Proof. Any set open in the product topology which contains $((\emptyset), 0)$ must contain a set of the form $G \times H$ where $(\emptyset) \in G$ and $\mathbf{0} \in H$ and G and H are open in the new interval topologies on L and M respectively. By Lemma 1, G contains $\Psi(B_n)$ for all but a finite number of n. Suppose that G contains $\Psi(B_j)$. By Lemma 2, $B_j \cap H \neq \emptyset$. Let $f_j \in B_j \cap H$, then $\Psi(f_j) \in G$ and $(\Psi(f_j), f_j) \in G \times H$. Yet $(\Psi(f_j), f_j) \in F$. Hence $(G \times H) \cap F \neq \emptyset$. Every open set containing $((\emptyset), \mathbf{0})$ must contain a member of F, which implies that $((\emptyset), \mathbf{0})$ is in the closure of F.

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On the modular relation in atomistic lattices

by

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1. Introduction. S. Maeda [6] and [7] as well as R. Wille [9] have recently investigated various types of atomistic lattices. Basically, Wille was concerned with upper continuous atomistic lattices equipped with some type of closure operator, while Maeda investigated modular and dual modular pairs in atomistic lattices. Our goal here is to extend and to some degree attempt to unify these two theories.

In an effort to make the paper fairly self-contained, we introduce our basic terminology and prove a few preliminary theorems in § 2. In § 3 we introduce the concept of a *finite-statisch* lattice and extend Wille's theory [9] to this class of lattice. In § 4 we discuss modularity in atomistic lattices, and relate the work of S. Maeda to that of Wille. Finally, in § 5 we list a few open questions.

2. Basic terminology. As much as possible our terminology and notation will follow that of Wille [9]. A noteable exception, however, is that rather than using Wille's symbolism, we will use the symbols $\cup$ and $\cap$ to denote set union and set intersection.

DEFINITION 2.1. A lattice L with 0 is called *atomistic* if every element of L is the join of a family of atoms.

DEFINITION 2.2. A non-empty subset T of a lattice L is called *increasing* (see [9], Definition 1.3, p. 5) if $x, y \in T$ implies the existence of an element z of T such that $x \vee y \leqslant z$. In symbols, the notation $x_a \uparrow x$ will denote the fact that $\{x_a\}$ is an increasing subset with join x. If $x_a \uparrow x$ and $x_a \wedge y \uparrow x \wedge y$ for all $y \in L$, then $\{x_a\}$ is called *a continuous increasing subset of L*.

In a lattice with atoms, let ax denote the set of atoms dominated by x. The next lemma then provides a useful characterization of continuous increasing subsets of an atomistic lattice.

LEMMA 2.3. *Let $x_\beta \uparrow x$ in an atomistic lattice L. Then $\{x_\beta\}$ is continuous if and only if* $ax = \bigcup_\beta (ax_\beta)$.

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Proof. Let $\{x_\beta\}$ be continuous and p an atom under x. Then $x_\beta \wedge \wedge p \uparrow x \wedge p = p$ implies $p \leqslant x_\beta$ for some index β. It follows that $ax = \bigcup_\beta (ax_\beta)$. Suppose, conversely, that $ax = \bigcup_\beta (ax_\beta)$, and let $y \in L$. Then $x \wedge y$ is an upper bound for $\{x_\beta \wedge y\}$. If $a < x \wedge y$, then there exists an atom p such that $p \leqslant x \wedge y$ but $p \nleqslant a$. Now $p \leqslant x$ implies $p \leqslant x_\beta$ for some index β, so $p \leqslant x_\beta \wedge y$. It follows that a is not an upper bound for $\{x_\beta \wedge y\}$. We conclude that $x_\beta \wedge y \uparrow x \wedge y$, so $\{x_\beta\}$ is continuous.

A complete lattice L is called *upper continuous* if every increasing subset of L is continuous. Following Wille [9], we agree to call an upper continuous atomistic lattice a *geometric* lattice.

DEFINITION 2.4. A set A of atoms of a lattice L with 0 is called a *linear set* if p contained in the join of a finite number of elements of A implies $p \in A$ for any atom p of L.

Given the lattice L, let $\mathfrak{G}(L)$ denote the set of linear subsets of L. By [9], Satz 3.1, p. 16, $\mathfrak{G}(L)$, when partially ordered by set inclusion, forms a geometric lattice. If L is a complete atomistic lattice, the mapping $x \to ax$ is a complete Λ-monomorphism of L into $\mathfrak{G}(L)$.

DEFINITION 2.5. A geometric lattice G is called *topological* if G possesses a *closure operator* $a \to \bar{a}$ satisfying: (1) $a \leqslant \bar{a} = \bar{\bar{a}}$ for all $a \in G$. (2) $\bar{0} = 0$. (3) If $\{p_i\}$ is a finite set of atoms then $\bigvee_i \bar{p}_i = \overline{\bigvee_i p_i}$. (4) $a \leqslant b$ implies $\bar{a} \leqslant \bar{b}$. If (4) is replaced by (4′), $\bar{a} \vee \bar{b} = \overline{a \vee b}$, then G is called a *classical topological geometric lattice*. Finally, if (4) is replaced by (4″), $\bar{a} \vee p = \overline{a \vee p}$ for any atoms p, then G is called a *semiclassical topological geometric lattice*. In any case, let $\mathfrak{A}(G)$ denote the set of closed elements of G; i.e., $\mathfrak{A}(G) = \{x \in G\colon \bar{x} = x\}$. Notice that $\mathfrak{A}(G)$ is a complete lattice with respect to the join and meet operations

$$\bar{x} \sqcup \bar{y} = \overline{\bar{x} \vee \bar{y}} \quad \text{and} \quad \bar{x} \sqcap \bar{y} = \bar{x} \wedge \bar{y}\,.$$

By [9], Satz 3.2, p. 17, if L is a complete atomistic lattice, then a closure operator $A \to \bar{A}$ can be defined on $\mathfrak{G}(L)$ by the formula

$$\bar{A} = \bigcap (X\colon A \subseteq X \in aL) = a(\bigvee A)\,.$$

Then $\mathfrak{G}(L)$ becomes a topological geometric lattice with $\mathfrak{A}(\mathfrak{G}(L)) \cong L$. By [9], Satz 3.3, p. 17, if G is a topological geometric lattice, then $\mathfrak{G}(\mathfrak{A}(G))$ is a complete atomistic lattice with $\mathfrak{G}(\mathfrak{A}(G)) \cong G$.

DEFINITION 2.6. A lattice L with 0 is called a *section-semicomplemented* or an *SSC-lattice* if $a < b$ implies the existence of an element x such that $0 < x \leqslant b$ and $a \wedge x = 0$.

In a lattice L with 0, let $F(L)$ denote the set of elements that may be expressed as the join of a finite (possibly empty) family of atoms.

We say that e *covers* f in L and write $e \succ f$ in case $e > f$ and $e \geqslant x \geqslant f$ implies $x = e$ or $x = f$. We introduce the *covering property* (C) as follows:

(C) *If p is an atom and $p \nleqslant a$, then $p \vee a \succ a$.*

Following S. Maeda [7], we call L an AC-*lattice* if it is an atomistic lattice with the covering property (C). A *matroid lattice* may then be defined to be an upper continuous AC-lattice. It will prove illuminating to show that in a fairly large class of lattices $F(L)$ is an ideal of L and $\mathfrak{G}(L)$ is isomorphic to the lattice of ideals of $F(L)$. First we need some additional terminology.

DEFINITION 2.7. In a lattice L the pair (a, b) is called a *modular pair*, denoted $M(a, b)$, if $x \leqslant b \Rightarrow x \vee (a \wedge b) = (x \vee a) \wedge b$. Dually, (a, b) is called a *dual modular pair*, in symbols $M^*(a, b)$, if $x \geqslant b \Rightarrow x \wedge (a \vee b) = (x \wedge a) \vee b$. The lattice L is called *M-symmetric* if $M(a, b) \Rightarrow M(b, a)$ for all $a, b \in L$, and *M^*-symmetric* if $M^*(a, b) \Rightarrow M^*(b, a)$. A lattice L with 0 is called *weakly modular* if $a \wedge b \neq 0 \Rightarrow M(a, b)$.

LEMMA 2.8. *Let L be a weakly modular* SSC-*lattice with* 1 *in which the covering property* (C) *holds. If* 1 *is the join of a finite number of atoms, then every element of L is the join of a finite number of atoms.*

Proof. Let $1 = p_1 \vee p_2 \vee \ldots \vee p_n$ where the $\{p_i\}$ are distinct atoms. If $a < 1$, then $a \ngeqslant$ all p_i, so way assume that $a \ngeqslant p_1$. Then by (C), $p_1 \vee \vee p_2, p_1 \vee p_3, \ldots, p_1 \vee p_n$ are atoms in $[p_1, 1] = \{x \in L\colon x \geqslant p_1\}$. Since L is weakly modular, $[p_1, 1]$ is modular and by [4], Hilfsatz 2.11, p. 78, $[p_1, 1]$ is an atomic complemented modular lattice with height at most $n-1$. If $b \leqslant c \leqslant a$ and $p_1 \vee b = p_1 \vee c$, then $b \leqslant c \leqslant p_1 \vee b$. Since $p_1 \nleqslant a$, clearly $p_1 \nleqslant b$, so $p_1 \vee b \succ b$. Since $p_1 \nleqslant c$, this forces $c = b$, so $b < c < a \Rightarrow p_1 \vee \vee b < p_1 \vee c$. If now $a \geqslant b_1 > b_2 > \ldots > b_n$, then $p_1 \vee b_1 > p_1 \vee b_2 > \ldots > p_1 \vee b_n$ in $[p_1, 1]$. But this contradicts the fact that the height of $[p_1, 1]$ is at most $n-1$. It follows that $[0, a]$ has no properly increasing or decreasing chains of length $\geqslant n$, so $[0, a]$ is atomic. By SSC, $[0, a]$ is atomistic, and since it has finite height, a must be the join of a finite number of atoms.

LEMMA 2.9. *Let L be a lattice with* 0 *in which $F(L)$ is an ideal. Then $\mathfrak{G}(L) \cong I(F(L))$, the lattice of ideals of $F(L)$.*

Proof. If $A \in \mathfrak{G}(L)$, let $I(A)$ denote the ideal generated by the elements of A. If J is an ideal of $F(L)$, let aJ be the set of atoms of J. Then $I\colon \mathfrak{G}(L) \to I(F(L))$ and $a\colon I(F(L)) \to \mathfrak{G}(L)$ are isotone.

Note that $I(aJ)$ is the ideal generated by the atoms of J. Hence $I(aJ) \leqslant J$. If $x \in J$ then x is the join of a finite number of atoms of J and $x \in I(aJ)$. Hence $I(aJ) = J$.

If $A \in \mathfrak{G}(L)$, then $p \in A \Rightarrow p \in I(A) \Rightarrow p \in aI(A)$. Thus $A \subseteq aI(A)$. If $p \in aI(A)$ then p is an atom and $p \in I(A) \Rightarrow p$ is contained in the join of finitely many elements of A, so $p \in A$. Thus $A = aI(A)$, completing the proof.

THEOREM 2.10. *If L is an* AC-*lattice or a weakly modular* SSC-*lattice in which the covering property* (C) *holds, then* $\mathfrak{G}(L) \cong I(F(L))$.

Proof. In view of Lemma 2.9 we need only show $F(L)$ to be an ideal of L. If L is an AC-lattice, the assertion is contained in [6], Lemma 3, p. 167, If L is a weakly modular SSC-lattice in which (C) holds, apply Lemma 2.8.

Remark 2.11. Since the properties of being distributive, modular, or weakly modular are inherited from L, first by $F(L)$, and then by $I(F(L))$, [9], Satz 3.5 and 3.6, pp. 18–19 (except for the assertion regarding semimodularity) now follow immediately.

3. Statisch and finite-statisch lattices.

DEFINITION 3.1. If $\{x_a\colon a \in I_1\}$ and $\{y_\beta\colon \beta \in I_2\}$ are increasing subsets of a lattice L, then their "join", $\{x_a \vee y_\beta\colon a \in I_1,\ \beta \in I_2\}$, is again an increasing subset of L. Following Wille [9], Definition 3.3, p. 20, we agree to call a complete lattice L *statisch* if the join of every pair of continuous increasing subsets is again continuous. We further agree to call a complete atomistic lattice *finite-statisch* if for any continuous increasing subset $\{x_a\colon a \in I\}$, $\{x_a \vee p\colon a \in I\}$ is continuous for any atom p of L.

As a direct analogue of [9], Satz 3.8, p. 21, we have

THEOREM 3.2. *A complete atomistic lattice is finite-statisch if and only if p, q atoms with $p \leqslant q \vee a$ implies the existence of a finite number of atoms $\{p_i\} \leqslant a$ such that $p \leqslant q \vee (\bigvee_i p_i)$.*

Proof. Let L be finite-statisch. Given $a \in L$, let $\mathfrak{E}(a)$ denote the set of all finite sets of atoms under a. Given $E \in \mathfrak{E}(a)$, let $a_E = \bigvee \{r\colon r \in E\}$. In view of Lemma 2.3 $\{a_E\colon E \in \mathfrak{E}(a)\}$ is a continuous increasing family, so $\{a_E \vee q\colon E \in \mathfrak{E}(a)\}$ is continuous. But now $a_E \vee q \uparrow a \vee q \geqslant p$, so by Lemma 2.3, there exists $E \in \mathfrak{E}(a)$ such that $p \leqslant a_E \vee q$ as claimed.

Suppose conversely that p, q atoms with $p \leqslant q \vee a$ implies $p \leqslant a \vee \vee (\bigvee_i p_i)$ for some finite set $\{p_i\}$ of atoms under a. Let $\{x_\beta\colon \beta \in I\}$ be a continuous increasing subset with join x. Let q be an atom. If $p \leqslant q \vee x$ then there exist finitely many atoms $\{p_i\} \leqslant x$ such that $p \leqslant q \vee (\bigvee_i p_i)$. Since $\{x_\beta\colon \beta \in I\}$ is increasing, there must exist an index $\beta' \in I$ such that $\bigvee_i p_i \leqslant x_{\beta'}$. Then $p \leqslant q \vee x_{\beta'}$. It is immediate that $a(x \vee q) = \bigcup [a(x_\beta \vee q)]$ so $\{x_\beta \vee q\colon \beta \in I\}$ is continuous.

We now attempt to relate semi-classical topological geometric lattices with finite-statisch lattices in a manner analogous to [9], Satz 3.4, p. 18. A key item is provided by

LEMMA 3.3. *Let G be a topological geometric lattice. An increasing subset $\{\overline{x}_\beta\}$ of $\mathfrak{A}(G)$ is continuous if and only if $\bigvee_\beta \overline{x}_\beta = \bigsqcup_\beta \overline{x}_\beta$.*

Proof. If $\{\overline{x}_\beta\}$ is continuous, then by Lemma 2.3, $a(\bigsqcup_\beta \overline{x}_\beta) = \bigsqcup_\beta a\overline{x}_\beta$. It is immediate that ($p$ an atom) $p \leqslant \bigsqcup_\beta \overline{x}_\beta$ implies $p \leqslant \bigvee_\beta \overline{x}_\beta$ so $\bigsqcup_\beta \overline{x}_\beta \leqslant \bigvee_\beta \overline{x}_\beta$. The reverse inequality always attains. If, on the other hand, $\bigsqcup_\beta \overline{x}_\beta = \bigvee_a \overline{x}_\beta$ since G is upper continuous, it is clear that $a(\bigsqcup_\beta \overline{x}_\beta) = \bigcup_\beta (a\overline{x}_\beta)$, so by Lemma 2.3, $\{\overline{x}_\beta\}$ is continuous.

THEOREM 3.4. *If L is a finite-statisch lattice then $\mathfrak{G}(L)$ is a semi-classical topological geometric lattice. If G is a semi-classical topological geometric lattice, then $\mathfrak{A}(G)$ is finite-statisch.*

Proof. Let L be a finite-statisch lattice. By [9], Satz 3.2, p. 17, $\mathfrak{G}(L)$ is a topological geometric lattice and $L \cong \mathfrak{A}(\mathfrak{G}(L))$. Since L is finite-statisch, this forces $\mathfrak{A}(\mathfrak{G}(L))$ to be finite-statisch. Letting "$\sqcup$" and "$\sqcap$" represent the lattice operations in $\mathfrak{A}(\mathfrak{G}(L))$, we see that if $\{\overline{p_0}\} \leqslant \overline{A} \sqcup \{\overline{p}\}$, there must exist finitely many atoms $\{q_i\}$ such that $\{\overline{q_i}\} \leqslant \overline{A}$ and

$$\{\overline{p_0}\} \leqslant (\bigsqcup_i \{\overline{q_i}\}) \sqcup \{\overline{p}\} = (\bigvee_i \{q_i\}) \sqcup \{p\}$$
$$= \{r \in L\colon r \text{ an atom, } r \leqslant (\bigvee_i q_i) \vee p\}.$$

Hence $p_0 \leqslant (\bigvee_i q_i) \vee p$, $\{\overline{p_0}\} \leqslant \overline{A} \vee \{\overline{p}\}$, and consequently $\overline{A} \sqcup \{\overline{p}\} = \overline{A} \vee \{\overline{p}\}$. It follows that $\overline{A} \sqcup \{\overline{p}\} = \overline{A \vee \{p\}}$, so $\mathfrak{G}(L)$ is semi-classical.

Suppose now that G is a semi-classical topological geometric lattice. If $\{\overline{x}_a\}$ is a continuous increasing subset of $\mathfrak{A}(G)$, then by Lemma 3.3, $\bigvee_a \overline{x}_a = \bigsqcup_a \overline{x}_a$. It follows that for any atom p,

$$\bigsqcup_a (\overline{x}_a \sqcup \overline{p}) = (\bigsqcup_a \overline{x}_a) \sqcup \overline{p} = (\bigvee_a \overline{x}_a) \sqcup \overline{p} = (\bigvee_a \overline{x}_a) \vee \overline{p}.$$

For an atom $x \leqslant \bigsqcup_a (\overline{x}_a \sqcup \overline{p}) = (\bigvee_a x_a) \vee \overline{p}$, by upper continuity of G, there exists an index a such that $x \leqslant \overline{x}_a \vee \overline{p}$. It follows from Lemma 2.3 that $\{\overline{x}_a \sqcup \overline{p}\}$ is continuous in $\mathfrak{A}(G)$.

THEOREM 3.5. *If L is a finite-statisch* AC-*lattice then $\mathfrak{G}(L)$ is a semi-classical topological matroid lattice. If G is a semi-classical topological matroid lattice, then $\mathfrak{A}(G)$ is a finite-statisch* AC-*lattice.*

Proof. If follows easily from [1], Lemma 3, p. 197, that if L is a finite-statisch AC-lattice, then $F(L)$ is an ideal such that $a \in F(L) \Rightarrow [0, a]$ is of finite height. It is easy to show that this forces $I(F(L))$ to be a matroid lattice, so by Theorem 2.10, $\mathfrak{G}(L)$ is a matroid lattice.

If G is a semi-classical topological matroid lattice, then $\mathfrak{A}(G)$ is finite-statisch by Theorem 3.4. If $\overline{p} \nleqslant \overline{a}$ in $\mathfrak{A}(G)$, then $\overline{p} \sqcup \overline{a} = \overline{p} \vee \overline{a}$ covers $\overline{a}$ in G, hence in $\mathfrak{A}(G)$.

Combining these two theorems we have:

THEOREM 3.6. *With respect to the functions $\mathfrak{G}$ and $\mathfrak{A}$ one can identify the following classes of lattices:*

(i) *finite-statisch lattices with semi-classical topological geometric lattices,*

(ii) *finite-statisch* AC-*lattices with semi-classical topological matroid lattices.*

4. Modularity in AC-lattices. Following S. Maeda [7], p. 108, let us call a lattice L with 0 *finite-modular* if $a \in F(L)$ implies $M(b, a)$ for every $b \in L$. This condition turns out to be intimately related to the *lower covering property*

(LC) If $a\vee b \gtrdot a$ *then* $b \gtrdot a\wedge b$.

LEMMA 4.1. *Let L be an* AC-*lattice in which the lower covering property* (LC) *holds. Then $M^*(a, q)$ holds for all $a \in L$ and all atoms q of L.*

Proof. Let q be an atom under b. Then for arbitrary $a \in L$, $b\wedge(a\vee q) \geqslant (b\wedge a)\vee q$. We must establish $b\wedge(a\vee q) \leqslant (b\wedge a)\vee q$. If $q\leqslant a$, there is nothing to prove, so assume $q \nleqslant a$. Set $d = b\wedge(a\vee q)$. Then $a\leqslant a\vee d \leqslant a\vee q$. If $a\vee d = a$, then $q\leqslant d\leqslant a$, a contradiction. Thus $a\vee d = a\vee q$, so $a\vee d \gtrdot a$. By (LC), $d\gtrdot a\wedge d = a\wedge b\wedge(a\vee q) = a\wedge b$. Since $q\nleqslant a$ and $q\leqslant d$, it follows that $q\vee(a\wedge b) = (q\vee a)\wedge b$.

THEOREM 4.2. *For an* AC-*lattice L*, TAE:

(i) *L is finite-modular.*

(ii) *L is M^*-symmetric.*

(iii) *$M^*(a, q)$ holds for all $a \in L$ and all atoms q of L.*

(iv) *The lower covering property* (LC) *holds.*

Proof. (i) $\Rightarrow$ (ii). [6], Lemma 4, p. 168.

(ii) $\Rightarrow$ (iii). In an AC-lattice it is easy to show that $M^*(q, a)$ holds for all $a\in L$ and all atoms q.

(iii) $\Rightarrow$ (iv). This is Lemma 4.1.

(iv) $\Rightarrow$ (i). See [6], Theorem 1, p. 167.

Since the next theorem merely restates [7], Theorems 5.1 and 5.2 in the context of Wille [9], its proof will be omitted.

THEOREM 4.3. *If L is a complete finite-modular* AC-*lattice, then $\mathfrak{G}(L)$ is a semi-classical topological modular matroid lattice. If G is a semi-classical topological modular matroid lattice, then $\mathfrak{A}(G)$ is a finite-modular* AC-*lattice.*

In view of the above theorem, it seems natural to ask if every finite-modular AC-lattice may be embedded in a *complete* finite-modular AC-lattice. We will answer this question in the affirmative by showing that the completion by cuts of a finite-modular AC-lattice is itself a finite-modular AC-lattice.

Given a subset S of a lattice L, let S^* denote the set of upper bounds of S, and S^+ its set of lower bounds. Following Birkhoff [1], p. 127, let us call an ideal I of L *closed* if $I = (I^*)^+$. The completion by cuts of L is then defined to be the set of closed ideals of L, partially ordered by set inclusion; it will be denoted by $\bar{L}$. Notice that $\bar{L}$ is a complete lattice and $x\to J_x = \{y\in L\colon y\leqslant x\}$ embeds L as a sublattice of $\bar{L}$, so as to preserve any existing suprema or infima of subsets of L. We shall make use of the fact that if I, J are closed ideals of L, then $I = J$ if and only if $I^* = J^*$. The interested reader is referred to [1], pp. 126–127, for further details.

LEMMA 4.4. *Let L be a lattice with 0. Then $M(x, y)$ in L implies $M(J_x, J_y)$ in $\bar{L}$.*

Proof. Let $I < J_y$. We must prove that $(I\vee J_x)\cap J_y = I\vee J_{x\wedge y}$. We will do this by showing that a is an upper bound for $(I\vee J_x)\cap J_y$ if and only if it is an upper bound for $I\vee J_{x\wedge y}$. Any upper bound for $(I\vee J_x)\cap J_y$ is evidently an upper bound for $I\vee J_{x\wedge y}$. On the other hand, let a be an upper bound for $I\vee J_{x\wedge y}$. Then $a\geqslant x\wedge y$ and $a \in I^*$. Since $y \in I^*$, $a\wedge y\in I^*$. Hence $(a\wedge y)\vee x \in (I\vee J_x)^*$ and $[(a\wedge y)\vee x]\wedge y \in [(I\vee J_x)\cap J_y]^*$. But using $M(x, y)$ in L, we have

$$[(a\wedge y)\vee x]\wedge y = (a\wedge y)\vee(x\wedge y) = a\wedge y\,,$$

since $a\geqslant x\wedge y$. Hence $a\geqslant a\wedge y \in [(I\vee J_x)\cap J_y]^*$, as desired.

THEOREM 4.5. *If L is a finite-modular* AC-*lattice, so is $\bar{L}$.*

Proof. (1) We first establish that $\bar{L}$ is an AC-lattice. Note first that $I\in\bar{L}$ is an atom if and only if $I = J_p$ for some atom p of L. If $J_p\leqslant J$ in L, then $M(J_p, J)$ is clear. If $J_p\nleqslant J$ then $J_p\cap J = (0)$. It follows that p is not contained in all upper bounds of J. Hence there exists an $x\in J^*$ such that $p\wedge x = 0$. Now $M(p, x)$ in L implies $M(J_p, J_x)$ in $\bar{L}$, and since $J\leqslant J_x$, it is immediate that $M(J_p, J)$ in $\bar{L}$. Since $\bar{L}$ is clearly atomistic, it follows from [6], Lemma 1, p. 166, that $\bar{L}$ is an AC-lattice.

(2) We now establish that $\bar{L}$ is finite-modular. In view of [7], Lemma 2.2, p. 108, it suffices to show that $M(I, J_{p\vee q})$ holds in $\bar{L}$ for all atoms p, q of L. We must therefore show that $(0) < K < J_{p\vee q}$ implies $(K\vee I)\cap J_{p\vee q} = K\vee(I\cap J_{p\vee q})$. Now $(0)<K<J_{p\vee q}$ implies that $K = J_r$ for some atom $r\leqslant p\vee q$. Hence we must show that if $r\leqslant p\vee q$ is an atom, then

$$(*)\qquad (J_r\vee I)\cap J_{p\vee q} = J_r\vee(I\cap J_{p\vee q})\,.$$

Case 1: $p\vee q\in I$. Then both sides of $(*)$ equal $J_{p\vee q}$.

Case 2: $I\cap J_{p\vee q} = J_t$ for some atom $t\leqslant p\vee q$. If $r = t$, then $r\in I$, so

$$(J_r\vee I)\cap J_{p\vee q} = I\cap J_{p\vee q} = J_r = J_r\vee(I\cap J_{p\vee q})\,.$$

If $r\neq t$, then $r\vee t = p\vee q$ in L, and

$$J_{p\vee q}\geqslant (J_r\vee I)\cap J_{p\vee q}\geqslant J_r\vee(I\cap J_{p\vee q}) = J_r\vee J_t = J_{p\vee q}\,.$$

Case 3: $I\cap J_{p\vee q} = 0$. Then there exists an upper bound x of I such that $(p\vee q)\wedge x < p\vee q$. If $(p\vee q)\wedge x = 0$, stop here. If not, then $(p\vee q)\wedge x = s$ for some atom s. Now $s\leqslant$ all upper bounds of I would imply $s\in I$, a contradiction, so there is a $y\in I^*$ such that $s\wedge y = 0$. Then

$x \wedge y \in I^*$ and $(p \vee q) \wedge (x \wedge y) = 0$. In any event, I has an upper bound z such that $(p \vee q) \wedge z = 0$. Then $M(z, p \vee q)$ in L implies $M(J_z, J_{p \vee q})$ in $\bar{L}$. Since $I \leqslant J_z$ and $J_z \cap J_{p \vee q} = (0)$, this clearly implies $M(I, J_{p \vee q})$ in $\bar{L}$.

Following the terminology of Grätzer and Schmidt [2], p. 30, we call an ideal S of a lattice L *standard* if $(I \vee S) \cap J = (I \cap J) \vee (S \cap J)$ for all ideals I, J of L. By [2], Theorem 2, p. 30, the ideal S is standard if and only if $S \vee J_x = \{s \vee x_1 \colon s \in S, x_1 \leqslant x\}$ for any $x \in L$. Interestingly enough, we now prove:

THEOREM 4.6. *If L is a finite-modular* AC-*lattice, then $F(L)$ is a standard ideal.*

Proof. In view of [2], Theorem 2, p. 30, it suffices to show that $b \leqslant a \vee x$ with $a \in F(L)$ implies $b = (b \wedge x) \vee a_1$ for some $a_1 \in F(L)$.

(1) Suppose that $b \leqslant x \vee p$, p an atom. If $b \leqslant x$ we may write $b = (b \wedge x) \vee 0$ with $0 \in F(L)$, so assume $b \nleqslant x$. Then $b \wedge x < b$ and if q is an atom such that $q \leqslant b$, $q \nleqslant x \wedge b$, then $q \nleqslant x$ so $x \vee q = x \vee p$. Then $b = b \wedge (x \vee p) = b \wedge (x \vee q) = (b \wedge x) \vee q$ with $q \in F(L)$.

(2) If $b \leqslant x \vee p_1 \vee \ldots \vee p_n$ with $n > 1$ and each p_i an atom, set $y = x \vee \vee p_1 \vee \ldots \vee p_{n-1}$. Then by (1), $b \leqslant y \vee p_n$ implies $b = (b \wedge y) \vee r$ for some atom r. By induction we may assume that since $b \wedge y \leqslant x \vee p_1 \vee \ldots \vee p_{n-1}$, there exists an element a of $F(L)$ such that

$$b \wedge y = (b \wedge y \wedge x) \vee a = (b \wedge x) \vee a\,.$$

Hence $b = (b \wedge y) \vee r = (b \wedge x) \vee a \vee r$ with $a \vee r \in F(L)$.

In a survey paper on the lattice-theoretic approach to geometry, B. Jónsson has shown ([3], Theorem 4.4, p. 193) that a geometry is *strongly planar* ([3], Definition 4.2, p. 193) if and only if its lattice of subspaces L is a matroid lattice satisfying

(SP) *For any atoms p, q, r of L and any element a of L, the conditions $p \leqslant q \vee a$ and $r \leqslant a$ jointly imply the existence of an atom $s \leqslant a$ such that $p \leqslant q \vee r \vee s$.*

In view of this, an atomistic lattice is often called *strongly planar* if it satisfies (SP). We are now able to state

THEOREM 4.7. *If L is a complete, strongly planar* AC-*lattice, then $\mathfrak{G}(L)$ is a semi-classical topological weakly modular matroid lattice. If G is a semi-classical topological weakly modular matroid lattice, then $\mathfrak{A}(G)$ is a strongly planar* AC-*lattice.*

Proof. Let L be a complete strongly planar AC-lattice. Then if $a \in F(L)$, $[0, a]$ is a strongly planar matroid lattice. By [5], Theorem 2.19, p. 105, $[0, a]$ is weakly modular. Hence $F(L)$ is weakly modular. Since this clearly implies that $I(F(L))$ is weakly modular, we may apply Theorem 2.10 to conclude that $\mathfrak{G}(L)$ is weakly modular.

Let G be a semi-classical topological weakly modular matroid lattice. Let p, q, r be points with $\bar{p} \leqslant \bar{q} \sqcup \bar{a}$, $r \leqslant \bar{a}$. Since G is semi-classical, $p \leqslant q \vee \bar{a}$ in G. If $p = r$ or $q = r$, take $s = p$. Then $p \leqslant q \vee r \vee s$ with $s \leqslant \bar{a}$ as desired. If $p \neq r$ and $q \neq r$, then $p \vee r$, $q \vee r$ are atoms in $[r, 1]$ and $r \vee p \leqslant (q \vee r) \vee \bar{a}$. Since $[r, 1]$ is modular, there exists an element l such that $l \succ r$ and $p \leqslant q \vee r \vee l$ (see [9], Satz 3.11, p. 22). Since $l \succ r$, if s is an atom such that $s \leqslant l$ but $s \neq r$, then $l = r \vee s$. But then $p \leqslant q \vee r \vee s$ with $s \leqslant a$. This shows $\mathfrak{A}(G)$ to be strongly planar. By Theorem 3.5, it is an AC-lattice.

Combining Theorems 4.3 and 4.7, we have

THEOREM 4.8. *With respect to the functions $\mathfrak{G}$ and $\mathfrak{A}$ one can identify the following classes of lattices:*

(1) *complete finite-modular* AC-*lattices with semi-classical topological modular matroid lattices.*

(2) *complete strongly planar* AC-*lattices with semi-classical topological weakly modular matroid lattices.*

Remark 4.9. Making use of Theorem 4.3 and 4.7, the following unpublished result of S. Maeda may be obtained: "*An* AC-*lattice L is strongly planar if and only if for every atom p of L, $[p, 1]$ is a finite-modular* AC-*lattice.*" It then follows from Theorem 4.2 that an AC-lattice L is strongly planar if and only if $a \wedge b \neq 0$, $M^*(a, b) \Rightarrow M^*(b, a)$. It also follows from this and Theorem 4.5 that the completion by cuts of a strongly planar AC-lattice is strongly planar.

Finally, we have a quick look at modularity in a statisch AC-lattice.

THEOREM 4.10. *A statisch* AC-*lattice is M^*-symmetric if and only if it is modular, it is strongly planar if and only if it is weakly modular.*

Proof. In view of Theorem 2.10 and [9], Satz 3.9, p. 21, L may be regarded as a sublattice of $I(F(L))$. The theorem now follows immediately from Theorems 4.3 and 4.7.

LEMMA 4.11. *Let G be a semi-classical topological matroid lattice. Then $M(\bar{a}, \bar{b})$ in $\mathfrak{A}(G)$ implies $M(\bar{a}, \bar{b})$ in G.*

Proof. Let $c \in G$, $c \leqslant \bar{b}$. Then $c \vee (\bar{a} \wedge \bar{b}) \leqslant (c \vee \bar{a}) \wedge \bar{b}$. If p is an atom and $p \leqslant (c \vee \bar{a}) \wedge \bar{b}$, then by upper continuity, there is a finite element $c_1 \leqslant c$ such that $p \leqslant (c_1 \vee \bar{a}) \wedge \bar{b}$. Thus $c_1 = \bar{c}_1$, and using $M(\bar{a}, \bar{b})$ in $\mathfrak{A}(G)$,

$$(c_1 \vee \bar{a}) \wedge \bar{b} = (\bar{c}_1 \sqcup \bar{a}) \sqcap \bar{b} = \bar{c}_1 \sqcup (\bar{a} \wedge \bar{b}) = c_1 \vee (\bar{a} \wedge \bar{b})\,,$$

so $p \leqslant c_1 \vee (\bar{a} \wedge \bar{b})$ in G. Thus $p \leqslant c_1 \vee (\bar{a} \wedge \bar{b}) \leqslant c \vee (\bar{a} \wedge \bar{b})$. If follows that $(c \vee \bar{a}) \wedge \bar{b} = c \vee (\bar{a} \wedge \bar{b})$, so $M(\bar{a}, \bar{b})$ holds in G.

THEOREM 4.12. *Every statisch* AC-*lattice is M-symmetric.*

Proof. In view of [9], Satz 3.10, p. 22, we may regard L as the lattice of closed elements of a classical topological matroid lattice. Let

$\bar{a}, \bar{b} \in L$ with $M(\bar{a}, \bar{b})$. By Lemma 4.11, $M(\bar{a}, \bar{b})$ in G, and by [8], Theorem 1, $M(\bar{b}, \bar{a})$ holds in G. It follows that $M(\bar{b}, \bar{a})$ in L.

Remark 4.13. By an obvious modification of the above lemma one can show that in any finite-statisch AC-lattice, if $\bar{a}$ or $\bar{b}$ is finite then $M(\bar{a}, \bar{b}) \Rightarrow M(\bar{b}, \bar{a})$.

5. Some open questions. We close by listing a few open questions that have suggested themselves during the writing of this paper.

1. Is every finite-modular AC-lattice M-symmetric?

2. In [7], S. Maeda calls a lattice L a DAC-*lattice* in case both L and its dual are AC-lattices, and shows ([7], Theorem 2.1, p. 108) that every DAC-lattice is a finite-modular AC-lattice. Can every M-symmetric, finite-modular AC-lattice be embedded in a DAC-lattice?

3. Is Remark 4.13 valid for an arbitrary AC-lattice?

4. Is $F(L)$ a standard ideal for L an arbitrary AC-lattice? What if L is a matroid lattice?

5. In a finite-modular AC-lattice, by [6], Lemma 4, p. 168, $M^*(a, b)$ is equivalent to the assertion that p an atom, $p \leqslant a \vee b$ implies the existence of atoms $q \leqslant a$, $r \leqslant b$ such that $p \leqslant q \vee r$. In an arbitrary AC-lattice, what does it mean to say that $p \leqslant a \vee b$, p an atom, implies the existence of finite elements $a_1 \leqslant a$, $b_1 \leqslant b$ such that $p \leqslant a_1 \vee b_1$?

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a-adic completions of Noetherian lattice modules *

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§ 1. Introduction. Several years ago R. P. Dilworth [1] began a study of the ideal theory of commutative rings in an abstract setting. Since the investigation was to be purely ideal-theoretic, he chose to study a lattice with a commutative multiplication. Many of Dilworth's ideas have since been extended and several new concepts have been introduced ([2], [3]). In particular, E. W. Johnson [3] has introduced the notions of a Noetherian lattice module and a completion of a Noetherian lattice module. The purpose of this paper is to generalize the methods used in [3] and to extend some of the results. For undefined terms concerning Noetherian lattices, the reader is referred to [1] and [3].

The basic concepts are introduced in § 2. In § 3 the a-adic pseudometric is introduced. If M is an L-module, then, for each element a of L, a distance function, d_a, can be defined on M. This distance function d_a is called the a-*adic pseudometric* on M. Theorem 3.10 gives necessary and sufficient conditions for d_a to be a metric. Assuming that d_a is a metric, the set of all Cauchy sequences is divided into classes by an equivalence relation, and M^* is used to denote this set. The concepts of a regular Cauchy sequence and a completely regular Cauchy sequence are given in § 4. It is shown (Theorem 4.14) that each element of M^* has a unique completely regular representative. In § 5 the extension of elements from M to M^* is defined. For A in M, the extension of A to M^* is denoted by AM^*. A lattice structure is developed for M^* and in § 6 it is shown that M^* satisfies the ascending chain condition (Theorem 6.3) under the hypothesis that L is a Noetherian lattice and M is a Noetherian L-module. The a-adic completion of M is defined (Definition 6.5). A contraction of elements of M^* to M is introduced (Definition 7.1) in § 7. For A in M^*, its contraction to M is denoted by $A \cap M$. It is shown that $A = AM^* \cap M$ for all A in M (Proposition 7.2).

The remainder of the paper is concerned with the particular case where L is a local Noetherian lattice and M is a Noetherian L-module. In § 8 a connection between the different metrics on M^* is determined (Theorem 8.12 and Corollary 8.13). In § 9 p-adic completions of lattice

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