On bimeasurable images of universally measurable sets

by

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Denote by $\mathfrak{B}_B$ the set of Borel subsets of a Borel subset B of the set R of real numbers, and denote by $\mathfrak{B}$ the set of Borel subsets of R. A (real-valued) Borel measurable function f defined on an element G of $\mathfrak{B}$ is said to be *bimeasurable* [3] if $f(B) \in \mathfrak{B}$ for each $B \in \mathfrak{B}_G$.

Notice that if f is a bimeasurable function on a Borel set G and H is a Borel subset of G, then the restriction $f|H$ of f to H is a bimeasurable function on H.

R. Purves has established [3] the following characterization of bimeasurable functions.

Let $G \in \mathfrak{B}$ and let f be a Borel measurable function defined on G. A necessary and sufficient condition that f be bimeasurable is that there are at most countably many values v in the range of f such that $f^{-1}(v) = \{x \in G;\ f(x) = v\}$ is uncountable.

A subset E of R is said to be a *universally measurable set* if $\mu^*(E) = \mu_*(E)$ for each non-atomic probability measure μ on $\mathfrak{B}$, and E is said to be *universal null set* if $\mu^*(E) = 0$ for each such μ.

It is well known that a necessary and sufficient condition in order that a subset E of R be a universal null set is that each homeomorphism φ of R carries E onto a set $\varphi(E)$ of Lebesgue measure zero. However, the author has shown [1] that, subject to the continuum hypothesis, there exists a universal null subset E of the interval $I = [0, 1]$ and a continuous function h of bounded variation on I such that the image $h(E)$ of E is not a universal null set.

The purpose of this note is to show that a bimeasurable function f on R carries a universally measurable set onto a universally measurable set and, hence, carries a universal null set onto a universal null set. To this end, it will be convenient to have the following lemmas.

LEMMA 1. *If G is an element of $\mathfrak{B}$, E is a universally measurable subset of G, and g is a one to one Borel measurable function on G, then $g(E)$ is a universally measurable set.*

Proof. Let λ be a non-atomic probability measure on $\mathfrak{B}$. Let μ be defined on the elements H of $\mathfrak{B}_G$ by $\mu(H) = \lambda\big(f(H)\big)$. Since E is a universally measurable set, for each $\varepsilon > 0$, there exist Borel subsets U and V of G such that $U \subset E \subset V$ and $\mu(V) - \mu(U) < \varepsilon$. Then $f(U)$ and $f(V)$ are Borel sets satisfying $f(U) \subset f(E) \subset f(V)$ and $\lambda(f(V)) - \lambda(f(U)) < \varepsilon$. Hence $f(E)$ is a universally measurable set.

Lemma 2, the analogous result for universal null sets, follows from the proof of Lemma 1.

LEMMA 2. *If G is an element of $\mathfrak{B}$, E is a universal null subset of G, and g is g one to one Borel measurable function on G, then $g(E)$ is a universal null set.*

In the course of establishing Theorem 1, it will be convenient to denote by J the set of irrational numbers.

THEOREM 1. *If E is a universally measurable subset of R and f is a bimeasurable function on R, then $f(E)$ is a universally measurable set.*

Proof. Let $B = R - (S_1 \cup S_2 \cup S_3)$ where S_1 is the set of rational numbers, $S_2 = \{f^{-1}(x);\ x \in S_1\}$, and $S_3 = \{x;\ f^{-1}(f(x))$ is uncountable$\}$. Then B is a Borel subset of J, $f(B)$ is a Borel subset of J, and the restriction $g = f|B$ of f to B is semi-regular: for each x in R, $g^{-1}(x)$ is a countable set. Hence it follows from page 243 of [2] that there is a sequence $\{B_i\}$ of pairwise disjoint Borel subsets of B such that $\bigcup B_i = B$ and the restrictions $g_i = g|B_i$ are one to one Borel measurable functions. Lemma 1 tells us that each $g_i(E \cap B_i)$ is a universally measurable set. Thus, since $f(R-B)$ is a countable set,

$$f(E) = f(E \cap B) \cup f(E - B) = \{\bigcup g_i(E \cap B_i)\} \cup f(E - B)$$

is a universally measurable set.

Applying Lemma 2 instead of Lemma 1 yields Theorem 2.

THEOREM 2. *If E is a universal null subset of R and f is a continuous bimeasurable function on R, then $f(E)$ is a universal null set.*

Banach's characterization of CBV functions implies that the CBV function h constructed in [1] satisfies $m(\{x;\ h^{-1}(x)$ is uncountable$\}) = 0$, where m denotes the Lebesgue measure. An examination of the construction for h shows that there is a perfect set P such that $h^{-1}(x)$ is uncountable if $x \in P$.

References

[1] R. B. Darst, *A CBV image of a universal null set need not be a universal null set*, to appear in Fund. Math.

[2] N. Lusin, *Leçons sur les ensembles analytiques et leurs applications*, Paris 1930.

[3] R. Purves, *Bimeasurable functions*, Fund. Math. 58 (1966), pp. 149–157.

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On discontinuous additive functions

by

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One of the classical problems of analysis is this:

Let T be a set on the real line R or, more generally, in the n-dimensional Euclidean space R^n, and let f be a real-valued function which is defined in R^n and *additive*, i.e. satisfies Cauchy's functional equation:

$$f(x+y) = f(x) + f(y) \tag{1}$$

for $x, y \in R^n$. Suppose that f is upper-bounded on T. What conditions upon the set T imply the continuity of f?

The same problem may be stated for functions which are defined in some convex domain $\Delta \subset R^n$ and satisfy Jensen's inequality (2) instead of (1):

$$f\left(\frac{x+y}{2}\right) \leqslant \frac{f(x)+f(y)}{2} \tag{2}$$

for $x, y \in \Delta$. Such functions will be referred to as *Q-convex*. This expression is justified by the observation that they satisfy also the inequality

$$f\big((1-a)x + ay\big) \leqslant (1-a)f(x) + af(y) \tag{3}$$

for $x, y \in \Delta$, a rational, $0 \leqslant a \leqslant 1$; and the latter is an immediate consequence of the generalized Jensen formula

$$f\left(\frac{x_1 + \ldots + x_q}{q}\right) \leqslant \frac{1}{q}\big(f(x_1) + \ldots + f(x_q)\big) \tag{4}$$

for $x_1, \ldots, x_q \in \Delta$, $q = 1, 2, 3, \ldots$, which may be found in any textbook on convex functions, e.g. [2]. In order to obtain (3) for $a = p/q$, $p = 0, 1, \ldots, q$, it suffices to set in (4) $x_1 = \ldots = x_{q-p} = x$, $x_{q-p+1} = \ldots \ldots = x_q = y$.

R. Ger and M. Kuczma introduce in [1] the following set classes:

A set $T \subset R^n$ belongs to the class $\mathfrak{A}$ iff every Q-convex function $f\colon \Delta \to R$, $T \subset \Delta \subset R^n$, upper-bounded on T, is continuous in Δ.

A set $T \subset R^n$ belongs to the class $\mathfrak{B}$ iff every additive function $f\colon R^n \to R$, upper-bounded on T, is continuous.