\prod X_i$ $(2 \leqslant i \leqslant k+1)$ which agrees with $P_2 \circ f$ at x and on some subsequence Q of Q_1. Now the product map $g \times h$ is continuous from (X, U) into $\prod X_i$ $(1 \leqslant i \leqslant k+1)$ and agrees with f at x and on Q.

Proof of Theorem. Now for the case $n = 1$, the theorem is trivial. Assume $n > 1$, and $\prod X_i$ $(1 \leqslant i \leqslant n)$ is not coherent with respect to the copies of (X, U) in $\prod X_i$ $(1 \leqslant i \leqslant n)$. There exists a subset K of $\prod X_i$ such that K is not closed, and for every copy A of (X, U) in $\prod X_i$, $A \cap K$ is closed in A. There is a sequence Q_1 of distinct points of K converging to a point x not in K. Since Q_1 forms an infinite set, there is a projection map P_j of $\prod X_i$ onto X_j such that $P_j(Q_1)$ is infinite. Now for some subsequence Q_2 of Q_1, $x \cup Q_2$ is the graph of a function f from $P_j(x) \cup P_j(Q_2)$ into $\prod X_i$ $(1 \leqslant i \leqslant n,\ i \neq j)$. By Lemma there exists a continuous map g from X_j into $\prod X_i$ $(1 \leqslant i \leqslant n,\ i \neq j)$ such that g agrees with f at $P_j(x)$ and on some subsequence of $P_j(Q_2)$. Now the graph G of g in $\prod X_i$ $(1 \leqslant i \leqslant n)$ is a copy of (X, U), but $G \cap K$ is not closed in G. This is a contradiction.

COROLLARY 3.4.1. *If for $1 \leqslant i \leqslant n$, X_i is a copy of an upper special, homosequential space (X, U), and $\prod X_i$ $(1 \leqslant i \leqslant n)$ is sequential, then $\prod X_i$ is upper special.*

Proof. Let A be a copy of (X, U) which is an h-retract of $\prod X_i$. Let Z be the collection of homeomorphic images of A in $\prod X_i$. By Theorem 3.4, $\prod X_i$ is coherent with respect to Z. Since every homeomorphism is an identification map, $\prod X_i$ is anti-related to A by Theorem 3.2. By Theorem 3.3, $\prod X_i$ is upper special.

THEOREM 3.5. *Let L denote the class of all sequential spaces. If (X, U) is generated and homosequential, then (X, U) is L-special.*

Proof. Suppose not. There exists $(X, V) \in L$ such that $U \neq V$, and $C(U) = C(V)$. Since (X, U) is generated, $U \subseteq V$. Thus there exist $K \in V - U$ and a sequence $x_1, x_2, \ldots$ of distinct points of K converging to a point x of $X - K$ with respect to U. If (X, V) is discrete, then (X, V) is T_1-special (a trivial result). Thus $U \neq V$ implies that $C(U) \neq C(V)$ which is a contradiction. Thus (X, V) is not discrete, and there exists a sequence $y_1, y_2, \ldots$ of distinct points of X converging to a point y with respect to V. Since $U \subseteq V$, $y_1, y_2, \ldots$ converges to y with respect to U. Define f from $\{y, y_1, y_2, \ldots\}$ into X by $f(y) = x$, and for each i, $f(y_i) = x_i$. Since (X, U) is homosequential, there exists $g \in C(U) = C(V)$ such that $g(y) = x$, and g agrees with f on some subsequence of $y_1, y_2, \ldots$ Since $K \in V$, and $x \notin K$, f fails to preserve a limit point with respect to V. This is a contradiction.

4. Special spaces.

THEOREM 4.1. Assume. *The space (X, U) is T_1-special and is not discrete. If $p, q, r, s \in X$ such that $p \neq q$ and $r \neq s$, then there exists $f \in C(U)$ such that $f(p) = r$, and $f(q) = s$. Then (X, U) is special.*

Proof. Suppose not. There exists a topology V on X such that $V \neq U$ and $C(V) = C(U)$. Since (X, U) is T_1-special, (X, V) is not T_1, and some point a is a limit point of a point b with respect to V. Fix $x \in X$. For each $y \in X$ there is an $f_y \in C(U) = C(V)$ such that $f_y(a) = x$, and $f_y(b) = y$. Then x is a limit point of each y distinct from x, and (X, V) is the indiscrete space. Thus $C(V)$ is the set X^X. Hence $C(U) = X^X$, and (X, U) is discrete. This is a contradiction.

THEOREM 4.2. Assume. *The space (X, U) is T_1-special. The space (A, W) is an h-retract of (X, U). The space (A, W) is special. The set $C(U, W)$ is point separating.*

Then (X, U) is special.

Proof. Suppose not. There exists (X, V) such that $U \neq V$, $C(U) = C(V)$ and some point a is a limit point of a point b with respect to V. By Theorem 3.1 (II), (A, W) is a subspace of (X, V). Then there exists a function f in $C(U, W) = C(V, W)$ such that $f(a) \neq f(b)$. Thus $f(a)$ is a limit point of $f(b)$ in (A, W). This contradicts the fact that (A, W) is T_1.

THEOREM 4.3. *If X is an infinite set, a is an infinite cardinal number and $(X, X/a)$ is not discrete, then $(X, X/a)$ is special.*

Proof. Assume (X, U) is T_1, and $C(U) = C(X/a)$.

LEMMA 1. $X/a \subseteq U$.

Proof. $(X, X/a)$ is generated.

LEMMA 2. *If $A \in U - X/a$ and $B \subseteq A$, then $B \in U$.*

Proof. Let $p \in X - A$. For every $x \in A - B$ define $f_x(y) = y$ if $y \neq x$, and $f_x(x) = p$. By Theorem 1.2 for every $x \in A - B$, $f_x \in C(X/a) = C(U)$. Thus $f_x^{-1}(X - A)$ is open in (X, U). Now $X - B = \bigcup f_x^{-1}(X - A)$ $(x \in A - B)$ is open in (X, U), and $B \in U$.

LEMMA 3. *If $A \in U - X/a$, $B \subseteq X - A$, and cardinal of $B \leqslant$ cardinal of A, then $B \in U$.*

Proof. Let h be a bijection of a subset C of A onto B. Define $f(x) = h(x)$ if $x \in C$, $f(x) = h^{-1}(x)$ if $x \in B$ and $f(x) = x$ if $x \in X - (B \cup C)$. By Theorem 1.2, $f \in C(X/a) = C(U)$. By Lemma 2, $C \in U$. Thus $B = f^{-1}(C) \in U$.

LEMMA 4. *If $A \in U - X/a$ and $B \subseteq X$ such that cardinal of $B \leqslant$ cardinal of A, then $B \in U$.*

Proof. $B = (A \cap B) \cup [(X - A) \cap B]$. By Lemma 2 and Lemma 3, $B \in U$.

Now if $U = 2^X$, then (X, U) is T_1-special, $X/a = U$, and $(X, X/a)$ is discrete. This is a contradiction. Thus there is a least cardinal number b such that for some $A \subseteq X$, cardinal of $A = b$, and $A \notin U$. By Lemma 1, $a \leqslant b$. By definition, $X/b \subseteq U$. If $A \in U$ and cardinal of $A \geqslant b$, by Lemma 4 every subset of X with cardinal b is in U. This contradicts the definition of b. Hence $A \in U$ implies that $A \in X/b$, and $U = X/b$.

Assume there exists $A \in U - X/a$, and let $p \in A$. Define $f(x) = p$ if $x \in A$, and $f(x) = x$ if $x \in X - A$. By Theorem 1.2, $f \in C(X/b) = C(U) = C(X/a)$. Hence $A = f^{-1}(p) \in X/a$. This is a contradiction, and $U \subseteq X/a$. Thus by Lemma 1, $U = X/a$, and $(X, X/a)$ is T_1-special. Due to Theorem 1.2 and Theorem 4.1 it is clear that $(X, X/a)$ is special.

THEOREM 4.4. *If (X, U) is a non-discrete, cofinite space, and for $1 \leqslant i \leqslant n$, X_i is a copy of (X, U), then $\prod X_i$ $(1 \leqslant i \leqslant n)$ is special.*

Proof. It is clear from Theorem 1.2 that (X, U) is homosequential. By Theorem 1.6, $\prod X_i$ is generated. Thus by Corollary 3.4.1 to show that $\prod X_i$ is T_1-special it is sufficient to show that $\prod X_i$ is sequential. The case $n = 1$ is trivial. Assume $\prod X_i$ $(1 \leqslant i \leqslant k)$ is sequential. For $1 \leqslant j \leqslant k+1$, let P_j denote the projection map of $\prod X_i$ $(1 \leqslant i \leqslant k+1)$ onto X_j. For each $x \in \prod X_i$ $(1 \leqslant i \leqslant k+1)$ let $K_x = \bigcup P_j^{-1}(P_j(x))$ $(1 \leqslant j \leqslant k+1)$, and let $R_x = \prod X_i$ $(1 \leqslant i \leqslant k+1) - K_x$. Then for every $x \in \prod X_i$ $(1 \leqslant i \leqslant k+1)$, K_x is closed and R_x is open. Now suppose $A \subseteq \prod X_i$ $(1 \leqslant i \leqslant k+1)$ and y is a limit point of A in the complement of A. If y is a limit point of $A \cap K_y$, then there exists j such that y is a limit point of $P_j^{-1}(P_j(y)) \cap A$. Since $P_j^{-1}(P_j(y))$ is a copy of $\prod X_i$ $(1 \leqslant i \leqslant k)$, there exists a sequence of distinct points of $A \cap P_j^{-1}(P_j(y))$ which converges to y. Now suppose y is not a limit point of $A \cap K_y$. Then y is a limit point of $A \cap R_y$. Let $y_1 \in A \cap R_y$. Since $y_1 \notin K_y$, $y \in R_{y_1}$. Thus there exists $y_2 \in A \cap R_y \cap R_{y_1}$. Similarly, $y \in R_{y_1} \cap R_{y_2}$ and there exists $y_3 \in A \cap R_y \cap R_{y_1} \cap R_{y_2}$. Inductively there exists $y_1, y_2, \ldots$ such that for every i, $y_{i+1} \in A \cap R_y \cap \cap R_{y_1} \cap \ldots \cap R_{y_i}$. Since each $y_i \notin R_{y_i}$, $y_1, y_2, \ldots$ is a sequence of distinct points of A. Also for every i, $y_{i+1} \notin K_{y_1} \cup K_{y_2} \cup \ldots \cup K_{y_i}$. Thus for $1 \leqslant j \leqslant k+1$, $P_j(y_{i+1}) \notin \{P_j(y_1), P_j(y_2), \ldots, P_j(y_i)\}$. Thus $P_j(y_1), P_j(y_2), \ldots$ is a sequence of distinct points of X_j. Since each X_j is cofinite, each $P_j(y_1), P_j(y_2), \ldots$ converges to $P_j(y)$. Hence $y_1, y_2, \ldots$ converges to y. Thus $\prod X_i$ $(1 \leqslant i \leqslant k+1)$ is sequential.

Now $\prod X_i$ $(1 \leqslant i \leqslant n)$ is T_1-special. Since X_1 is a special retract of $\prod X_i$ $(1 \leqslant i \leqslant n)$ and the continuous functions from $\prod X_i$ to X_1 are point separating (because X_1 is a copy of (X, U), and the projection maps to (X, U) are point separating), $\prod X_i$ is special by Theorem 4.2.

Notation. Let (I, E) denote the closed unit interval with the standard topology.

By simple application of the Tietze Extension Theorem we have that (I, E) is generated and homosequential [2].

THEOREM 4.5. *The space (I, E) is full and special.*

Proof. By Theorem 2.4 to show that (I, E) is full it is sufficient to show that for every $x \in I$ and for every $A \subseteq I - x$ such that x is a limit point of A, A is x-large over $C(E)$. There is a monotone sequence $x_1, x_2, \ldots$ of distinct points of A converging to x. Without loss of generality suppose $x_1, x_2, \ldots$ is increasing. Let $a_1, b_1, a_2, b_2, \ldots$ be an increasing sequence of distinct points of I converging to x. Define $f\colon I \to I$ by $f([a_i, b_i]) = x_i$ for each i, $f([0, a_1]) = x_1$, $f(x) = x$, $f(1) = 1$, f is linear in $[x, 1]$, f is linear in each $[b_i, a_{i+1}]$. Then $f \in C(E)$, f is $x - x$, and $f^{-1}(x_1, x_2, \ldots)$ is the set $([0, b_1] \cup [a_2, b_2] \cup [a_3, b_3] \cup \ldots)$. Thus $f^{-1}(A)$ contains this set. Similarly there is a $g \in C(E)$ such that g is $x - x$ and $g^{-1}(A)$ contains the set $([0, a_1] \cup \cup [b_1, a_2] \cup [b_2, a_3] \cup \ldots)$. Thus $f^{-1}(A) \cup g^{-1}(A) = [0, x)$. If $x = 1$, $f^{-1}(A) \cup \cup g^{-1}(A) = I - x$. Otherwise $x < 1$, and there exists a homeomorphism h of I onto I such that $h^{-1}([0, x)) = (x, 1]$. Then $f^{-1}(A) \cup g^{-1}(A) \cup \cup (f \circ h)^{-1}(A) \cup (g \circ h)^{-1}(A) = I - x$. Thus A is x-large over $C(E)$, and (I, E) is full. Thus (I, E) is upper special. Since (I, E) is generated, (I, E) is T_1-special. By Theorem 4.1, (I, E) is special.

It is interesting to note that with the help of Corollary 3.4.1 a proof that n-cells are special could be constructed. Then this result would be similar to that of Theorem 4.4. However, we have the following more general result:

THEOREM 4.6. *Every locally connected, separable metric continuum is special.*

Proof. Let (X, U) be a locally connected, separable metric continuum. If X is singleton, then (X, U) is special. Assume X is not singleton. Then (X, U) contains a retract copy (I', E') of (I, E) ([8], 5.1, p. 36). There is a continuous map f of (I', E') onto (X, U) ([8], 4.1, p. 33). Now f is an identification map because (I', E') is compact ([2], 1.4, p. 121). Thus (X, U) is anti-related to (I', E'). By Theorem 3.3, (X, U) is upper special. By Theorem 1.9, (X, U) is generated. Thus (X, U) is T_1-special. By Theorem 4.2, (X, U) is special.

THEOREM 4.7. *If (X, U) is locally Euclidean and T_2, then (X, U) is special.*

Proof. By Theorem 1.9, (X, U) is generated. Let (I', E') denote a retract copy of (I, E) in (X, U). By Theorem 3.3 to show that (X, U) is T_1-special it is sufficient to show that (X, U) is anti-related to (I', E'). Let Z denote the collection of all continuous images of (I', E') in (X, U). For each $(A, T) \in Z$ every continuous map of (I', E') onto (A, T) is an identification map because (I', E') is compact. Thus by Theorem 3.2 it is sufficient to show that (X, U) is coherent with respect to Z. But every n-cell in (X, U) is in Z, and (X, U) is coherent with respect to its n-cells. Thus (X, U) is T_1-special. Now $C(U, E')$ is point separating because (X, U) is completely regular and T_2. Thus by Theorem 4.2, (X, U) is special.

THEOREM 4.8. *If (X, U) is the space of a CW-complex, then (X, U) is special.*

Proof. The space (X, U) is T_2, normal, and coherent with respect to the set of all copies of n-cells in (X, U) ([5], pp. 215-216). Thus by Theorem 1.9, (X, U) is generated. Each n-cell A in (X, U) is a retract of (X, U) because A is AR (normal) ([2], 5.2, p. 151). Also each n-cell is special by Theorem 4.6. Now by Theorem 3.3, (X, U) is T_1-special. Let B be a retract arc in (X, U). The continuous maps of (X, U) into B are point separating because (X, U) is T_2 and completely regular. Thus by Theorem 4.2, (X, U) is special.

THEOREM 4.9. *If (X, U) is a zero dimensional metric space, then (X, U) is special or discrete.*

Proof. The space (X, U) is generated [4]. Suppose (X, U) is not discrete, (X, V) is T_1, and $C(U) = C(V)$. Then $U \subseteq V$. Assume $K \in V - U$. Since $K \notin U$, there exists a sequence $x_1, x_2, \ldots$ of distinct points of K converging to a point x in $X - K$ with respect to U. Let N_1 be an open and closed set in (X, U) with diameter less than 1 and which contains x but not x_1. Let N_2 be an open and closed set in (X, U) with diameter less than $\frac{1}{2}$ and which contains x and such that $N_2 \subseteq N_1$, and $N_1 - N_2$ contains a point y_1 of $x_1, x_2, \ldots$ Continuing inductively we have $N_1, N_2, \ldots$ such that for each i, N_i is open and closed in (X, U) with diameter less than $1/i$, $N_{i+1} \subseteq N_i$, and $N_i - N_{i+1}$ contains a point y_i of $x_1, x_2, \ldots$ Define $f(z) = x_1$ if $z \in X - N_1$, $f(z) = y_i$ if $z \in N_i - N_{i+1}$, and $f(x) = x$. Now $f \in C(U) = C(V)$, and $f^{-1}\{x_1, x_2, \ldots\} \supseteq f^{-1}\{y_1, y_2, \ldots\} = X - x$. Thus $f^{-1}(K) = X - x$, and $(X - x) \in V$. Hence x is an isolated point of (X, V). Define $g(z) = z$ for each $z \in X - x$, and $g(x) = x_1 \neq x$. Then $g \in C(V)$, but $g \notin C(U)$. This is a contradiction. Thus $U = V$, and (X, U) is T_1-special. To show that (X, U) is special if it is not discrete let $p, q, r, s \in X$ such that $p \neq q$, and $r \neq s$. Let N be an open and closed set in (X, U) which contains p but not q. Define $f(z) = r$ if $z \in N$ and $f(z) = s$ if $z \in X - N$. Then $f \in C(U)$, $f(p) = r$ and $f(q) = s$. By Theorem 4.1, (X, U) is special.

THEOREM 4.10. *If (X, U) is a subspace of the real line, then (X, U) is special or discrete.*

Proof. If (X, U) contains an arc, (X, U) is generated by Theorem 1.9. If (X, U) does not contain an arc, then (X, U) is generated because it is zero dimensional [4]. Let W denote the collection of all arcs and all non-discrete, zero dimensional retracts of (X, U). Then every member of W is a special retract of (X, U). To show that (X, U) is T_1-special it is sufficient to show that (X, U) is coherent with repsect to W. Suppose not. There exists $K \subseteq X$ such that $K \notin U$ and for each $A \in W$, $K \cap A$ is closed in A. Let p be a limit point of K not in K. Assume without loss of generality that $x_1, x_2, \ldots$ is an increasing sequence of distinct points of K converging to p. If for some real number $r < p$, the closed interval $[r, p]$ is in X, then $K \cap [r, p]$ contains every number in $x_1, x_2, \ldots$ larger than r. Thus $K \cap [r, p]$ is not closed in $[r, p]$. This is a contradiction because $[r, p]$ is in W. Hence there is a real number y_1 such that $x_1 < y_1 < p$ and $y_1 \notin X$. Also there is a real y_2 such that $y_1 < z_1 < y_2 < p$ for some $z_1 \in \{x_1, x_2, \ldots\}$, and $y_2 \notin X$. Inductively, there exist real numbers $y_1, y_2, \ldots$ such that for each i, $y_i < y_{i+1}$, some member z_i of $x_1, x_2, \ldots$ is between y_i and y_{i+1}, $y_i < p$, and $y_i \notin X$. Define $f(x) = x_1$ if $x \in X$ and $x < y_1$, $f(x) = z_i$ if $x \in X$ and $y_i < x < y_{i+1}$, $f(x) = p$ if $x \in X$ and $x > p$. Then $\{p, x_1, z_1, z_2, \ldots\}$ is a member of W. But $(K \cap \{p, x_1, z_1, z_2, \ldots\}) \supseteq \{x_1, z_1, z_2, \ldots\}$ and not p. Thus $(K \cap \{p, x_1, z_1, z_2, \ldots\})$ is not closed in $\{p, x_1, z_1, z_2, \ldots\}$. This is a contradiction. Hence (X, U) is coherent with respect to W, and (X, U) is T_1-special. To show that (X, U) is special if it is not discrete, note that either (X, U) contains a retract arc and Theorem 4.2 applies, or (X, U) is zero dimensional and Theorem 4.9 applies.

We now seek sufficient conditions on a subset F of X^X that there exists one and only one topology U on X such that $C(U) = F$. Assume X is infinite, and consider the following five properties:

(I) The collection F contains 1_X and all constant maps in X^X, and if $f, g \in F$, then $f \circ g \in F$.

(II) No finite union of F-fibers has a finite, non-empty complement in X.

(III) If $K \subseteq X$, and $p \in X - K$ such that every finite covering of K by F-fibers covers p, then there exist $f_1, f_2, \ldots, f_n \in F$ such that $\bigcup f_i^{-1}(K)$ has a finite, non-empty complement in X.

(IV) If $p, q, r, s \in X$, $p \neq q$, and $r \neq s$, then there exists $f \in F$ such that $f(p) = r$, and $f(q) = s$.

(V) If $p, q \in X$ with $p \neq q$, then X is covered by some finite set of F-fibers no one of which contains both p and q.

Now F is *type I* if and only if (I), (II), (III) and (IV) hold; and F is *type II* if and only if F is type I, and (V) holds. It is easy to check that if F is type I (or type II), then F is contained in a maximal type I (or type II) subset of X^X.

THEOREM 4.11. *If F is a maximal type I subset of X^X (where X is infinite), then there exists a topology U on X such that (X, U) is special, and $F = C(U)$.*

Proof. Let U denote the topology on X for which the set of F-fibers is a subbasis. By Theorem 1.1, (X, U) is generated. Suppose (for contradiction) p is an isolated point in (X, U). Since $X-p$ is closed, $X-p$ [illegible] inite

union of F-fibers. This contradicts (II). Thus (X, U) has no isolated points. To show that (X, U) is full suppose (X, V) is a space such that $U \subseteq V$, and $C(U) \subseteq C(V)$. If $v \in V - U$, then there exists $x \in X$ such that x is a limit point of v with respect to U, and $x \notin v$. Then every finite cover of v by F-fibers covers x. By (III) there exist $f_1, f_2, \ldots, f_n \in F \subseteq C(U) \subseteq C(V)$ such that $\bigcup f_i^{-1}(v)$ has a finite, non-empty complement in X. Since $\bigcup f_i^{-1}(v)$ is closed in (X, V), (X, V) has isolated points. Thus (X, U) is full. Thus (X, U) is upper special and, hence, T_1-special. By (IV) and Theorem 4.1, (X, U) is special. That $C(U)$ satisfies (I) is trivial, and (II) results from the fact that (X, U) has no isolated points. Since $F \subseteq C(U)$, Condition (IV) holds for $C(U)$. Since (X, U) is generated, and the $C(U)$-fibers form a subbasis for U, (III) holds for $C(U)$. Now $F \subseteq C(U)$, and $C(U)$ is type I implies that $F = C(U)$ because F is maximal type I.

THEOREM 4.12. *If F is a maximal type II subset of X^X, then there exists a topology U on X such that (X, U) is special, $F = C(U)$, and (X, U) is T_2 and regular.*

Proof. The proof that (X, U) is special and $F = C(U)$ is similar to the one for the previous theorem. The proof that (X, U) is T_2 is similar to one given for Theorem 1.12. By Theorem 1.11, (X, U) is regular.

5. S-admissible classes. If h is a bijection of X onto Y, then for each $f \in X^X$ define $h'(f)$ to be $h \circ f \circ h^{-1}$. Then h induces the bijection h' of X^X onto Y^Y. If (X, U) is a space, then $C(U)$ is a semigroup with respect to the operation of composition. An *S-admissible* class of topological spaces is a class Q of spaces such that if (X, U), $(Y, V) \in Q$, then a bijection h of X onto Y is a homeomorphism of (X, U) onto (Y, V) if and only if h' restricted to $C(U)$ is an isomorphism of $C(U)$ onto $C(V)$ [3]. It is known [3] that a class Q of spaces is S-admissible if and only if for every pair (X, U) and (Y, V) of spaces in Q with a bijection h of X onto Y such that $h'(C(U)) = C(V)$, h is a homeomorphism of (X, U) onto (Y, V).

THEOREM 5.1. *Let Q be a class of spaces containing all homeomorphic images of members of Q. Then Q is S-admissible if and only if every member of Q is Q-special.*

Proof. Assume Q is S-admissible. Suppose $(X, U) \in Q$, and let $(X, V) \in Q$ such that $C(U) = C(V)$. Then $(1_X)'(C(U)) = C(V)$. Thus 1_X is a homeomorphism of (X, U) onto (X, V). Hence $V = U$. It follows that (X, U) is Q-special.

Now assume that every member of Q is Q-special. Let (X, U), $(Y, V) \in Q$, and suppose h is a bijection of X onto Y such that $h'(C(U)) = C(V)$. Let V' be the topology on Y such that h is a homeomorphism of (X, U) onto (Y, V'). Then $(Y, V') \in Q$, and $h'(C(U)) = C(V')$. Thus $C(V) = C(V')$. Since every member of Q is Q-special, $V = V'$.

Corollary 5.1.1. *Let J be the class of all spaces (X, U) such that $C(U)$ satisfies $C(U)$-1, $C(U)$-2 and $C(U)$-3, and U is the $C(U)$-collection. Then J is S-admissible.*

Corollary 5.1.2. *If the class L contains all homeomorphic images of its members, then the class of all L-special spaces is S-admissible.*

Corollary 5.1.3. *The class of all special spaces is S-admissible, and the class of all T_1-special spaces is S-admissible.*

Theorem 5.2. *Let Q be any class of spaces which contains all homeomorphic images of members of Q. Let Q' be the subclass of Q consisting of members (X, U) of Q such that if $(X, V) \in Q$ for which $C(U) \subseteq C(V)$, then $U \subseteq V$. Then Q' is S-admissible.*

Proof. It is simple to show that Q' contains all homeomorphic images of its members. Now let $(X, U) \in Q'$, and suppose $(X, V) \in Q'$ with $C(U) = C(V)$. By definition of Q', $U \subseteq V$, and $V \subseteq U$. Thus $U = V$, and (X, U) is Q'-special. Now by Theorem 5.1, Q' is S-admissible.

Theorem 5.3. *If the class Q of Theorem 5.2 contains all products of its members and all retract subspaces of its members, then the class Q' contains all products of its members.*

Proof. For every $b \in B$ let (X_b, U_b) be a member of Q'. Then the product (X, U) of all members of $\{(X_b, U_b)\colon b \in B\}$ is in Q. To show that (X, U) is in Q' let $(X, V) \in Q$ such that $C(U) \subseteq C(V)$. We may consider each (X_b, U_b) to be a subspace of (X, U) playing the role of an "axis". This requires embedding maps, but homeomorphic images of members of Q' are in Q'. For each $b \in B$ let P_b be the projection map of (X, U) onto (X_b, U_b). For each b let U'_b be the topology that X_b inherits from V. Since $P_b \in C(U) \subseteq C(V)$, (X_b, U'_b) is a retract of (X, V). Hence $(X_b, U'_b) \in Q$. To show that $C(U_b) \subseteq C(U'_b)$ we simply note that $C(U) \subseteq C(V)$, and the continuous maps of retract spaces are just the restrictions of the continuous maps of their super spaces. Now $U_b \subseteq U'_b$ because $(X_b, U_b) \in Q'$. The set of all inverses under projection maps of closed sets in the (X_b, U_b) spaces is a subbasis for the closed sets in (X, U). Since each $P_b \in C(V)$, and each $U_b \subseteq U'_b$, this subbasis is contained in V. Thus $U \subseteq V$, and $(X, U) \in Q'$.

It is clear that if Q in Theorem 5.3 is the class of all T_1-spaces, then Q' is the class of all generated spaces.

Definition 5.1. If $f, g \in X^X$, let $E(f, g) = \{x \in X\colon f(x) = g(x)\}$. A space (X, U) is an *equalizer space* if and only if (X, U) is T_2, and $\{E(f, g)\colon f, g \in C(U)\}$ is a subbasis for U.

Earlier mention was made of S^*-spaces. Now a space (X, U) is an *S-space* if and only if the collection of fixed point sets for members of $C(U)$ is a basis for U, and (X, U) is T_2 [3]. Since the fixed point set for a function f in X^X is $E(f, 1_X)$, and a fiber $f^{-1}(x)$ is the set $E(f, x')$ where x' denotes the constant map of X into x, every S-space, every T_2, S^*-space and every T_2, generated space is an equalizer space.

Theorem 5.4. *If Q in Theorem 5.3 is the class of all T_2-spaces, then every equalizer space is in Q'.*

Proof. If (X, U) is T_2, and $f, g \in C(U)$, then $E(f, g) \in U$. The theorem follows.

In conclusion, we might comment that it is painfully clear that the topic of this paper is in need of further study. It is hoped that what is here is sufficient to encourage investigation.

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