For $Y = L$, this map is 1_2. As the thing works componentwise, it suffices to look at the case $Y = A$. The function $t_A\colon X_1 \to X_2$ is $(g)\,U_P$ by definition.

We leave out the definition of Φ_P on morphisms of $(T)\mathcal{M}$ and, again, the remaining verifications that Φ_P^{-1} (or Φ_P) is one-one on morphisms and preserves their composition, and that the diagram (1) is commutative.

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The new interval topology on lattice products *

by

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Although the interval, ideal [3], and new interval [2] topologies all give the topology of the real line, only the latter two give the topology of the plane. That is, the ideal or new interval topology of the plane (ordered coordinate-wise) is equivalent to the product of the ideal or new interval topologies of the real line. It is reasonable to ask whether this property holds in the case of a finite product of chains or, more generally, a finite product of lattices.

Alo and Frink ([1], Theorem 2) proved that the ideal topology of a finite product of lattices is equivalent to the product of the ideal topologies of these lattices. However, for the new interval topology, they were forced to restrict the lattices to chains ([1], Theorem 9).

In this paper we show by means of a counterexample that the new interval topology on the finite product of lattices is not equal to the product of the new interval topology on the lattices. First, however, we prove two theorems that give conditions upon the individual lattices that insure the equivalence of the two topologies. These theorems, besides being of interest in themselves, give insight into the counterexample.

1. Definitions. The product order $\prod_{\alpha\in A} L_\alpha$ of an arbitrary number of lattices is defined coordinate-wise:

$$(a_\alpha) \leqslant (b_\alpha) \textit{ if and only if } a_\alpha \leqslant b_\alpha \textit{ for all } \alpha\,.$$

$\prod_{\alpha\in A} L_\alpha$ is a lattice under this order:

$$(a_\alpha)\vee(b_\alpha) = (a_\alpha\vee b_\alpha) \quad\text{and}\quad (a_\alpha)\wedge(b_\alpha) = (a_\alpha\wedge b_\alpha)$$

where $\vee$ $(\wedge)$ is the lattice supremum (infimum).

The interval topology is defined by taking as subbasic closed sets the closed rays $[a, +\infty) = \{x \mid x \geqslant a\}$ and $(-\infty, b] = \{x \mid x \leqslant b\}$. Clearly the intervals $[a, b] = \{x \mid a \leqslant x \leqslant b\}$ are closed in this topology.

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The new interval topology is derived by first defining the *closed bounded sets* as arbitrary intersections of finite unions of closed intervals $[a, b]$. Thus, a closed bounded set $B = \bigcap_{\alpha \in A} (\bigcup_{i=1}^{n_\alpha} [a_i, b_i])$ where n_α is finite for all α. Closed sets in the new interval topology are defined to be exactly those sets whose intersection with every closed bounded set is a closed bounded set. As Alo and Frink pointed out ([1], p. 1011), this is equivalent to requiring that a set be closed if and only if its intersection with every closed interval $[a, b]$ is a closed bounded set.

Because the order relation on a lattice is antisymmetric, the new interval topology is clearly T_1. Therefore, if $F \cap [a, b]$ is finite for all closed intervals $[a, b]$, F is closed in the new interval topology.

2. Theorems. To simplify the notation in the following theorems, they are proved for the product of two lattices. The generalization to a finite number is obvious.

Alo and Frink ([1], Theorem 8) have proved that if L is the direct product of any number of lattices L_j, then every set F of L which is closed in the cartesian product of the new interval topologies of the lattices L_j is also a closed set in the new interval topology of L. Therefore, to show that the two topologies are equivalent, it will suffice to show that every set closed in the new interval topology is closed in the product topology.

THEOREM 1. *Let the new interval topologies of both of the lattices L and M satisfy the first axiom of countability. Then, if F is closed in the new interval topology of $L \times M$, F is closed in the product of the new interval topologies of L and M.*

Proof. The product of the new interval topologies is first countable since the new interval topologies of L and M are first countable. Let (a, b) be in the closure of F with respect to the product topology. We will show that $(a, b) \in F$. By the first axiom of countability, there exists a sequence $\{(a_n, b_n)\}_{n=1}^{\infty} \subseteq F$ which converges in the product topology to (a, b). If the sequence has only a finite number of distinct elements, then we are done. Therefore we assume that the sequence has an infinite number of distinct elements. Let $A = \{a_n\}_{n=1}^{\infty}$ and let $B = \{b_n\}_{n=1}^{\infty}$. Either A or B has an infinite number of distinct elements. Assume that A does and that $a \notin A$ (if $a \in A$, use $A - \{a\}$). We now show that there exists $c, d \in L$ such that $A \cap [c, d]$ is infinite. If for all c, d, $[c, d] \cap A$ were finite, then A would be closed. Now $\{(a_n, b_n)\}$ converges to (a, b), which implies that $\{a_n\}_{n=1}^{\infty}$ converges to a. This last fact together with the fact that A is closed implies that $a \in A$. A contradiction. Therefore, there exists c, d such that infinitely many $a_n \in [c, d]$. Denote the subsequence of $\{(a_n, b_n)\} \subseteq [c, d] \times M$ by $\{(a_n', b_n')\}_{n=1}^{\infty} = 1$. Let $B' = \{b_n'\}_{n=1}^{\infty}$.

Case 1: B' has an infinite number of distinct elements. As before, we assume $b \notin B'$. By the same reasoning as above, there exists $e, f \in M$ such that infinitely many members of B' are contained in $[e, f]$. Let I denote the interval $[c, d] \times [e, f]$, and let $\{(a_n'', b_n'')\}_{n=1}^{\infty}$ denote the subsequence of $\{(a_n', b_n')\}$ which is contained in I. F was assumed to be closed in the new interval topology of $L \times M$, so $F \cap I = \bigcap_{\alpha \in A} (\bigcup_{i=1}^{n_\alpha} [p_i, q_i])$ where n_α is finite for all α. This implies that $\{(a_n'', b_n'')\}_{n=1}^{\infty} \subseteq F \cap I \subseteq \bigcup_{i=1}^{n_\alpha} [p_i, q_i]$ for all α. Since n_α is finite, there exists an $m \leqslant n_\alpha$ such that a subsequence of $\{(a_n'', b_n'')\}$ is contained in $[p_m, q_m]$. This subsequence converges to (a, b) in the product topology, and $[p_m, q_m]$ is closed in that topology (actually in both topologies); therefore $(a, b) \in [p_m, q_m] \subseteq \bigcup_{i=1}^{n_\alpha} [p_i, q_i]$. This is true for all α, hence $(a, b) \in \bigcap_{\alpha \in A} (\bigcup_{i=1}^{n_\alpha} [p_i, q_i]) = F \cap I \subseteq F$.

Case 2: B' has a finite number of distinct elements.

Let $e = \bigwedge \{b_n'\}$, $f = \bigvee \{b_n'\}$. Letting $(a_n'', b_n'') = (a_n', b_n')$, the same reasoning can be applied.

COROLLARY 2. *Let the new interval topologies of both of the lattices L and M satisfy the first axiom of countability. Then the product of the new interval topologies of L and M is equivalent to the new interval topology of $L \times M$.*

Proof. Theorem 1 and Alo and Frink [1], Theorem 8.

The following theorem and corollary are necessary for the proof of Theorem 5.

THEOREM 3. *Let L be a lattice with the new interval topology. Let $a, b \in L$ such that $a < b$. Then the relative topology on $[a, b]$ is equal to the new interval topology on $[a, b]$ which is in turn equal to the interval toplogy on $[a, b]$.*

Proof. Since $[a, b]$ is a lattice with a greatest and a least element, the new interval topology on $[a, b]$ is equal to the interval topology of $[a, b]$ by the first half of Theorem 1 of Birkhoff [2]. A subbasis for the closed sets of this topology is $\{[a, x], [x, b] \mid a \leqslant x \leqslant b\}$. Since $[a, x]$, $[x, b]$ are closed in L, the interval topology is clearly contained in the relative topology.

Let F be closed in L. $F \cap [a, b]$ is closed in the relative topology. By definition of the new interval topology,

$$F \cap [a, b] = \bigcap_{\alpha \in A} (\bigcup_{i=1}^{n_\alpha} [a_i, b_i]).$$

But

$$F \cap [a,b] = F \cap [a,b] \cap [a,b]$$
$$= [\bigcap_{\alpha\in A} (\bigcup_{i=1}^{n_\alpha} [a_i, b_i])] \cap [a,b] = \bigcap_{\alpha\in A} [(\bigcup_{i=1}^{n_\alpha} [a_i, b_i]) \cap [a,b]]$$
$$= \bigcap_{\alpha\in A} (\bigcup_{i=1}^{n_\alpha} [a_i, b_i] \cap [a,b]) = \bigcap_{\alpha\in A} (\bigcup_{i=1}^{n_\alpha} [a_i \vee a, b_i \wedge b]) .$$

The sets $[a_i \vee a, b_i \wedge b]$ are closed in the interval topology, and since arbitrary intersections of finite unions of closed sets are closed, $F \cap [a,b]$ is closed in the interval topology of $[a,b]$. Thus the relative topology on $[a,b]$ is contained in the interval topology.

COROLLARY 4. *Let L and M be lattices. Let $[a,b]$, $[c,d]$ be closed intervals in L, M, respectively. The relative topology on $[a,b]\times[c,d]$ as a subset of $L\times M$ where $L\times M$ has the new interval topology is equal to the product of the relative topologies on $[a,b]$ and $[c,d]$ where both L and M have the new interval topology.*

Proof. By Theorem 3, the relative topology on $[a,b]\times[c,d]$ is equal to the interval topology on $[a,b]\times[c,d]$. Alo and Frink ([1], Theorem 3) proved that the interval topology on $[a,b]\times[c,d]$ is equal to the product of the interval topologies of $[a,b]$ and $[c,d]$. Using Theorem 3 on $[a,b]$ and $[c,d]$ again, the proof is complete.

The following theorem implies Theorem 9 of Alo and Frink [1].

THEOREM 5. *Let the new interval topologies of the lattices L_1 and L_2 have the property that for all $x \in L_i$:*

(1) *there exists a, b such that $a \leqslant x \leqslant b$,*

(2) *there exists an open set G in the new interval topology on L_i such that $x \in G \subseteq [a,b]$.*

Then if F is closed in the new interval topology of $L_1\times L_2$, F is closed in the product of the new interval topologies of L_1 and L_2.

Proof. Let F be closed in the new interval topology of $L_1\times L_2$. Let $x = (x_1, x_2) \notin F$. By hypothesis, there exist a_i, b_i, $i = 1, 2$, such that $a_i \leqslant x_i \leqslant b_i$, and there exist open sets G_i in the new interval topology of L_i such that $x_i \in G_i \subseteq [a_i, b_i]$ for $i = 1, 2$. Let

$$J = [a_1, b_1]\times[a_2, b_2] = [(a_1, a_2), (b_1, b_2)] .$$

$F \cap J$ is closed in the relative topology of J. By Corollary 4, the relative topology of J is equal to the product of the relative topologies on $[a_1, b_1]$ and $[a_2, b_2]$, and so $F \cap J$ is closed in the product of the relative topologies. Now $x \in J-(F\cap J)$ implies that there exists an open set G in the product of the relative topologies such that $x \in G \subseteq J-(F\cap J)$. It may be assumed that G is a basic open set. $G = G_1'\times G_2'$. $G_i' = G_i'' \cap [a_i, b_i]$ where G_i''

is open in the new interval topology of L_i, and $x_i \in G_i''$ for $i = 1, 2$. Now $G_i' \cap G_i = G_i'' \cap G_i \cap [a_i, b_i] = G_i'' \cap G_i$ for $i = 1, 2$. Moreover, G_i'', G_i are both open in the new interval topology on L_i, which implies that $G_i' \cap G_i$ is open in the new interval topology on L_i for $i = 1, 2$. In addition $x_i \in G_i' \cap G_i$, $i = 1, 2$. Therefore $x = (x_1, x_2) \in (G_1' \cap G_1)\times(G_2' \cap G_2) \subseteq G_1' \times G_2' = G \subseteq J-(F\cap J)$. $(G_1' \cap G_1)\times(G_2' \cap G_2)$ is open in the product of the new interval topologies on L_1 and L_2. It is disjoint from F and contains x. As this is true for all $x \notin F$, the complement of F is open in the product topology, which finishes the proof.

COROLLARY 6. *Let the new interval topologies of the lattices L_1 and L_2 satisfy the conditions of Theorem 5. Then the new interval topology of $L_1\times L_2$ is equivalent to the product of the new interval topologies of L_1 and L_2.*

Proof. Theorem 5 and Alo and Frink [1], Theorem 8.

3. Counterexample. In the following example, the new interval topology of $L\times M$ does not equal the product of the new interval topologies of L and M.

Let A be the lattice of finite subsets of real numbers, including the empty set $\emptyset$, ordered by inclusion. Let $L = A^N$, where N is the set of natural numbers. L is a lattice with the product order. Let M be the bounded (not necessarily continuous) functions on $[0,1]$ where $f \leqslant g$ if and only if $f(x) \leqslant g(x)$ for all x.

By Theorem 1 we know that at least one of our lattices must not be first countable under its new interval topology. The proof of Lemma 2 in the following yields the information which gives an easy proof that the lattice M is not first countable.

Similarly, by Theorem 5, one of the lattices must have a point which has no bounded neighborhood. Lemma 2 also implies that no neighborhood of the function identically 0 in M is bounded.

Indeed, the lattice M was chosen precisely because it has these properties. The lattice L was chosen because it permitted the construction of a set F such that every bounded interval in $L\times M$ contained only a finite number of members of F.

Because the product topology is contained in the topology of the product ([1], Theorem 8), we must exhibit a set F which is closed in the new interval topology on the product but not in the product of the new interval topologies. To construct F, we need

$$B_n = \{f \in M \mid f(t) = 0 , \ t \neq t_1, t_2, \ldots, t_n;\ f(t_i) = n , \ i = 1, 2, \ldots, n\} .$$

A function in B_n will be denoted by f_n. Now, to $f_n \in B_n$ we associate a unique element $\Psi(f_n)$ in L as described below. Since the cardinality of the singletons in A and the cardinality of B_n are both equal to the continuum, each $f_n \in B_n$ can be associated with a unique singleton in A. Define $\Psi(f_n)$

to be the sequence which consists of the empty set except at the nth position and at the nth position it is the singleton associated with f_n. Let $F_n = \{(\Psi(f_n), f_n) \mid f_n \in B_n\}$. Let $F = \bigcup_{n=1}^{\infty} F_n$. Then

(1) *F is closed in the new interval topology.*

We show that $F \cap I$ is finite for any closed interval I. Let $I = [(a, f), (b, g)] = [a, b] \times [f, g]$. If $(\Psi(f_n), f_n) \in I$, then $\Psi(f_n) \in [a, b]$ and $f_n \in [f, g]$.

(a) As g is bounded, there exists a natural number m such that $g \leqslant \boldsymbol{m}$ (where $\boldsymbol{x}$ denotes the constant function whose value is x). This implies that $[f, g] \cap B_n = \emptyset$ for $n > m$. Thus $F \cap I \subseteq \bigcup_{n=1}^{m} F_n$.

(b) Recall that L is a countable product of A's. If $\Psi(f_i)$ is contained in $[a, b]$, then $[\Psi(f_i)]_i \in [a_i, b_i]$. But any given closed interval $[a_i, b_i]$ in A contains at most a finite number of singletons. Therefore there are only a finite number of $f_i \in B_i$ whose images $\Psi(f_i) \in [a, b]$. Thus for all i, $F_i \cap I$ is finite.

(c) By (a) $F \cap I \subseteq \bigcup_{n=1}^{m} F_n$, hence $F \cap I \subseteq \bigcup_{n=1}^{m} (F_n \cap I)$. By (b) each $F_n \cap I$ is a finite set, and therefore $F \cap I$ is a finite set.

(2) *F is not closed in the product topology.*

LEMMA 1. *If G is an open set in the new interval topology on L which contains $(\emptyset)$, the sequence which is identically the empty set, then G must contain $\Psi(B_n)$ for all but a finite number of n.*

Proof. We will show that if P is a closed set in the new interval topology on L such that for infinitely many n's $P \cap \Psi(B_n) \neq \emptyset$, then $(\emptyset) \in P$. Let P be as described. For each n such that $P \cap \Psi(B_n) \neq \emptyset$, pick exactly one element $g_n \in B_n$ such that $\Psi(g_n) \in P$. Let S be the set of all the $\Psi(g_n)$'s. Let $b = \bigvee S$; since in any one coordinate i only one $[\Psi(g_n)]_i \neq \emptyset$. b exists and is in L. Since P is closed, $P \cap [(\emptyset), b] = \bigcap_{\alpha \in A} (\bigcup_{i=1}^{n_\alpha} [a_i, c_i])$ where n_α is finite for all α. Therefore $P \cap [(\emptyset), b] \subseteq \bigcup_{i=1}^{n_\alpha} [a_i, c_i]$ for all α. By the definition of b, $S \subseteq [(\emptyset), b]$; moreover, $S \subseteq P$. Therefore $S \subseteq P \cap [(\emptyset), b] \subseteq \bigcup_{i=1}^{n_\alpha} [a_i, c_i]$. Because n_α is finite, there exists an $m \leqslant n_\alpha$ such that at least two (in fact an infinite number) of members of S are in $[a_m, c_m]$.

$$\Psi(g_k) \in [a_m, c_m] \quad \text{implies that} \quad (a_m)_p = \emptyset, \ p \neq k,$$
$$\Psi(f_l) \in [a_m, c_m] \quad \text{implies that} \quad (a_m)_p = \emptyset, \ p \neq l.$$

But $k \neq l$. Hence $(a_m) = (\emptyset)$. Therefore $(\emptyset) \in [a_m, c_m] \subseteq \bigcup_{i=1}^{n_\alpha} [a_i, c_i]$. This is true for all α, which implies that $(\emptyset) \in \bigcap_{\alpha \in A} (\bigcup_{i=1}^{n_\alpha} [a_i, c_i]) = P \cap [(\emptyset), b] \subseteq P$.

LEMMA 2. *If G is an open set in the new interval topology on M which contains the constant function $\boldsymbol{0}$, then $G \cap B_n \neq \emptyset$ for all n.*

Proof. We will show that a closed set in M, which contains all of B_n for some n must contain $\boldsymbol{0}$. Let Q be a closed set such that $B_q \subseteq Q$ for some fixed q. For $f \in B_q$, let $\tau(f) = \{t \mid f(t) = q\}$. There exists at least a countably infinite number of elements in B_q — call those elements f^j, $j = 1, 2, \ldots$ — such that $\tau(f^j) \cap \tau(f^i) = \emptyset$, $j \neq i$. Since Q is closed in the new interval topology on M, $Q \cap [\boldsymbol{0}, \boldsymbol{q}] = \bigcap_{\alpha \in A} (\bigcup_{i=1}^{n_\alpha} [h_i, k_i])$ where the n_α are finite for all α. Now $f^j \in [\boldsymbol{0}, \boldsymbol{q}]$ for all j. Therefore $\{f^j\}_{j=1}^{\infty} \subseteq Q \cap [\boldsymbol{0}, \boldsymbol{q}] \subseteq \bigcup_{i=1}^{n_\alpha} [h_i, k_i]$. As n_α is finite, there exists an $m \leqslant n_\alpha$ such that at least two (in fact infinitely many) of the f^j's are contained in $[h_m, k_m]$.

$$f^j \in [h_m, k_m] \text{ implies that } h_m(t) \leqslant 0 \text{ for } t \notin \tau(f^j),$$
$$f^i \in [h_m, k_m] \text{ implies that } h_m(t) \leqslant 0 \text{ for } t \notin \tau(f^i).$$

But $\tau(f^j) \cap \tau(f^i) = \emptyset$. Therefore $h_m \leqslant \boldsymbol{0}$. On the other hand, $f^j \in [h_m, k_m]$ implies that $\boldsymbol{0} \leqslant k_m$. Hence $\boldsymbol{0} \in [h_m, k_m]$. This is true for all α, thus $\boldsymbol{0} \in \bigcap_{\alpha \in A} (\bigcup_{i=1}^{n_\alpha} [h_i, k_i]) = Q \cap [\boldsymbol{0}, \boldsymbol{q}] \subseteq Q$.

Note. The proof of Lemma 2 shows that every open set containing $\boldsymbol{0}$ must contain all but a finite number of elements from B_1. Suppose there existed a countable base U_n, $n = 1, 2, \ldots$, for the open sets containing $\boldsymbol{0}$. Clearly

$$\bigcap_{\substack{U \text{ open} \\ 0 \in U}} U = \bigcap_{n=1}^{\infty} U_n, \quad \text{and} \quad \bigcap_{\substack{U \text{ open} \\ 0 \in U}} U = \{\boldsymbol{0}\},$$

since the space is T_1. Thus

$$B_1 = (M - \bigcap_{\substack{U \text{ open} \\ 0 \in U}} U) \cap B_1 = (M - \bigcap_{n=1}^{\infty} U_n) \cap B_1$$
$$= [\bigcup_{n=1}^{\infty} (M - U_n)] \cap B_1 = \bigcup_{n=1}^{\infty} (B_1 - U_n).$$

But $B_1 - U_n$ is finite for every n, and therefore the last union is countable. A contradiction.

PROPOSITION. *$((\emptyset), \boldsymbol{0})$ is not in F, yet it is in the closure of F in the product of the new interval topologies on L and M. This proves* (2).

Proof. Any set open in the product topology which contains $((\emptyset), 0)$ must contain a set of the form $G\times H$ where $(\emptyset)\in G$ and $0\in H$ and G and H are open in the new interval topologies on L and M respectively. By Lemma 1, G contains $\Psi(B_n)$ for all but a finite number of n. Suppose that G contains $\Psi(B_j)$. By Lemma 2, $B_j\cap H\neq\emptyset$. Let $f_j\in B_j\cap H$, then $\Psi(f_j)\in G$ and $(\Psi(f_j), f_j)\in G\times H$. Yet $(\Psi(f_j), f_j)\in F$. Hence $(G\times H)\cap F\neq\emptyset$. Every open set containing $((\emptyset), 0)$ must contain a member of F, which implies that $((\emptyset), 0)$ is in the closure of F.

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On the modular relation in atomistic lattices

by

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1. Introduction. S. Maeda [6] and [7] as well as R. Wille [9] have recently investigated various types of atomistic lattices. Basically, Wille was concerned with upper continuous atomistic lattices equipped with some type of closure operator, while Maeda investigated modular and dual modular pairs in atomistic lattices. Our goal here is to extend and to some degree attempt to unify these two theories.

In an effort to make the paper fairly self-contained, we introduce our basic terminology and prove a few preliminary theorems in § 2. In § 3 we introduce the concept of a *finite-statisch* lattice and extend Wille's theory [9] to this class of lattice. In § 4 we discuss modularity in atomistic lattices, and relate the work of S. Maeda to that of Wille. Finally, in § 5 we list a few open questions.

2. Basic terminology. As much as possible our terminology and notation will follow that of Wille [9]. A noteable exception, however, is that rather than using Wille's symbolism, we will use the symbols $\cup$ and $\cap$ to denote set union and set intersection.

DEFINITION 2.1. A lattice L with 0 is called *atomistic* if every element of L is the join of a family of atoms.

DEFINITION 2.2. A non-empty subset T of a lattice L is called *increasing* (see [9], Definition 1.3, p. 5) if $x, y\in T$ implies the existence of an element z of T such that $x\vee y\leqslant z$. In symbols, the notation $x_\alpha\uparrow x$ will denote the fact that $\{x_\alpha\}$ is an increasing subset with join x. If $x_\alpha\uparrow x$ and $x_\alpha\wedge y\uparrow x\wedge y$ for all $y\in L$, then $\{x_\alpha\}$ is called a *continuous increasing subset of L*.

In a lattice with atoms, let ax denote the set of atoms dominated by x. The next lemma then provides a useful characterization of continuous increasing subsets of an atomistic lattice.

LEMMA 2.3. *Let $x_\beta\uparrow x$ in an atomistic lattice L. Then $\{x_\beta\}$ is continuous if and only if $ax = \bigcup_\beta (ax_\beta)$.*

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