$\bar{a}, \bar{b} \in L$ with $M(\bar{a}, \bar{b})$. By Lemma 4.11, $M(\bar{a}, \bar{b})$ in G, and by [8], Theorem 1, $M(\bar{b}, \bar{a})$ holds in G. It follows that $M(\bar{b}, \bar{a})$ in L.

Remark 4.13. By an obvious modification of the above lemma one can show that in any finite-statisch AC-lattice, if $\bar{a}$ or $\bar{b}$ is finite then $M(\bar{a}, \bar{b}) \Rightarrow M(\bar{b}, \bar{a})$.

5. Some open questions. We close by listing a few open questions that have suggested themselves during the writing of this paper.

1. Is every finite-modular AC-lattice M-symmetric?

2. In [7], S. Maeda calls a lattice L a DAC-*lattice* in case both L and its dual are AC-lattices, and shows ([7], Theorem 2.1, p. 108) that every DAC-lattice is a finite-modular AC-lattice. Can every M-symmetric, finite-modular AC-lattice be embedded in a DAC-lattice?

3. Is Remark 4.13 valid for an arbitrary AC-lattice?

4. Is $F(L)$ a standard ideal for L an arbitrary AC-lattice? What if L is a matroid lattice?

5. In a finite-modular AC-lattice, by [6], Lemma 4, p. 168, $M^*(a, b)$ is equivalent to the assertion that p an atom, $p \leqslant a \vee b$ implies the existence of atoms $q \leqslant a$, $r \leqslant b$ such that $p \leqslant q \vee r$. In an arbitrary AC-lattice, what does it mean to say that $p \leqslant a \vee b$, p an atom, implies the existence of finite elements $a_1 \leqslant a$, $b_1 \leqslant b$ such that $p \leqslant a_1 \vee b_1$?

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a-adic completions of Noetherian lattice modules *

by

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§ 1. Introduction. Several years ago R. P. Dilworth [1] began a study of the ideal theory of commutative rings in an abstract setting. Since the investigation was to be purely ideal-theoretic, he chose to study a lattice with a commutative multiplication. Many of Dilworth's ideas have since been extended and several new concepts have been introduced ([2], [3]). In particular, E. W. Johnson [3] has introduced the notions of a Noetherian lattice module and a completion of a Noetherian lattice module. The purpose of this paper is to generalize the methods used in [3] and to extend some of the results. For undefined terms concerning Noetherian lattices, the reader is referred to [1] and [3].

The basic concepts are introduced in § 2. In § 3 the a-adic pseudometric is introduced. If M is an L-module, then, for each element a of L, a distance function, d_a, can be defined on M. This distance function d_a is called the a-*adic pseudometric* on M. Theorem 3.10 gives necessary and sufficient conditions for d_a to be a metric. Assuming that d_a is a metric, the set of all Cauchy sequences is divided into classes by an equivalence relation, and M^* is used to denote this set. The concepts of a regular Cauchy sequence and a completely regular Cauchy sequence are given in § 4. It is shown (Theorem 4.14) that each element of M^* has a unique completely regular representative. In § 5 the extension of elements from M to M^* is defined. For A in M, the extension of A to M^* is denoted by AM^*. A lattice structure is developed for M^* and in § 6 it is shown that M^* satisfies the ascending chain condition (Theorem 6.3) under the hypothesis that L is a Noetherian lattice and M is a Noetherian L-module. The a-adic completion of M is defined (Definition 6.5). A contraction of elements of M^* to M is introduced (Definition 7.1) in § 7. For A in M^*, its contraction to M is denoted by $A \cap M$. It is shown that $A = AM^* \cap M$ for all A in M (Proposition 7.2).

The remainder of the paper is concerned with the particular case where L is a local Noetherian lattice and M is a Noetherian L-module. In § 8 a connection between the different metrics on M^* is determined (Theorem 8.12 and Corollary 8.13). In § 9 p-adic completions of lattice

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intervals are investigated. It is shown that the L-module $[AM^*, BM^*]$ with the p-adic metric is the p-adic completion of the Noetherian L-module $[A, B]$ (Theorem 9.5). In § 10 residuals in L^* are studied. For A, B in M, Theorem 10.3 establishes that $(A \wedge B)M^* = AM^* \wedge BM^*$ and Theorem 10.4 establishes that $(A : B)L^* = AM^* : BM^*$.

§ 2. Definitions and notations.

DEFINITION 2.1. A lattice L is said to be *multiplicative* if L is a complete lattice and if there is defined on L a commutative, associative multiplication which distributes over arbitrary joins such that the one element of L is the identity for the multiplication.

DEFINITION 2.2. Let L be a multiplicative lattice. A *left lattice module over L*, or simply an *L-module*, is defined to be a complete lattice M together with a function $f: L \times M \to M$ which satisfies the following four conditions:

(2.1) $f(ab, A) = f(a, f(b, A))$
for all a, b in L and A in M;

(2.2) $f(\bigvee \{a_\alpha | \ \alpha \in D\}, \ \bigvee \{B_\beta | \ \beta \in E\}) = \bigvee \{f(a_\alpha, B_\beta) | \ \alpha \in D$ and $\beta \in E\}$,
for all nonempty families $\{a_\alpha | \ \alpha \in D\} \subseteq L$ and $\{B_\beta | \ \beta \in E\} \subseteq M$, where D and E are arbitrary index sets;

(2.3) $f(I, A) = A$ *for all A in M, where I is the one element of L;*

(2.4) $f(0, A) = 0$ *for all A in M.*

For each a in L and B in M, the element $f(a, B)$ of M will be denoted by aB. Elements of L will generally be denoted by small letters $a, b, c, \ldots$, except that the least and greatest elements of L will be denoted by 0 and I, respectively. Elements of M will generally be denoted by capital letters $A, B, C, \ldots$, except that the least and greatest elements of M will be denoted by 0_M and $\mathcal{M}$, respectively. When no confusion is possible, 0 will be used in place of 0_M. In the remainder of this section, L is a multiplicative lattice and M is an L-module.

Just as in the case of modules over a commutative ring with a unit element, there are natural residual operations associated with an L-module M. The three residual operations used in this paper are defined below. The existence and uniqueness of these residual operations are consequences of conditions (2.1)–(2.4) in Definition 2.2.

DEFINITION 2.3. For all a, b in L and A, B in M,

(i) $a : b$ is defined to be the greatest element c in L such that $cb \leqslant a$;

(ii) $A : b$ is defined to be the greatest element C in M such that $bC \leqslant A$; and,

(iii) $A : B$ is defined to be the greatest element a in L such that $aB \leqslant A$.

Some of the more important properties of residuation and multiplication by scalars are listed below. The proofs are straightforward and will be omitted. For all a, b, c, a_i in L and A, B, C in M,

(2.5) *If* $a \leqslant b$, *then* $aB \leqslant bB$.

(2.6) *If* $A \leqslant B$, *then* $aA \leqslant aB$.

(2.7) *If* $a \leqslant b$ *and* $A \leqslant B$, *then* $aA \leqslant bB$.

(2.8) $A \leqslant B$ *if and only if* $B : A = I$.

(2.9) $A \leqslant A : a$.

(2.10) $A \leqslant A : I$.

(2.11) $a \leqslant aA : A$.

(2.12) $(A : a) \vee (B : a) \leqslant (A \vee B) : a$.

(2.13) $(A : C) \vee (B : C) \leqslant (A \vee B) : C$.

(2.14) $B : ab = (B : a) : b$.

(2.15) $B : aA = (B : a) : A$.

(2.16) $(B : a) : A = (B : A) : a$.

(2.17) $(B \wedge C) : A = (B : A) \wedge (C : A)$.

(2.18) $(B \wedge C) : a = (B : a) \wedge (C : a)$.

(2.19) $A : (B \vee C) = (A : B) \wedge (A : C)$.

(2.20) $A : (a \vee b) = (A : a) \wedge (A : b)$.

(2.21) $(A \wedge B) : B = A : B$.

(2.22) $A : (A \vee B) = A : B$.

(2.23) *If* $a \vee c = b \vee c = I$, *then* $ab \vee c = I$.

(2.24) *If* $a \vee c = I$, *then* $(a \wedge b) \vee c = b \vee c$.

(2.25) $(\bigvee_1^n a_i)^{k_1 + \ldots + k_n} \leqslant \bigvee_1^n a_i^{k_i}$, *where each k_i is a positive integer.*

(2.26) $(a \wedge b)B \leqslant aB \wedge bB$.

(2.27) $ab \leqslant a \wedge b$.

(2.28) $a(A \wedge B) \leqslant aA \wedge aB$.

(2.29) $b(a : b) = (a : b)b \leqslant a$.

(2.30) $a(A : a) \leqslant A$.

(2.31) $(A : B)B \leqslant A$.

In [1] R. P. Dilworth introduced principal elements. This definition was generalized by E. W. Johnson in [3].

Definition 2.4. Let A be an element of M. A is said to be *meet principal* in case $\big(b\wedge(B:A)\big)A = bA\wedge B$, for all b in L and B in M. A is said to be *join principal* in case $b\vee(B:A) = (bA\vee B):A$, for all b in L and B in M. A is said to be *principal* in case A is both meet principal and join principal.

Definition 2.5. M is said to be *principally generated* if each element of M is the join (finite or infinite) of principal elements of M. M is said to be *Noetherian* if M satisfies the ascending chain condition, is modular, and is principally generated.

Definition 2.6. Let L be modular. Then, with respect to the multiplication in L, L can be regarded as an L-module. If L is a Noetherian L-module, then L will be called a *Noetherian lattice*.

This definition of a Noetherian lattice was given in [3]. It is equivalent, however, to the definitions given in [1] and [2], and results from these papers will be used without special comment.

Definition 2.7. Let W be a lattice. Let A and B be elements of W such that $A\leqslant B$. Then the set $\{D\in W|\ A\leqslant D\leqslant B\}$ is a sublattice of W and will be denoted by $[A, B]$.

Remark 2.8. Let M be an L-module. Let A and B be elements of M such that $A\leqslant B$. *Then $[A, B]$ is naturally an L-module.*

Proof. Since M is an L-module, M is a complete lattice, and hence it is easily seen that $[A, B]$ is a complete lattice. For a in L and C in $[A, B]$, define $a\circ C = aC\vee A$. Since $A\leqslant a\circ C = aC\vee A\leqslant B$, it follows that "$\circ$" is a function from $L\times[A, B]$ into $[A, B]$. Properties (2.1) through (2.4) of Definition 2.2 are easily verified and hence $[A, B]$ is an L-module, q.e.d.

Remark 2.9. If M is a Noetherian L-module, it is easily verified that $[A, B]$ *is a Noetherian L-module with the above multiplication where A and B are elements of M such that $A\leqslant B$* (see [1]).

§ 3. a-adic pseudometric. Throughout this section M is an L-module.

Proposition 3.1. *For all A, B in M and a in L, if there exists a nonnegative integer n such that $A\vee a^n\mathcal{M} = B\vee a^n\mathcal{M}$, then $A\vee a^m\mathcal{M} = B\vee a^m\mathcal{M}$, for all nonnegative integers $m<n$.*

Proof. Assume that there exists a nonnegative integer n such that $A\vee a^n\mathcal{M} = B\vee a^n\mathcal{M}$ and let m be a nonnegative integer which is less than n. Then $a^n\leqslant a^m$, and hence $a^n\mathcal{M}\leqslant a^m\mathcal{M}$. It follows that

$$\begin{aligned} A\vee a^m\mathcal{M} &= (A\vee a^n\mathcal{M})\vee a^m\mathcal{M}\\ &= (B\vee a^n\mathcal{M})\vee a^m\mathcal{M}\\ &= B\vee a^m\mathcal{M}\end{aligned}$$

which completes the proof.

Corollary 3.2. *For all A, B in M and a in L, if there exists a nonnegative integer n such that $A\vee a^n\mathcal{M}\neq B\vee a^n\mathcal{M}$, then $A\vee a^m\mathcal{M}\neq B\vee a^m\mathcal{M}$, for all nonnegative integers $m>n$.*

Corollary 3.3. *For all A, B in M and a in L, there exists at most one nonnegative integer n such that $A\vee a^n\mathcal{M} = B\vee a^n\mathcal{M}$ and*

$$A\vee a^{n+1}\mathcal{M}\neq B\vee a^{n+1}\mathcal{M}\,.$$

Definition 3.4. Let a be an element of L. For (A, B) in $M\times M$ define

(3.1) $f_a(A, B) = n$ if there exists a nonnegative integer n such that $A\vee a^n\mathcal{M} = B\vee a^n\mathcal{M}$, and $A\vee a^{n+1}\mathcal{M}\neq B\vee a^{n+1}\mathcal{M}$;

(3.2) $f_a(A, B) = +\infty$ if no such nonnegative integer n exists.

Notice that f_a is well defined by Corollary 3.3 and that f_a is in fact defined on the Cartesian product $M\times M$ since it is always the case that $A\vee a^0\mathcal{M} = \mathcal{M} = B\vee a^0\mathcal{M}$, for each pair (A, B) in $M\times M$.

Definition 3.5. Let a be an element of L. For (A, B) in $M\times M$, define

$$d_a(A, B) = 2^{-f_a(A,B)} \quad \text{if} \quad f_a(A, B)\neq+\infty; \tag{3.3}$$

$$d_a(A, B) = 0 \quad \text{if} \quad f_a(A, B) = +\infty\,. \tag{3.4}$$

We now let "a" be an arbitrary (but fixed) element of L, and denote d_a from this point on simply by d.

Remark 3.6. For all A, B in M, the following are equivalent for any nonnegative integer n:

$$d(A, B)\leqslant 2^{-n}; \tag{3.5}$$

$$A\vee a^n\mathcal{M} = B\vee a^n\mathcal{M}\,. \tag{3.6}$$

Proof. Assume first that (3.5) holds. Then $f_a(A, B)\geqslant n$, and hence $A\vee a^n\mathcal{M} = B\vee a^n\mathcal{M}$. Conversely, if (3.6) holds, then $f_a(A, B)\geqslant n$. It follows that $d(A, B)\leqslant 2^{-n}$, q.e.d.

Proposition 3.7. *For all A, B in M, the following hold:*

$$d(A, B)\geqslant 0\,. \tag{3.7}$$

$$d(A, B) = d(B, A)\,. \tag{3.8}$$

$$\textit{If } A = B\,,\ \textit{then } d(A, B) = 0\,. \tag{3.9}$$

Proof. Clearly $d(A, B)\geqslant 0$. From the definition of f_a it immediately follows that $f_a(A, B) = f_a(B, A)$. Consequently $d(A, B) = d(B, A)$. To prove (3.9), assume $A = B$. Then clearly $A\vee a^n\mathcal{M} = B\vee a^n\mathcal{M}$, for all nonnegative integers n. Hence, $f_a(A, B) = +\infty$ and consequently $d(A, B) = 0$, q.e.d.

The following proposition shows that the function d satisfies the strong triangle inequality.

PROPOSITION 3.8. *For all A, B and C in M,*

$$d(A, B) \leqslant \max\{d(A, C), d(C, B)\}.$$

Proof. It may be assumed without loss of generality that

$$d(C, B) = \max\{d(A, C), d(C, B)\}.$$

There are now two cases to consider. First, assume that $d(C, B) = 0$. Then $d(A, C) = 0$. It follows by Remark 3.6 that $A \vee a^n\mathcal{M} = B \vee a^n\mathcal{M} = C \vee a^n\mathcal{M}$, for all nonnegative integers n. This implies $f_a(A, B) = +\infty$, and thus $d(A, B) = 0$.

For the second case, assume $d(C, B) \neq 0$. Hence, there exists a nonnegative integer n such that $d(C, B) = 2^{-n}$. It follows that $d(A, C) \leqslant d(C, B) = 2^{-n}$. This implies $A \vee a^n\mathcal{M} = B \vee a^n\mathcal{M} = C \vee a^n\mathcal{M}$ by Remark 3.6.

Consequently, $d(A, B) \leqslant 2^{-n}$, q.e.d.

THEOREM 3.9. *Let L be a multiplicative lattice, let M be an L-module, and let a be an element of L. Then (M, d_a) is a pseudometric space.*

Proof. This follows from Propositions 3.7 and 3.8.

The pseudometric d_a is called the *a-adic pseudometric* on M.

The following theorem gives a necessary and sufficient condition for the a-adic pseudometric on M to be a metric.

THEOREM 3.10. *Let M be an L-module and let a be an element of L. The a-adic pseudometric on M is a metric if and only if $C = \bigwedge_n (C \vee a^n\mathcal{M})$ for each C in M.*

Proof. To show that the a-adic pseudometric is a metric, it need only be shown that $d_a(A, B) = 0$ implies $A = B$.

Assume $C = \bigwedge_n \{C \vee a^n\mathcal{M}\}$ for all C in M. Let A and B be elements of M such that $d_a(A, B) = 0$. Then $A \vee a^n\mathcal{M} = B \vee a^n\mathcal{M}$ for all nonnegative integers n. Consequently, $A = \bigwedge_n (A \vee a^n\mathcal{M}) = \bigwedge_n (B \vee a^n\mathcal{M}) = B$ and thus d_a is a metric. Conversely, assume that d_a is a metric and let A be an element of M. Since

$$\begin{aligned} A \vee a^k\mathcal{M} &\leqslant [\textstyle\bigwedge_i (A \vee a^i\mathcal{M}] \vee a^k\mathcal{M} \\ &\leqslant (A \vee a^k\mathcal{M}) \vee a^k\mathcal{M} \\ &\leqslant A \vee a^k\mathcal{M}, \end{aligned}$$

for every nonnegative integer k, it follows that

$$A \vee a^k\mathcal{M} = [\textstyle\bigwedge_i (A \vee a^i\mathcal{M})] \vee a^k\mathcal{M},$$

for every nonnegative integer k. Hence, $d_a(A, \bigwedge_i (A \vee a^i\mathcal{M})) = 0$ and thus $A = \bigwedge_i (A \vee a^i\mathcal{M})$, q.e.d.

COROLLARY 3.11. *Let a be an element of L and let A, B be elements of M such that $A \leqslant B$. If the a-adic pseudometric on M is a metric, then the a-adic pseudometric on $[A, B]$ is a metric.*

§ 4. Completely regular representatives. Throughout this section, M is an L-module, and "a" is an arbitrary (but fixed) element of L such that the a-adic pseudometric on M is a metric. As in § 3 the single letter d will be used to denote this distance function.

PROPOSITION 4.1. *The function $(Y, Z) \to Y \vee Z$ of $M \times M \to M$ is uniformly continuous.*

Proof. Let d' denote the usual product metric on the product space $M \times M$. Let r be a positive real number and choose n to be the least nonnegative integer k such that $2^{-k} < r$.

Assume $d'((A, B), (C, D)) < r$. Then, $d(A, C) \leqslant 2^{-n}$ and $d(B, D) \leqslant 2^{-n}$. This implies $A \vee a^n\mathcal{M} = C \vee a^n\mathcal{M}$ and $B \vee a^n\mathcal{M} = D \vee a^n\mathcal{M}$ (Remark 3.6). Thus $(A \vee B) \vee a^n\mathcal{M} = (C \vee D) \vee a^n\mathcal{M}$ and hence $d(A \vee B, C \vee D) \leqslant 2^{-n} < r$, q.e.d.

Remark 4.2. If the a-adic pseudometric on L is also a metric, then, in a manner similar to that used in the proof of Proposition 4.1, it may be shown that the function $(x, Y) \to xY$ of $L \times M \to M$ is uniformly continuous.

DEFINITION 4.3. Let $C(M)$ denote the set of all Cauchy sequences of elements of M. For $\langle A_i \rangle$, $\langle B_i \rangle$ in $C(M)$, define $\langle A_i \rangle \sim \langle B_i \rangle$ if and only if $\lim_{i\to\infty} d(A_i, B_i) = 0$. It is easy to establish that $\sim$ is an equivalence relation on $C(M)$ and we let M^* denote the set of all equivalence classes of $C(M)$ determined by this equivalence relation.

The next theorem establishes a connection between the definition of the equivalence relation $\sim$ and the definition of the metric on M.

THEOREM 4.4. *Let $\langle A_i \rangle$ and $\langle B_i \rangle$ be Cauchy sequences of elements of M. Then the following three statements are equivalent:*

(4.1) $$\langle A_i \rangle \sim \langle B_i \rangle;$$

(4.2) $$\lim_{i\to\infty} d(A_i, B_i) = 0;$$

(4.3) *For each nonnegative integer n,*

$$A_i \vee a^n\mathcal{M} = B_i \vee a^n\mathcal{M},$$

for all sufficiently large positive integers i.

Proof. (4.1) and (4.2) are equivalent by the definition of $\sim$, so it is sufficient to show that (4.2) and (4.3) are equivalent. To prove that (4.2) implies (4.3), let n be a nonnegative integer. Since $d(A_i, B_i) \to 0$ as $i \to +\infty$, there exists a positive integer m such that $d(A_i, B_i) < 2^{-n}$, for all integers $i \geqslant m$. Hence, it follows (Remark 3.6) that $A_i \vee a^n\mathcal{M} = B_i \vee a^n\mathcal{M}$ for all integers $i \geqslant m$. Thus (4.3) holds.

Conversely, assume that (4.3) holds and let r be a positive real number. Choose n to be the least nonnegative integer k such that $2^{-k} < r$. Hence, $2^{-n} < r$. Now, by hypothesis, there is a positive integer m such that $A_i \vee a^n\mathcal{M} = B_i \vee a^n\mathcal{M}$, for all integers $i \geqslant m$. Hence, it follows that $d(A_i, B_i) \leqslant 2^{-n} < r$ for all integers $i \geqslant m$. Thus (4.2) holds, q.e.d.

COROLLARY 4.5. *Let B be an element of M^* and let the Cauchy sequence $\langle B_i \rangle$ be a representative of B. Then any subsequence of $\langle B_i \rangle$ is a representative of B.*

COROLLARY 4.6. *Let B be an element of M^* and let the Cauchy sequence $\langle B_i \rangle$ be a representative of B. Then $\langle B_i \vee a^i\mathcal{M} \rangle$ is a Cauchy sequence and is a representative of B.*

Proof. It is a straightforward computation to show that $\langle B_i \vee a^i\mathcal{M} \rangle$ is a Cauchy sequence. For the second part, let n be a nonnegative integer. Then

$$B_i \vee a^n\mathcal{M} = B_i \vee (a^i\mathcal{M} \vee a^n\mathcal{M}) = (B_i \vee a^i\mathcal{M}) \vee a^n\mathcal{M}$$

for all integers $i \geqslant n$. Therefore, $\langle B_i \rangle \sim \langle B_i \vee a^i\mathcal{M} \rangle$ by Theorem 4.4, q.e.d.

DEFINITION 4.7. Let $\langle B_i \rangle$ be a Cauchy sequence of elements of M. $\langle B_i \rangle$ is said to be *regular* in case $B_i \vee a^i\mathcal{M} = B_{i+1} \vee a^i\mathcal{M}$ for all positive integers i. $\langle B_i \rangle$ is said to be *completely regular* in case $B_i = B_{i+1} \vee a^i\mathcal{M}$ for all positive integers i.

Remark 4.8. It follows immediately from Definition 4.7 that if $\langle B_i \rangle$ is completely regular, then $B_i \geqslant a^i\mathcal{M}$, for all positive integers i, and $\langle B_i \rangle$ is decreasing in the sense that $B_i \geqslant B_{i+1}$ for all positive integers i. Furthermore, a completely regular sequence is a regular sequence.

THEOREM 4.9. *Let $\langle B_i \rangle$ be a Cauchy sequence in M. Then the following three statements are equivalent:*

(4.4) *$\langle B_i \rangle$ is a regular Cauchy sequence;*

(4.5) *For each positive integer n,*

$$B_n \vee a^n\mathcal{M} = B_m \vee a^n\mathcal{M} \quad \textit{for all integers } m \geqslant n;$$

(4.6) *For each positive integer n,*

$$d(B_n, B_m) \leqslant 2^{-n} \quad \textit{for all integers } m \geqslant n.$$

Proof. It is an immediate consequence of Remark 3.6 that (4.5) is equivalent to (4.6). (4.5) obviously implies (4.4). To show that (4.4) implies (4.5), it is sufficient by induction to show: If $B_n \vee a^n\mathcal{M} = B_{n+i} \vee a^n\mathcal{M}$, then $B_n \vee a^n\mathcal{M} = B_{n+i+1} \vee a^n\mathcal{M}$. But this is an immediate consequence of the following:

$$\begin{aligned} B_n \vee a^n\mathcal{M} &= B_{n+i} \vee a^n\mathcal{M} \\ &= (B_{n+i} \vee a^{n+i}\mathcal{M}) \vee a^n\mathcal{M} \\ &= (B_{n+i+1} \vee a^{n+i}\mathcal{M}) \vee a^n\mathcal{M} \\ &= B_{n+i+1} \vee a^n\mathcal{M} \end{aligned}$$

since $\langle B_i \rangle$ is a regular Cauchy sequence, q.e.d.

For completely regular Cauchy sequences we have the following theorem.

THEOREM 4.10. *Let $\langle B_i \rangle$ be a Cauchy sequence in M. Then, the following three statements are equivalent:*

(4.7) *$\langle B_i \rangle$ is completely regular;*

4.8) *For each positive integer n,*

$$B_n = B_m \vee a^n\mathcal{M} \quad \textit{for all integers } m > n;$$

(4.9) *For each positive integer n,*

$$B_n \geqslant a^n\mathcal{M} \textit{ and } d(B_n, B_m) \leqslant 2^{-n}, \quad \textit{for all integers } m > n.$$

Proof. Since (4.8) is equivalent to (4.9) by Remark 3.6, and since (4.8) clearly implies (4.7), we need only show that (4.7) implies (4.8). To show this, it is sufficient by induction to show: If $B_n = B_{n+i} \vee a^n\mathcal{M}$, then $B_n = B_{n+i+1} \vee a^n\mathcal{M}$. This is, however, an immediate consequence of the following:

$$\begin{aligned} B_n &= B_{n+i} \vee a^n\mathcal{M} \\ &= (B_{n+i+1} \vee a^{n+i}\mathcal{M}) \vee a^n\mathcal{M} \\ &= B_{n+i+1} \vee a^n\mathcal{M} \end{aligned}$$

since $\langle B_i \rangle$ is a completely regular Cauchy sequence, q.e.d.

LEMMA 4.11. *Let B be an element of M^* and let $\langle B_i \rangle$ be a representative of B. Then there exists a subsequence $\langle E_i \rangle$ of $\langle B_i \rangle$ such that $\langle E_i \rangle$ is a regular representative of B.*

Proof. This is a straightforward computation, q.e.d.

LEMMA 4.12. *Let $\langle B_i \rangle$ be a regular Cauchy sequence in M. Then $\langle B_i \vee a^i\mathcal{M} \rangle$ is a completely regular Cauchy sequence.*

Proof. Since $\langle B_i \rangle$ is a regular Cauchy sequence and since $a^{i+1}\mathcal{M} \leqslant a^i\mathcal{M}$, it follows that

$$B_i \vee a^i\mathcal{M} = B_{i+1} \vee a^i\mathcal{M} = (B_{i+1} \vee a^{i+1}\mathcal{M}) \vee a^i\mathcal{M},$$

for all positive integers i, q.e.d.

Corollary 4.13. *Let B be an element of M^* and let $\langle B_i\rangle$ be a regular representative of B. Then $\langle B_i \vee a^i\mathcal{M}\rangle$ is a completely regular representative of B.*

The following theorem establishes the existence and the uniqueness of completely regular representatives of elements of M^*.

Theorem 4.14. *Let A be an element of M^*. Then A has a completely regular representative. Furthermore, this completely regular representative is uniquely determined by A.*

Proof. The existence follows from Lemma 4.11 and Corollary 4.13. Now, assume that $\langle A_i\rangle$ and $\langle B_i\rangle$ are two such completely regular representatives of A. Then $\langle A_i\rangle \sim \langle B_i\rangle$ and hence, for all positive integers n,

$$B_n = B_i \vee a^n\mathcal{M} = A_i \vee a^n\mathcal{M} = A_n$$

for all positive integers i sufficiently large by Theorems 4.4 and 4.10. In particular, $A_n = B_n$ for every natural number n. This shows the uniqueness, q.e.d.

§ 5. Extensions. Throughout this section M is an L-module, and a is an arbitrary (but fixed) element of L such that the a-adic pseudometric on M is a metric. As before, d shall denote the a-adic pseudometric on M, and M^* shall denote the collection of equivalence classes determined by d.

Definition 5.1. Let A be an element of M. The equivalence class in M^* determined by the Cauchy sequence $\langle A_i\rangle$, where $A_i = A$ for all positive integers i, is defined to be the *extension of A to M^**, or simply A *extended*, and will be denoted by AM^*. For $N \subseteq M$, define $NM^* = \{AM^* \mid A \in N\}$.

Remark 5.2. Let A be an element of M. Then, it follows from Definition 5.1 and results from § 4 that the Cauchy sequence $\langle A \vee a^i\mathcal{M}\rangle$, $i = 1, 2, \ldots$, is the completely regular representative of AM^*.

Proposition 5.3. *Let A and B be elements of M. $AM^* = BM^*$ if and only if $A = B$.*

Proof. If $A = B$, then clearly $AM^* = BM^*$. Conversely, assume $AM^* = BM^*$. Then $\langle A_i\rangle \sim \langle B_i\rangle$, where $A_i = A$ and $B_i = B$, for all positive integers i. Hence,

$$0 = \lim_{i\to\infty} d(A_i, B_i) = \lim_{i\to\infty} d(A, B) = d(A, B) .$$

Consequently, $A = B$, q.e.d.

In view of the above proposition, it should be observed that the extension mapping $A \to AM^*$ of $M \to M^*$ is one-to-one and hence M is imbedded in M^* (as a set).

Definition 5.4. Let A and B be elements of M. Define

$$AM^* \vee BM^* = (A \vee B)M^* .$$

Definition 5.5. Let A and B be elements of M^*. Let $\langle A_i\rangle$, $\langle B_i\rangle$ be representatives of A and B, respectively. Define

$$d_a^*(A, B) = \lim_{i\to\infty} d(A_i, B_i) .$$

The following are well known ([4], p. 196).

(5.1) *d_a^* is well defined;*

(5.2) *d_a^* is a metric on M^*;*

(5.3) *(M^*, d_a^*) is a complete metric space;*

(5.4) *MM^* is dense in M^*;*

(5.5) *the extension map $A \to AM^*$ of $M \to M^*$ is an isometry.*

Combining these properties with Theorem 4.1 we obtain

(5.6) *$MM^* \times MM^*$ is dense in $M^* \times M^*$;*

(5.7) *the map $(AM^*, BM^*) \to AM^* \vee BM^*$ of $MM^* \times MM^* \to M^*$ is uniformly continuous.*

Consequently, there exists a uniformly continuous extension of this function "$\vee$" to $M^* \times M^*$ ([5], p. 118). The image of (A, B) in $M^* \times M^*$ under this extension will be denoted by $A \vee B$. This extension function is uniquely determined by the metric. One very important property of this extension is given below for future reference:

(5.8) *Let (A, B) be an element of $M^* \times M^*$. Let $\langle A_i\rangle$ and $\langle B_i\rangle$ be representatives of A and B, respectively. Then,*

$$A \vee B = \lim_{i\to\infty} \langle (A_i \vee B_i)M^*\rangle = \lim_{i\to\infty} \langle A_i M^* \vee B_i M^*\rangle .$$

Note that these limits always exist since the sequence $\langle (A_i \vee B_i)M^*\rangle$ is a Cauchy sequence (because of the uniform continuity and the isometry) and since M^* is a complete metric space. Also, note that if $\langle D_i\rangle$ and $\langle E_i\rangle$ are arbitrary representatives of A and B, respectively, then $\langle D_i \vee E_i\rangle$ is a representative of $A \vee B$.

Following the procedure established earlier with respect to d and d_a, d_a^* will be denoted from this point on simply by d^*.

Proposition 5.6. *Let A and B be elements of M^*. Let $\langle A_i\rangle$ and $\langle B_i\rangle$ be representatives of A and B, respectively. Then:*

(5.9) *$d^*(A, B) = 0$; or,*

(5.10) *there exists natural numbers n and j such that $d^*(A, B) = d(A_i, B_i) = 2^{-n}$ for all integers $i \geqslant j$.*

Proof. Assume $d^*(A, B) \neq 0$. Then $\lim_{i\to\infty} d(A_i, B_i) \neq 0$. Thus, there exists a natural number m such that

$$0 \neq d(A_i, B_i) = 2^{-f_a(A_i, B_i)}$$

for all integers $i \geqslant m$. Since $\lim_{i\to\infty} d(A_i, B_i)$ exists, it follows that there exists a natural number $j \geqslant m$ such that

$$d(A_i, B_i) = 2^{-f_a(A_j, B_j)}$$

for all integers $i \geqslant j$. Set $f_a(A_j, B_j) = n$. Then

$$d^*(A, B) = \lim_{i\to\infty} d(A_i, B_i) = d(A_i, B_i) = 2^{-n}$$

for all integers $i \geqslant j$, q.e.d.

Proposition 5.7. *Let A and B be elements of M^*. Let $\langle A_i\rangle$ and $\langle B_i\rangle$ be the completely regular representatives of A and B, respectively. Then, the sequence $\langle A_i \vee B_i\rangle$ is the completely regular representative of $A \vee B$.*

Proof. We know that the sequence $\langle A_i \vee B_i\rangle$ is a representative of $A \vee B$ (see (5.8)). Thus, by the uniqueness of the completely regular representative (Theorem 4.14), it is sufficient to show that the Cauchy sequence $\langle A_i \vee B_i\rangle$ is completely regular. Since the sequences $\langle A_i\rangle$ and $\langle B_i\rangle$ are completely regular, it follows that

$$\begin{aligned} A_i \vee B_i &= (A_{i+1} \vee a^i \mathcal{M}) \vee (B_{i+1} \vee a^i \mathcal{M}) \\ &= (A_{i+1} \vee B_{i+1}) \vee a^i \mathcal{M}, \end{aligned}$$

for all nonnegative integers i. Thus the sequence $\langle A_i \vee B_i\rangle$ is completely regular, q.e.d.

Definition 5.8. Let A and B be elements of M^*. Define

$$A \leqslant B \quad \textit{if and only if } A \vee B = B .$$

It can be shown that this definition establishes a partial order on M^* relative to which the operation $\vee$ in M^* is the least upper bound operation.

The following two propositions establish useful relations between the order in M and the order in M^*.

Proposition 5.9. *Let A and B be elements of M^* such that $A \leqslant B$. Let the sequences $\langle A_i\rangle$ and $\langle B_i\rangle$ be the completely regular representatives of A and B, respectively. Then, $A_i \leqslant B_i$ for all positive integers i.*

Proof. Since $A \leqslant B$, it follows that $A \vee B = B$. By Proposition 5.7, $\langle A_i \vee B_i\rangle$ is the completely regular representative of $A \vee B$. Hence, $A_i \vee B_i = B_i$, for all positive integers i, by the uniqueness of completely regular representatives (Theorem 4.14). Thus $A_i \leqslant B_i$, for all positive integers i, q.e.d.

Proposition 5.10. *Let A and B be elements of M^*. If there exists Cauchy sequences $\langle A_i\rangle$ and $\langle B_i\rangle$ such that:*

(5.11) *$\langle A_i\rangle$ is a representative of A;*

(5.12) *$\langle B_i\rangle$ is a representative of B; and,*

(5.13) *$A_i \leqslant B_i$, for all integers i which are sufficiently large;*

then $A \leqslant B$ in M^*.

Proof. Assume that $\langle A_i\rangle$ and $\langle B_i\rangle$ are representatives of A and B, respectively, and that $A_i \leqslant B_i$ for all sufficiently large integers i. Then, it follows that $A_i \vee B_i = B_i$ for all sufficiently large integers i. This implies that

$$A \vee B = \lim_{i\to\infty} \langle (A_i \vee B_i) M^*\rangle = \lim_{i\to\infty} \langle B_i M^*\rangle = B$$

by (5.8). Hence, $A \leqslant B$ by definition, q.e.d.

Corollary 5.11. *Let A and B be elements of M such that $A \leqslant B$. Then $AM^* \leqslant BM^*$.*

Corollary 5.12. *OM^* is the least element of M^* and $\mathcal{M}M^*$ is the greatest element of M^*.*

If, in addition to the conditions given at the beginning of this section, it is further assumed that the a-adic pseudometric on L is a metric, then the following development is possible.

Definition 5.13. For bL^* in LL^* and AM^* in MM^*, define

$$bL^* \cdot AM^* = (bA)M^* .$$

By combining Remark 4.2 with (5.1)–(5.5) we obtain

(5.14) *$LL^* \times MM^*$ is dense in $L^* \times M^*$;*

(5.15) *the map $(bL^*, AM^*) \to bL^* \cdot AM^*$ of $LL^* \times MM^* \to M^*$ is uniformly continuous.*

In a manner similar to that developed for "$\vee$", the function "$\cdot$" has a unique extension to $L^* \times M^*$. The image of (b, A) in $L^* \times M^*$ under this extension will be denoted by $b \cdot A$, or simply by bA. An important property of this function is given below for future reference.

(5.16) *Let (b, A) be an element of $L^* \times M^*$. Let $\langle b_i\rangle$ and $\langle A_i\rangle$ be representatives of b and A, respectively. Then,*

$$bA = \lim_{i\to\infty} \langle (b_i A_i) M^*\rangle = \lim_{i\to\infty} \langle b_i L^* \cdot A_i M^*\rangle .$$

Note that these limits always exist. Also, note, that if $\langle e_i\rangle$ and $\langle E_i\rangle$ are arbitrary representatives of b and A, respectively, then $\langle e_i E_i\rangle$ is a representative of bA.

PROPOSITION 5.14. *Let b be an element of L^* and let A be an element of M^*. Let $\langle b_i\rangle$ and $\langle A_i\rangle$ be the completely regular representatives of b and A, respectively. Then the sequence $\langle b_iA_i\rangle$ is a regular representative of bA.*

Proof. We know that the sequence $\langle b_iA_i\rangle$ is a representative of bA. Hence, it need only be shown that the Cauchy sequence $\langle b_iA_i\rangle$ is regular. Now, since $\langle b_i\rangle$ and $\langle A_i\rangle$ are completely regular Cauchy sequences, it follows that $b_i = b_{i+1}\vee a^i$ and $A_i = A_{i+1}\vee a^i\mathcal{M}$ for all positive integers i. Therefore,

$$b_iA_i\vee a^i\mathcal{M} = (b_{i+1}\vee a^i)(A_{i+1}\vee a^i\mathcal{M})\vee a^i\mathcal{M} = b_{i+1}A_{i+1}\vee a^i\mathcal{M}$$

for all positive integers i, q.e.d.

By applying Corollary 4.13 we obtain the following result.

COROLLARY 5.15. *Let b be an element of L^* and let A be an element of M^*. Let $\langle b_i\rangle$ and $\langle A_i\rangle$ be the completely regular representatives of b and A, respectively. Then, the sequence $\langle b_iA_i\vee a^i\mathcal{M}\rangle$ is the completely regular representative of bA.*

§ 6. a-adic completions. Throughout this section L is a Noetherian lattice, M is a Noetherian L-module, a is an arbitrary (but fixed) element of L such that the a-adic pseudometric on M is a metric, and M^* is the collection of equivalence classes determined by the metric.

LEMMA 6.1. ([3], Lemma 0.4). *Let b be an element of L, let A be an element of M, and let $\langle B_i\rangle$ be a sequence of elements of M satisfying*

$$b^iA \geqslant B_i \geqslant B_{i+1} \geqslant bB_i$$

for all positive integers i. Then there exists a positive integer n such that $B_{n+i} = b^iB_n$ for all nonnegative integers i.

LEMMA 6.2. *Let $C(1)\leqslant C(2)\leqslant \ldots$ be an ascending chain of elements of M^* and let $\langle C(n,j)\rangle$, $j = 1, 2, \ldots$, be the completely regular representative of $C(n)$, $n = 1, 2, \ldots$ Then,*

(6.1) $\quad C(n, i+1)\wedge a^i\mathcal{M} \geqslant C(n, i+2)\wedge a^{i+1}\mathcal{M} \geqslant a[C(n, i+1)\wedge a^i\mathcal{M}]$

for all positive integers n and nonnegative integers i.

Proof. Let n be a positive integer. Then

$$C(n, i+1) = C(n, i+2)\vee a^{i+1}\mathcal{M}$$

for all nonnegative integers i, since the Cauchy sequence $\langle C(n,j)\rangle$, $j = 1, 2, \ldots$, is completely regular. Thus

$$C(n, i+1)\wedge a^i\mathcal{M} \geqslant C(n, i+2)\wedge a^{i+1}\mathcal{M}$$

for all nonnegative integers i. Furthermore,

$$\begin{aligned} aC(n, i+1) &= a[C(n, i+2)\vee a^{i+1}\mathcal{M}] \\ &= aC(n, i+2)\vee a^{i+2}\mathcal{M} \\ &\leqslant C(n, i+2) \end{aligned}$$

for all nonnegative integers i. It follows that

$$\begin{aligned} a[C(n, i+1)\wedge a^i\mathcal{M}] &\leqslant aC(n, i+1)\wedge a^{i+1}\mathcal{M} \\ &\leqslant C(n, i+2)\wedge a^{i+1}\mathcal{M} \end{aligned}$$

for all nonnegative integers i, q.e.d.

THEOREM 6.3. *M^* satisfies the ascending chain condition.*

Proof. We use a modification of an argument due to E. W. Johnson [3]. Let $C(1)\leqslant C(2)\leqslant \ldots$ be an ascending chain of elements of M^* and let $\langle C(n,j)\rangle$, $j = 1, 2, \ldots$, be the completely regular representative of $C(n)$, $n = 1, 2, \ldots$ Since

$$C(1,1)\leqslant C(2,1)\leqslant \ldots \leqslant C(n,1)\leqslant C(n+1,1)\leqslant \ldots$$

by Proposition 5.9, and since M satisfies the ascending chain condition, there exists a natural number N such that

(6.2) $$C(n, 1) = C(N, 1)$$

for all natural numbers $n\geqslant N$. From Lemma 6.2 we obtain

$$\begin{aligned} a^i\mathcal{M} &\geqslant C(n, i+1)\wedge a^i\mathcal{M} \\ &\geqslant C(n, i+2)\wedge a^{i+1}\mathcal{M} \\ &\geqslant a[C(n, i+1)\wedge a^i\mathcal{M}] \end{aligned}$$

for all positive integers n and nonnegative integers i. Consequently,

$$\begin{aligned} a^i\mathcal{M} &\geqslant \bigvee_n \{C(n, i+1)\wedge a^i\mathcal{M}\} \\ &\geqslant \bigvee_n \{C(n, i+2)\wedge a^{i+1}\mathcal{M}\} \\ &\geqslant \bigvee_n \{a[C(n, i+1)\wedge a^i\mathcal{M}]\} \\ &= a[\bigvee_n \{C(n, i+1)\wedge a^i\mathcal{M}\}] \end{aligned}$$

for all nonnegative integers i by Definition 2.2. Now, for each positive integer i, set

$$B_i = \bigvee_n \{C(n, i+1)\wedge a^i\mathcal{M}\}\,.$$

It follows from above that $a^i\mathcal{M}\geqslant B_i\geqslant B_{i+1}\geqslant aB_i$ for all positive integers i. Thus, there exists a positive integer m (Lemma 6.1) such that

(6.3) $$B_{m+i} = a^iB_m$$

for all nonnegative integers i. It may clearly be assumed that $m\geqslant N$.

Now, fix the positive integer i and observe that

$$C(1, i+1)\wedge a^i\mathcal{M}\leqslant \ldots \leqslant C(n, i+1)\wedge a^i\mathcal{M}\leqslant \ldots$$

is an ascending chain of elements of M (Proposition 5.9). Hence, for each i, $1 \leqslant i \leqslant m$, there exists a natural number n_i such that

$$B_i = C(n_i, i+1) \wedge a^i \mathcal{M}.$$

Set $k = \max\{n_1, n_2, \ldots, n_m, N\}$. Then

$$B_i = C(k, i+1) \wedge a^i \mathcal{M}$$

for all integers i such that $1 \leqslant i \leqslant m$. It follows that

$$C(k+j, i+1) \wedge a^i \mathcal{M} = C(k, i+1) \wedge a^i \mathcal{M} \tag{6.4}$$

for all integers i such that $1 \leqslant i \leqslant m$ and all nonnegative integers j.

Now, let i be a nonnegative integer. Then,

$$\begin{aligned} B_{m+i} &= a^i B_m \\ &= a^i [C(k, m+1) \wedge a^m \mathcal{M}] \\ &\leqslant C(k, m+i+1) \wedge a^{m+i} \mathcal{M} \\ &\leqslant B_{m+i} \end{aligned}$$

by (6.3). Hence

$$C(k, m+i+1) \wedge a^{m+i} \mathcal{M} = B_{m+i}$$

for all nonnegative integers i. This implies

$$C(k+j, m+i+1) \wedge a^{m+i} \mathcal{M} = C(k, m+i+1) \wedge a^{m+i} \mathcal{M} \tag{6.5}$$

for all nonnegative integers i and all nonnegative integers j. By combining (6.4) and (6.5) we obtain

$$C(k+j, i+1) \wedge a^i \mathcal{M} = C(k, i+1) \wedge a^i \mathcal{M} \tag{6.6}$$

for all nonnegative integers i and all nonnegative integers j.

It shall now be shown that $C(k+i, n) = C(k, n)$ for all positive integers n and all nonnegative integers i. The proof shall be by induction on n. First, consider the case where $n = 1$. Since $k \geqslant N$, we have $C(k+i, 1) = C(k, 1)$ for all nonnegative integers i by (6.2). Hence, the case where $n = 1$ has been established. Now, let j be a positive integer greater than 1 and assume that $C(k+i, j) = C(k, j)$ for all nonnegative integers i. Let i be a nonnegative integer. Then,

$$\begin{aligned} C(k+i, j+1) &= C(k+i, j+1) \wedge C(k+i, j) \\ &= C(k+i, j+1) \wedge C(k, j) \\ &= C(k+i, j+1) \wedge [C(k, j+1) \vee a^j \mathcal{M}] \\ &= C(k, j+1) \vee [C(k+i, j+1) \wedge a^j \mathcal{M}] \\ &= C(k, j+1) \vee [C(k, j+1) \wedge a^j \mathcal{M}] \\ &= C(k, j+1) \end{aligned}$$

by the induction hypothesis, modularity in M, and (6.6). Consequently, $C(k+i, j+1) = C(k, j+1)$ for all nonnegative integers i. The induction is now complete. It follows that $C(k+i) = C(k)$ for all nonnegative integers i, q.e.d.

COROLLARY 6.4. *M^* is a complete lattice.*

Proof. Let C be an arbitrary nonempty collection of elements of M^*. By Theorem 6.3, M^* satisfies the ascending chain condition. It follows immediately from this that C has a least upper bound and a greatest lower bound, q.e.d.

DEFINITION 6.5. Let L be a Noetherian lattice, let M be a Noetherian L-module, let a be an element of L, let the a-adic pseudometric on M, d_a, be a metric, and let M^* be the completion of M determined by the metric d_a. Now, M^* may be made into an L-module by the following construction. Let b be an element of L, let A be an element of M^*, and let the sequence $\langle A_i \rangle$ be the completely regular representative of A; define bA to be that element of M^* determined by the Cauchy sequence $\langle bA_i \rangle$. [Observe that $b(AM^*) = (bA)M^*$ for every A in M.] It is easily verified that M^* becomes an L-module under this definition of multiplication. M^* now has the following properties:

(6.7) *M^* is an L-module;*

(6.8) *(M^*, d_a^*) is a complete metric space;*

(6.9) *MM^* is dense in M^*;*

(6.10) *If A and B are elements of M^*, and if $\langle A_i \rangle$ and $\langle B_i \rangle$ are representatives of A and B, respectively, then*

$$d_a^*(A, B) = \lim_{i\to\infty} d_a(A_i, B_i);$$

(6.11) *OM^* is the least element of M^*;*

(6.12) *$\mathcal{M}M^*$ is the greatest element of M^*;*

(6.13) *M^* satisfies the ascending chain condition;*

(6.14) *M^* is a complete lattice.*

This L-module M^* is defined to be the a-adic *completion* of the Noetherian L-module M.

Remark 6.6. It is clear that the a-adic completion of the Noetherian L-module M is uniquely determined up to a lattice isomorphism.

§ 7. Contractions. Throughout this section, L is a Noetherian lattice, M is a Noetherian L-module, a is an element of L such that the a-adic pseudometric on M is a metric, and M^* is the a-adic completion of M.

Definition 7.1. Let A be an element of M^* and let the sequence $\langle A_i\rangle$, $i=1,2,\ldots$, be the completely regular representative of A. $\bigwedge_n A_n$ is defined to be the *contraction* of A to M, or simply A *contracted*, and will be denoted henceforth by $A\frown M$.

Proposition 7.2. *Let A be an element of M. Then $A=AM^*\frown M$.*

Proof. From Remark 5.2 we know that the sequence $\langle A\vee a^i\mathcal{M}\rangle$, $i=1,2,\ldots$, is the completely regular representative of AM^*. Hence, $A=\bigwedge_i(A\vee a^i\mathcal{M})=AM^*\frown M$ by Theorem 3.10, q.e.d.

Corollary 7.3. *Let A be an element of M^*. Then $(A\frown M)M^*\frown M=A\frown M$.*

Corollary 7.4. *Let A be an element of M. Then $(AM^*\frown M)M^*=AM^*$.*

Proposition 7.5. *Let A be an element of M^*. Then $(A\frown M)M^*\leqslant A$.*

Proof. Let the sequence $\langle A_i\rangle$ be the completely regular representative of A. Since $A\frown M=\bigwedge_n A_n\leqslant A_i$ for all positive integers i, it follows that $(A\frown M)M^*\leqslant A$ by Proposition 5.10, q.e.d.

Proposition 7.6. *Let A and B be elements of M^* such that $A\leqslant B$. Then $A\frown M\leqslant B\frown M$.*

Proof. Let the sequences $\langle A_i\rangle$ and $\langle B_i\rangle$, $i=1,2,\ldots$, be the completely regular representatives of A and B, respectively. Then, $A_i\leqslant B_i$ for all positive integers i (Proposition 5.9). This implies $A\frown M=\bigwedge_n A_n\leqslant\bigwedge_n B_n=B\frown M$, q.e.d.

Corollary 7.7. *Let A and B be elements of M^*. Then*

$$(A\frown M)\vee(B\frown M)\leqslant(A\vee B)\frown M\,, \tag{7.1}$$

$$(A\wedge B)\frown M\leqslant(A\frown M)\wedge(B\frown M)\,. \tag{7.2}$$

Proposition 7.8. *Let A be an element of M and let B be an element of M^*. If there exists a Cauchy sequence $\langle B_i\rangle$, $i=1,2,\ldots$, of elements of M such that:*

$$\langle B_i\rangle \text{ is a representative of } B; \tag{7.3}$$

$$A\leqslant B_i \text{ for all sufficiently large } i; \tag{7.4}$$

then $A\leqslant B\frown M$.

Proof. Assume that there exists a Cauchy sequence $\langle B_i\rangle$, $i=1,2,\ldots$, satisfying (7.3) and (7.4). It may clearly be assumed without loss of generality that $\langle B_i\rangle$ is a regular Cauchy sequence and that $A\leqslant B_i$, for all positive integers i. Now, since $\langle B_i\rangle$ is a regular Cauchy sequence, $\langle B_i\vee a^i\mathcal{M}\rangle$ is the completely regular representative of B (Corollary 4.13). Since $A\leqslant B_i\leqslant B_i\vee a^i\mathcal{M}$ for all positive integers i, it follows that $A\leqslant\bigwedge_i(B_i\vee a^i\mathcal{M})=B\frown M$, q.e.d.

If, in addition to the conditions given at the beginning of this section, it is further assumed that the a-adic pseudometric on L is a metric and that L^* is the a-adic completion of L, considered as an L-module, then it is easily verified that M^* is an L^*-module and we have the following result.

Proposition 7.9. *Let b be an element of L^* and let B be an element of M^*. Then $(b\frown L)(B\frown M)\leqslant(bB)\frown M$.*

Proof. Let the sequences $\langle b_i\rangle$ and $\langle B_i\rangle$, $i=1,2,\ldots$, be the completely regular representatives of b and B, respectively. Then the Cauchy sequence $\langle b_iB_i\vee a^i\mathcal{M}\rangle$, $i=1,2,\ldots$, is the completely regular representative of bB (Corollary 5.15). Since $\bigwedge_n b_n\leqslant b_i$ and $\bigwedge_n B_n\leqslant B_i$ for all positive integers i, it follows that $(\bigwedge_n b_n)(\bigwedge_n B_n)\leqslant b_iB_i\leqslant b_iB_i\vee a^i\mathcal{M}$ for all positive integers i. This implies $(b\frown L)(B\frown M)=(\bigwedge_n b_n)(\bigwedge_n B_n)\leqslant\bigwedge_n(b_nB_n\vee a^n\mathcal{M})=(bB)\frown M$, q.e.d.

Remark 7.10. Let b be an element of L. Since M^* is an L-module and M^* is an L^*-module, it is natural to ask what is the relation between the b-adic pseudometric on M^* and the bL^*-adic pseudometric on M^*. It is a straightforward computation to show that these two pseudometrics are in fact equal on M^*. The proof will be omitted.

§ 8. The local case. Throughout the remainder of this paper, (L,p) is a local Noetherian lattice; M is a Noetherian L-module; L^* is the p-adic completion of L; and M^* is the p-adic completion of M. If A, B are elements of M such that $A\leqslant B$, then it is easily verified that $[A,B]$ is a Noetherian L-module. The proof is omitted.

For convenience we state the following results. The reader is referred to [3] for their proofs.

Theorem 8.1. ([3], Theorem 1.6). *For each A in M, the lattice $[pA,A]$ is finite dimensional.*

Corollary 8.2 ([3], Corollary 1.7). *Let $\langle A_i\rangle$ be any sequence of elements of M satisfying $A_{i+1}\leqslant A_i\vee p^i\mathcal{M}$ for all positive integers i. Then the sequence $\langle A_i\rangle$ is Cauchy.*

Corollary 8.3 ([3], Corollary 1.8). *Let B and C be elements of M^*. Let the sequences $\langle B_i\rangle$ and $\langle C_i\rangle$ be the completely regular representatives of B and C, respectively. Then the sequence $\langle B_i\wedge C_i\rangle$, $i=1,2,\ldots$, is a representative of $B\wedge C$.*

Corollary 8.4 ([3], Corollary 1.9). *M^* is modular.*

PROPOSITION 8.5 ([3], Lemma 1.10). *Let A and B be elements of M^*. Let the sequences $\langle A_i\rangle$ and $\langle B_i\rangle$, $i=1,2,\ldots$, be the completely regular representatives of A and B, respectively. Then the sequence $\langle A_i:B_i\rangle$, $i=1,2,\ldots$, is Cauchy and is a representative of $A:B$.*

THEOREM 8.6 ([3], Theorem 1.11). *Let $\langle A_i\rangle$ be a Cauchy sequence of principal elements of M and let B in M^* be the equivalence class determined by $\langle A_i\rangle$. Then B is a principal element of M^* (considered as an L^*-module).*

THEOREM 8.7 ([3], Theorem 1.12). *L^* is a Noetherian lattice and M^* is a Noetherian L^*-module.*

It is easily verified that L^* is in fact a local Noetherian lattice with unique maximal prime element pL^*. In the remainder of this paper, we shall use p^* to denote pL^*.

The following is actually a corollary to Lemma 6.1.

PROPOSITION 8.8 ([3], Corollary 0.5). *Let $b\neq I$ be an element of L, and let B be an element of M. Then $\bigwedge_n b^nB=0$.*

COROLLARY 8.9. *The p-adic pseudometric on M is a metric.*

Proof. Let A be an element of M. Consider the Noetherian L-module $[A,\mathcal{M}]$. Since $A=\bigwedge_n(p^n\circ\mathcal{M})=\bigwedge_n(A\vee p^n\mathcal{M})$ by Proposition 8.8, the desired result follows by Theorem 3.10, q.e.d.

THEOREM 8.10 ([3], Theorem 2.1). *M is a complete L-module with respect to the p-adic metric if and only if given any decreasing sequence $\langle B_i\rangle$, $i=1,2,\ldots$, of elements of M and any positive integer n, $B_i\leqslant(\bigwedge_j B_j)\vee p^n\mathcal{M}$ for all sufficiently large i.*

PROPOSITION 8.11. *Let A and B be elements of M^*. Then $(A\wedge B)\cap M=(A\cap M)\wedge(B\cap M)$.*

Proof. We have $(A\wedge B)\cap M\leqslant(A\cap M)\wedge(B\cap M)$ by (7.4). Hence, we need only establish the reverse inequality. Let the sequences $\langle A_i\rangle$ and $\langle B_i\rangle$, $i=1,2,\ldots$, be the completely regular representatives of A and B, respectively. Then $\langle A_i\wedge B_i\rangle$, $i=1,2,\ldots$, is a representative of $A\wedge B$ (Corollary 8.3). $(\bigwedge_n A_n)\wedge(\bigwedge_n B_n)\leqslant A_i\wedge B_i$ for all nonnegative integers i implies that $(A\cap M)\wedge(B\cap M)=(\bigwedge_n A_n)\wedge(\bigwedge_n B_n)\leqslant(A\wedge B)\cap M$ by Proposition 7.8, q.e.d.

Since M^* is an L-module, the p-adic pseudometric, d_p, is defined on M^*. Since (L^*,p^*) is a local Noetherian lattice and M^* is a Noetherian L^*-module (Theorem 8.7), the p^*-adic pseudometric on M^*, d_{p^*}, is a metric (Corollary 8.9). Since $p^*=pL^*$, it follows from the comments at the end of section 7 that d_p is a metric. The following result is essential to the later development of this paper.

THEOREM 8.12. *Let C and D be elements of M^*. Then, $d_p^*(C,D)=d_p(C,D)$.*

Proof. Since MM^* is dense in (M^*,d_p^*) by (6.8), it is sufficient to show that these two metrics agree on MM^*. This shall now be established. Let A and B be elements of M. Hence AM^* and BM^* are elements of MM^*. By using Proposition 7.2, a routine calculation shows that $AM^*\vee(p^n)(\mathcal{M}M^*)=BM^*\vee(p^n)(\mathcal{M}M^*)$ if and only if $A\vee P^n\mathcal{M}=B\vee p^n\mathcal{M}$ for each nonnegative integer n. Consequently, $d_p(AM^*,BM^*)=d_p(A,B)=d_p^*(AM^*,BM^*)$. It follows that $d_p(AM^*,BM^*)=d_p^*(AM^*,BM^*)$, for all elements A and B of M, q.e.d.

COROLLARY 8.13. *The three metrics d_p^*, d_p, and d_{p^*} are equal on M^*.*

This corollary plays an important role in later theorems of this paper. In particular, (6.8) and (6.9) of Definition 6.5 apply to M^* with the metric d_{p^*} or the metric d_p.

§ 9. Completions of intervals.

DEFINITION 9.1. Let M be an L-module, let A, B be elements of M such that $A\leqslant B$, and let a be an element of L. Then, for all elements C and D in $[A,B]$, $d(C,D,a,A,B)$ is defined to be the a-adic distance between C and D considered as elements of the L-module $[A,B]$.

THEOREM 9.2. *Let A and B be elements of M such that $A\leqslant B$. Then the L-module $[AM^*,BM^*]$ is complete with respect to the p-adic metric.*

Proof. It was previously noted that d_p and d_{p^*} are equal on M^*. Similar remarks show that the corresponding metrics d_p and d_{p^*} are equal on the module $[AM^*,BM^*]$. Consequently, in order to show that $([AM^*,BM^*],d_p)$ is a complete metric space, it is sufficient to show that $([AM^*,BM^*],d_{p^*})$ is a complete metric space. This shall now be established.

Since (M^*,d_p^*) is a complete metric space, it follows that (M^*,d_{p^*}) is a complete metric space (Corollary 8.13). Hence, M^* is a complete L^*-module with respect to the p^*-adic metric. By Corollary 3.11 it follows that $([AM^*,BM^*],d_{p^*})$ is a metric space. It will now be shown that this metric space is complete. Let $\langle C_i\rangle$, $i=1,2,\ldots$, be a Cauchy sequence of elements of $([AM^*,BM^*],d_{p^*})$. It will be shown that there exists an element C in $[AM^*,BM^*]$ such that $C_i\to C$ as $i\to+\infty$. It may be assumed that the sequence $\langle C_i\rangle$ is completely regular (in the L^*-module $[AM^*,BM^*]$ with the p^*-adic metric). Since each C_i is an element of $[AM^*,BM^*]$, $\bigwedge_j C_j$ is an element of $[AM^*,BM^*]$.

Now, consider the sequence $\langle(p^*)^i(\mathcal{M}M^*)\wedge BM^*\rangle$, $i=1,2,\ldots$ Since this sequence satisfies the conditions of Lemma 6.1 (recall that M^* is a Noetherian L^*-module by Theorem 8.7), there exists a natural number n such that

$$(9.1)\qquad (p^*)^{n+i}(\mathcal{M}M^*)\wedge BM^* = (p^*)^i[(p^*)^n(\mathcal{M}M^*)\wedge BM^*]$$

for all nonnegative integers i.

It will be shown that

$$(9.2)\qquad C_i\to\bigwedge_j C_j \quad \text{as} \quad i\to+\infty .$$

Let ε be a positive real number. Choose m to be the least natural number k such that $2^{-k}<\varepsilon$. Since M^* is a complete L^*-module with respect to the p^*-adic metric, and since the sequence $\langle C_i\rangle$ is decreasing (Remark 4.8), it follows from Theorem 8.10 that there exists a natural number N such that

$$C_i\leqslant(\bigwedge_j C_j)\vee(p^*)^{n+m}(\mathcal{M}M^*),$$

for all nonnegative integers $i\geqslant N$. Hence

$$C_i\vee(p^*)^{n+m}(\mathcal{M}M^*) = (\bigwedge_j C_j)\vee(p^*)^{n+m}(\mathcal{M}M^*),$$

for all integers $i\geqslant N$. Consequently

$$(9.3)\qquad BM^*\wedge[C_i\vee(p^*)^{n+m}(\mathcal{M}M^*)] = BM^*\wedge[(\bigwedge_j C_j)\vee(p^*)^{n+m}(\mathcal{M}M^*)],$$

for all integers $i\geqslant N$. Since

$$BM^*\wedge[C_i\vee(p^*)^{n+m}(\mathcal{M}M^*)] = C_i\vee(p^*)^m[(p^*)^n(\mathcal{M}M^*)\wedge BM^*]$$

and since

$$BM^*\wedge[(\bigwedge_j C_j)\vee(p^*)^{n+m}(\mathcal{M}M^*)] = (\bigwedge_j C_j)\vee(p^*)^m[(p^*)^n(\mathcal{M}M^*)\wedge BM^*]$$

by modularity in M^* and (9.1), we have that

$$C_i\vee(p^*)^m[(p^*)^n(\mathcal{M}M^*)\wedge BM^*] = (\bigwedge_j C_j)\vee(p^*)^m[(p^*)^n(\mathcal{M}M^*)\wedge BM^*]$$

for all integers $i\geqslant N$, by (9.3). Now, let i be an integer such that $i\geqslant N$. Then

$$\begin{aligned} C_i\vee(p^*)^m\circ(BM^*) &= C_i\vee(p^*)^m[(p^*)^n(\mathcal{M}M^*)\wedge BM^*]\vee(p^*)^m(BM^*)\\ &= (\bigwedge_j C_j)\vee(p^*)^m[(p^*)^n(\mathcal{M}M^*)\wedge BM^*]\vee(p^*)^m(BM^*)\\ &= (\bigwedge_j C_j)\vee(p^*)^m\circ(BM^*).\end{aligned}$$

Hence,

$$C_i\vee(p^*)^m\circ(BM^*) = (\bigwedge_j C_j)\vee(p^*)^m\circ(BM^*),$$

for all integers $i\geqslant N$. It follows that

$$d(C_i, \bigwedge_j C_j, p^*, AM^*, BM^*)\leqslant 2^{-m}<\varepsilon,$$

for all integers $i\geqslant N$. This establishes (9.2), q.e.d.

PROPOSITION 9.3. *Let A and B be elements of M such that $A\leqslant B$. Then the map $C\to CM^*$ of $[A, B]$ with the p-adic metric to $[AM^*, BM^*]$ with the p-adic metric is an isometry.*

Proof. This is a straightforward computation and will be omitted.

THEOREM 9.4. *Let A and B be elements of M such that $A\leqslant B$. Then, the set $[A, B]M^*$ is dense in the L-module $[AM^*, BM^*]$ with the p-adic metric.*

Proof. Since the metrics d_p and d_{p^*} are equal on $[AM^*, BM^*]$, it is sufficient to prove that the set $[A, B]M^*$ is dense in the metric space $([AM^*, BM^*], d_{p^*})$. This shall now be established.

Let C be an element of $[AM^*, BM^*]$. Now, considering C as an element of M^*, let the sequence $\langle C_i\rangle$, $i=1,2,\ldots$, of elements of M be the completely regular representative of C determined by the p-adic metric on M. Since the sequence $\langle C_i\rangle$ is completely regular, it is decreasing (Remark 4.8). Hence, the sequence $\langle C_i\wedge B\rangle$, $i=1,2,\ldots$, is decreasing, and thus is a Cauchy sequence (Corollary 8.2). Since the sequence $\langle C_i\rangle$ is the completely regular representative of C, and since the sequence $\langle B\vee p^i\mathcal{M}\rangle$, $i=1,2,\ldots$, is the completely regular representative of BM^* (Remark 5.2), the sequence $\langle C_i\wedge(B\vee p^i\mathcal{M})\rangle$, $i=1,2,\ldots$, is a representative of $C\wedge BM^*$ ($=C$) by Corollary 8.3. Since $C_i\wedge(B\vee p^i\mathcal{M}) = (C_i\wedge B)\vee p^i\mathcal{M}$ for all positive integers i, and since $\langle C_i\wedge B\rangle\sim\langle(C_i\wedge B)\vee p^i\mathcal{M}\rangle$ by Corollary 4.6, it follows that the Cauchy sequence $\langle C_i\wedge B\rangle$ is a representative of C. Hence

$$(9.4)\qquad (C_i\wedge B)M^*\to C \quad \text{as} \quad i\to+\infty$$

with the d_p^* metric, and thus with the p^*-adic metric by Theorem 8.12.

Now, since C is an element of $[AM^*, BM^*]$, $AM^*\leqslant C\leqslant BM^*$. Since $\langle C_i\rangle$ is the completely regular representative of C, $\langle A\vee p^i\mathcal{M}\rangle$ is the completely regular representative of AM^*, and $\langle B\vee p^i\mathcal{M}\rangle$ is the completely regular representative of BM^* (Remark 5.2), we obtain $A\leqslant A\vee p^i\mathcal{M}\leqslant C_i\leqslant B\vee p^i\mathcal{M}$ for all positive integers i (Proposition 5.9). Consequently $A=A\wedge B\leqslant C_i\wedge B\leqslant B\wedge(B\vee p^i\mathcal{M}) = B$ for all integers i, and thus $C_i\wedge B$ is an element of $[A, B]$ for all positive integers i.

Since the sequence $\langle(p^*)^i(\mathcal{M}M^*)\wedge BM^*\rangle$, $i=1,2,\ldots$, satisfies the conditions of Lemma 6.1, there exists a natural number n such that

$$(9.5)\qquad (p^*)^{n+i}(\mathcal{M}M^*)\wedge BM^* = (p^*)^i[(p^*)^n(\mathcal{M}M^*)\wedge BM^*]$$

for all nonnegative integers i.

Now, let ε be a positive real number. Choose m to be the least natural number k such that $2^{-k}<\varepsilon$. Since $(C_i\wedge B)M^*\to C$ as $i\to+\infty$ with the p^*-adic metric, (9.4), there exists a natural number N such that

$$d\big((C_i\wedge B)M^*, C, p^*, OM^*, \mathcal{M}M^*\big)\leqslant 2^{-(m+n)}$$

for all integers $i \geqslant N$. Hence,

$$(C_i \wedge B)M^* \vee (p^*)^{n+m}(\mathcal{M}M^*) = C \vee (p^*)^{n+m}(\mathcal{M}M^*)$$

for all integers $i \geqslant N$. Consequently

$$(9.6)\qquad BM^* \wedge [(C_i \wedge B)M^* \vee (p^*)^{n+m}(\mathcal{M}M^*)] = BM^* \wedge [C \vee (p^*)^{n+m}(\mathcal{M}M^*)]$$

for all integers $i \geqslant N$. Since

$$BM^* \wedge [(C_i \wedge B)M^* \vee (p^*)^{n+m}(\mathcal{M}M^*)] = (C_i \wedge B)M^* \vee (p^*)^m[(p^*)^n(\mathcal{M}M^*) \wedge BM^*]$$

and since

$$BM^* \wedge [C \vee (p^*)^{n+m}(\mathcal{M}M^*)] = C \vee (p^*)^m[(p^*)^n(\mathcal{M}M^*) \wedge BM^*]$$

by modularity in M^* and (9.5), we have

$$(9.7)\qquad (C_i \wedge B)M^* \vee (p^*)^m[(p^*)^n(\mathcal{M}M^*) \wedge BM^*] = C \vee (p^*)^m[(p^*)^n(\mathcal{M}M^*) \wedge BM^*]$$

for all integers $i \geqslant N$, by (9.6). Now, let i be an integer such that $i \geqslant N$. Then,

$$\begin{aligned}(C_i \wedge B)M^* \vee (p^*)^m \circ (BM^*) &= (C_i \wedge B)M^* \vee (p^*)^m[(p^*)^n(\mathcal{M}M^*) \wedge BM^*] \vee (p^*)^m(BM^*)\\ &= C \vee (p^*)^m[(p^*)^n(\mathcal{M}M^*) \wedge BM^*] \vee (p^*)^m(BM^*)\\ &= C \vee (p^*)^m \circ (BM^*)\end{aligned}$$

by (9.7). Hence

$$(C_i \wedge B)M^* \vee (p^*)^m \circ (BM^*) = C \vee (p^*)^m \circ (BM^*)$$

for all integers $i \geqslant N$. Consequently,

$$d\big((C_i \wedge B)M^*, C, p^*, AM^*, BM^*\big) \leqslant 2^{-m} < \varepsilon$$

for all integers $i \geqslant N$, q.e.d.

THEOREM 9.5. *Let A and B be elements of M such that $A \leqslant B$. Then the L-module $[AM^*, BM^*]$ with the p-adic metric is the p-adic completion of the Noetherian L-module $[A, B]$.*

Proof. This result follows immediately from Theorem 9.2, Proposition 9.3, and Theorem 9.4, q.e.d.

§ 10. Residuation. The following theorem establishes the first basic fact about residuation in L^*. The general case will be proved later.

THEOREM 10.1. *Let A be an element of M and let B be a principal element of M. Then $(A:B)L^* = AM^* : BM^*$.*

Proof. Since $(A:B)B \leqslant A$, we have $(A:B)L^* \cdot BM^* = [(A:B)B]M^* \leqslant AM^*$ (Corollary 5.11). Thus $(A:B)L^* \leqslant AM^* : BM^*$. Hence, it is sufficient to establish that $(A:B)L^* \geqslant AM^* : BM^*$. This shall now be proven.

Let n be a nonnegative integer. Since LL^* is dense in L^*, there exists an element x of L such that $d_{p^*}(xL^*, AM^* : BM^*) \leqslant 2^{-n}$. Consequently $xL^* \vee (p^*)^n = (AM^* : BM^*) \vee (p^*)^n$. Since $(AM^* : BM^*)(BM^*) \leqslant AM^*$, it follows that

$$\begin{aligned}(xB)M^* &\leqslant [(xL^*) \vee (p^*)^n](BM^*)\\ &= [(AM^* : BM^*) \vee (p^*)^n](BM^*)\\ &= (AM^* : BM^*)(BM^*) \vee (p^*)^n(BM^*)\\ &\leqslant AM^* \vee (p^*)^n(BM^*)\\ &= (A \vee p^nB)M^* .\end{aligned}$$

Consequently, $xB = (xB)M^* \cap M \leqslant (A \vee p^nB)M^* \cap M = A \vee p^nB$ by Propositions 7.2 and 7.6. Hence, $x \leqslant (A \vee p^nB) : B = (A:B) \vee p^n$ since B is a principal element of M by hypothesis. It follows that $xL^* \leqslant [(A:B) \vee p^n]L^* = (A:B)L^* \vee (p^*)^n$. Consequently,

$$(AM^* : BM^*) \vee (p^*)^n = xL^* \vee (p^*)^n \leqslant (A:B)L^* \vee (p^*)^n .$$

Since n was an arbitrary nonnegative integer, we have

$$(AM^* : BM^*) \vee (p^*)^n \leqslant (A:B)L^* \vee (p^*)^n$$

for all nonnegative integers n. Now, since (L^*, p^*) is a local Noetherian lattice, we obtain

$$\begin{aligned}AM^* : BM^* &= \bigwedge_n \{(AM^* : BM^*) \vee (p^*)^n\}\\ &\leqslant \bigwedge_n \{(A:B)L^* \vee (p^*)^n\}\\ &= (A:B)L^* ,\end{aligned}$$

by Theorem 3.10, q.e.d.

COROLLARY 10.2. *Let A be an element of M and let B be a principal element of M. Then $(A \wedge B)M^* = AM^* \wedge BM^*$.*

Proof. Since B is a principal element of M, BM^* is a principal element of M^* (Theorem 8.6). Hence $AM^* \wedge BM^* = (AM^* : BM^*)(BM^*) = [(A:B)L^*](BM^*) = [(A:B)B]M^* = (A \wedge B)M^*$ by Definition 2.4 and Theorem 10.1, q.e.d.

THEOREM 10.3. *For all elements A and B of M, $(A \wedge B)M^* = AM^* \wedge BM^*$.*

Proof. Since M is a Noetherian L-module, there exists principal elements $C_1, C_2, \ldots, C_n$ such that $B = C_1 \vee \ldots \vee C_n \vee (A \wedge B)$. The theorem shall be proven by induction on n.

First, consider the case where $n=1$. Hence $B=C_1\vee(A\wedge B)$. If $C_1=0$, the result follows immediately. Since C_1 is a principal element of M, $C_1\vee(A\wedge B)$ is a principal element of the L-module $[A\wedge B, \mathcal{M}]$ (see comments at the beginning of § 1 of [3] and [1], Lemma 4.1, p. 488). Consequently,

$$\begin{aligned}(A\wedge B)M^* &= \big(A\wedge\big(C_1\vee(A\wedge B)\big)\big)M^*\\ &\Leftrightarrow \big(A\wedge\big(C_1\vee(A\wedge B)\big)\big)[A\wedge B, \mathcal{M}]^*\\ &= A[A\wedge B, \mathcal{M}]^*\wedge\big(C_1\vee(A\wedge B)\big)[A\wedge B, \mathcal{M}]^*\\ &= A[A\wedge B, \mathcal{M}]^*\wedge B[A\wedge B, \mathcal{M}]^*\\ &\Leftrightarrow AM^*\vee BM^*\end{aligned}$$

by Corollary 10.2 and Theorem 9.5. Hence, $(A\wedge B)M^*=AM^*\wedge BM^*$ and the theorem holds when $n=1$. Assume that the theorem holds for the case where $n=k$ and let $B=C_1\vee\ldots\vee C_{k+1}\vee(A\wedge B)$. Set $C=C_{k+1}\vee A$. Then $A\wedge B=(A\wedge B)\vee(A\wedge C_{k+1})=A\wedge\big((A\wedge B)\vee C_{k+1}\big)=A\wedge\big(B\wedge(C_{k+1}\vee A)\big)$ $=A\wedge(B\wedge C)$. Consequently, by modularity in M,

(10.1) $$C\wedge B=C_{k+1}\vee(A\wedge B)=C_{k+1}\vee\big(A\wedge(C\wedge B)\big)$$

and

(10.2) $$\begin{aligned}B&=C_1\vee\ldots\vee C_{k+1}\vee(A\wedge B)\\ &=C_1\vee\ldots\vee C_k\vee\big(C_{k+1}\vee(A\wedge B)\big)\\ &=C_1\vee\ldots\vee C_k\vee\big(B\wedge(C_{k+1}\vee A)\big).\end{aligned}$$

Now, consider A and $C\wedge B$. Since $C\wedge B=C_{k+1}\vee\big(A\wedge(C\wedge B)\big)$ by (10.1), the case $n=1$ applies, and hence $(A\wedge B)M^*=\big(A\wedge(C\wedge B)\big)M^*=AM^*\wedge$ $\wedge(C\wedge B)M^*$. Next, consider C and B. Since $B=C_1\vee\ldots\vee C_k\vee(C\wedge B)$ by (10.2), we have $(C\wedge B)M^*=CM^*\wedge BM^*$ by the induction hypothesis. Hence, $(A\wedge B)M^*=AM^*\wedge(C\wedge B)M^*=AM^*\wedge(CM^*\wedge BM^*)=AM^*\wedge BM^*$. The induction is now complete, q.e.d.

THEOREM 10.4. *For all elements A and B of M, $(A:B)L^*=AM^*:BM^*$.*

Proof. Let A and B be elements of M. Since M is a Noetherian L-module, there exists principal elements $C_1, \ldots, C_n$ of M such that $B=C_1\vee\ldots\vee C_n$. Consequently,

$$\begin{aligned}(A:B)L^*&=\big(A:(C_1\vee\ldots\vee C_n)\big)L^*\\ &=\big((A:C_1)\wedge\ldots\wedge(A:C_n)\big)L^*\\ &=(A:C_1)L^*\wedge\ldots\wedge(A:C_n)L^*\\ &=AM^*:(C_1M^*\vee\ldots\vee C_nM^*)\\ &=AM^*:(C_1\vee\ldots\vee C_n)M^*\\ &=AM^*:BM^*\end{aligned}$$

by (2.19), Theorems 10.1 and 10.3, q.e.d.

Remark 10.5. In view of Definition 5.4 and Theorem 10.3, it follows that the extension map $A\to AM^*$ of $M\to MM^*$ is a lattice isomorphism, i.e. M is lattice isomorphic to MM^* considered as a sublattice of M^*.

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