2.5. *The Kaplansky space $\varkappa(G)$ of an abelian l-group G need not be a T_0 space.*

Do 2.4 on two halves of a circle.

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Lawvere's elementary theories and polyadic and cylindric algebras

by

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*À la mémoire de Léon Leblanc
mon regretté ami et collègue*

Contents

Introduction *

In his short papers [8], [9], [10] and some talks, Lawvere has presented a new approach to the problem of the algebraization of first order Logic in which elementary theories become categories. It is the purpose of this paper to describe the exact relationship that the new approach bears to the older one constituted by the theory of polyadic and cylindric algebras. We hope thus to call attention to Lawvere's important contribution to Algebraic Logic. (Throughout the paper, we shall mean by "polyadic algebra", locally finite polyadic algebra with equality and a fixed infinite set of variables. "Cylindric algebra" has a similarly restricted meaning).

(*) This paper is an amended version of a paper read at the conference on the Construction of Models for Axiomatic Systems in Warsaw, August 26–September 1, 1968.

We express ourselves in categorical terms. Two main facts are established.

Fact 1 (Theorem 3.1) states that the category $\mathcal{C}$ of elementary theories and the category $\mathcal{P}$ whose objects are pairs (P, D) where P is a polyadic algebra and D a set of operations of P which is admissible in a certain sense (see end of Section 0) are equivalent provided an axiom (see L4 before Theorem 0.5) is added to the definition of theory to eliminate the repetitions among morphisms $A^n \to A$. It follows (Corollary 3.2) that the category of polyadic algebras is equivalent to the full subcategory of $\mathcal{C}$ whose objects are the theories in which, moreover, every definable operation is present.

Fact 2 asserts that the semantics functors which assign to an elementary theory its category of models and to a pair (P, D), the category of 2-valued representations of P are equivalent (modulo the equivalence of $\mathcal{P}$ and $\mathcal{C}$). This can perhaps in part be infered indirectly from Lawvere's statements in [8] and from the fact that polyadic and cylindric algebras are known (see, for instance, Henkin–Tarski [7] and Daigneault [1], [3]) to be equivalent to ordinary elementary theories. We give here though, a direct and complete account of these equivalences except for leaving out a rather large number of rather straightforward verifications.

Except for a Master's thesis [4] now being written by Donais, a student of ours, Lawvere's work on this subject has not yet, to our knowledge, received a full exposition. Hence, in the first part of Section 0, we find it necessary to recall the definitions and results of Lawvere in somewhat greater detail than in [8], [9], [10], but we leave out the proofs. The second part of Section 0 recalls the definition of a cylindric algebra and discusses briefly in polyadic terms the question of propositional variables and their quantifications which plays an essential role in Lawvere's approach. Fact 1 is established in Sections I, II and III. Functors θ^{-1}: $\mathcal{P} \to \mathcal{C}$ and θ: $\mathcal{C} \to \mathcal{P}$ are described in Sections I and II respectively and the fact that they are reciprocal equivalences is proved in Section III. Finally Fact 2 is discussed in the closing Section IV. [1]

0. Preliminaries

Except when otherwise indicated we shall write function symbols on the right of their arguments. Accordingly the product of functions or of morphisms will follow the so called diagrammatic notation by which a product such as $\varphi\psi$ means intentionally: first φ then ψ.

(1) We are thankful to Dana Schlomiuk who raised a question which led to the discovery of a mistake in a previous version of this paper where it was wrongly asserted that the category $\mathcal{C}$ is equivalent to that of polyadic algebras.

Elementary theories. We recall the (simplified) definition of "elementary theory" as given by Lawvere in [9]. Such a theory is essentially a small category T verifying certain conditions. More precisely it is a 5-tuple (T, A, L, V, F) where T is a category, A, L are distinguished objects of T and V, F are distinguished morphisms of T such that the conditions (L0) up to (L3), to be defined below, are satisfied. Here is the first condition:

(L0) Finite categorial products exist in T. In particular, the product of an empty set of objects exists and is noted 1; it has the property that for any $X \in |T|$ (the set of objects of T) there exists one and only one morphism $X \to 1$.

If X and Y are objects in T and $X \times Y$ is their product we shall denote by $\mathrm{pr}(X \times Y \to X)$, and sometimes by p_X, the projection of $X \times Y$ on X. Strictly speaking, the categorical product consists of the object $X \times Y$ together with p_X and p_Y. A special notation will be used for the ith projection $A^n \to A$ which we shall call v_{in} ($i \leqslant n$). The choice of the letter v is in view of the fact that these projections will be related to the individual variables $v_1, v_2, v_3, \ldots$ If φ_i: $Y \to X_i$ $(i = 1, 2)$ are two morphisms (in T) we denote by $\langle \varphi_1, \varphi_2 \rangle$ the unique morphism $Y \to X_1 \times X_2$, whose existence is asserted by the definition of product, such that $\varphi_i = \langle \varphi_1, \varphi_2 \rangle p_{X_i}$. We shall use the following notations

$$\check{v}_{in} = \langle v_{1n}, \ldots, v_{in} \rangle ,$$

$$\hat{v}_{in} = \langle v_{1n}, \ldots, v_{i-1,n} v_{i+1,n}, \ldots, v_{nn} \rangle .$$

The first of these retains the first i "components" whereas the second one forgets the ith component.

The next condition is:

(L1) V and F are morphisms $1 \to L$ such that, if we define, for each $X \in |T|$, $V_X = X \to 1 \overset{V}{\to} L$ and $F_X = X \to 1 \overset{F}{\to} L$, the maps $\langle 1_X, V_X \rangle$ and $\langle 1_X, F_X \rangle$ from X to $X \times L$ constitute the (categorical) coproduct of two "copies of X".

In short this means that $X \times L = X + X$. More exactly it means that, if φ_i: $X \to Y$ $(i = 1, 2)$ are two morphisms, there exists a unique morphism ψ: $X \times L \to Y$ such that $\langle 1_X, V_X \rangle \psi = \varphi_1$ and $\langle 1_X, F_X \rangle \psi = \varphi_2$.

The equation $X \times L = X + X$ generalizes to an equation $X \times L^k = 2^k \cdot X$ where the right member is the coproduct of 2^k "copies of X", k being a positive integer. To make this statement precise we introduce notations which we will retain throughout. Let $\{e_{ik} | \ 1 \leqslant i \leqslant 2^k\}$ be a fixed enumeration of 2^k, the set of all functions from the set $[1, k]$ of all integers j such that $1 \leqslant j \leqslant k$, to $2 = \{0, 1\}$. For each such function e and object X of T we define a function

$$e_X\colon\ [1, k] \to \{V_X, F_X\}$$

such that, for each $j \in [1, k]$, $(j)e_X = V_X$ if $(j)e = 1$, and $(j)e_X = F_X$ if $(j)e = 0$. Hence for each $e \in 2^k$ and $X \in |T|$, we have a morphism

$$e_X^{\#} = \langle (1)e_X, \ldots, (k)e_X \rangle\colon\ X \to L^k .$$

The precise meaning of the equation $X \times L^k = 2^k \cdot X$ is the following statement which is proved by induction on k and which is equivalent to (L1):

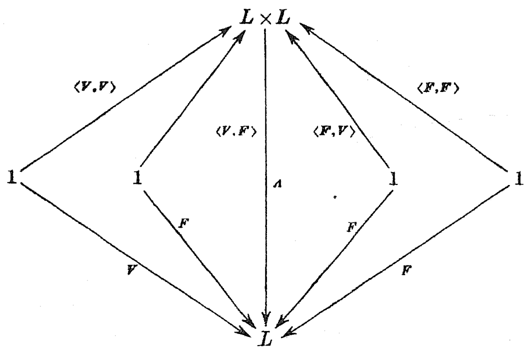
THEOREM 0.1. *The family of* 2^k *morphisms*

$$\{\langle 1_X, e_X^{\#} \rangle\colon\ X \to X \times L^k |\ e \in 2^k\}$$

is a coproduct.

(Caution: 2^k is used, depending on the context, to denote a set of functions and a natural number).

If $X = 1$, this means that $\{\langle (1)e_1, \ldots, (k)e_1 \rangle\colon 1 \to L^k |\ e \in 2^k\}$ is a coproduct. In particular, if $k = 2$, we can define, for instance, a morphism $\Lambda\colon L^2 \to L$ as the unique map such that the following diagram is commutative:



This corresponds to the usual truth-table for the conjonction Λ.

If $\mathcal{C}$ is a category and $X, Y \in |\mathcal{C}|$ the class of objects of $\mathcal{C}$, we shall denote by $\mathcal{C}(X, Y)$ the class of morphisms $X \to Y$ in $\mathcal{C}$. If X is an object of the theory, $T(X, L)$ becomes a Boolean algebra if, for instance, one defines the intersection Λ_X by the equation

$$\varphi_1 \Lambda_X \varphi_2 = \langle \varphi_1, \varphi_2 \rangle \Lambda$$

where, for $i = 1, 2$, $\varphi_i\colon X \to L$ and the right member is a composite of morphisms in $\mathcal{T}$. This assignment of Boolean algebra structures to the $T(X, L)$ is functorial in the sense that it makes the cofunctor $T(-, L)\colon \mathcal{T} \to \mathrm{Ens}$, the category of sets, which assigns the set $T(X, L)$ to X and

multiplication of the elements of $T(Y, L)$ on the left by φ to $\varphi\colon X \to Y$, factor through the category $\mathcal{B}$ of Boolean algebras. This means that

THEOREM 0.2. *If* $\varphi\colon X \to Y$, *the map* $\varphi^{\perp}\colon T(Y, L) \to T(X, L)$ *which assigns* $\varphi\psi$ *to* $\psi\colon Y \to L$ *is a homomorphism of Boolean algebras.*

This follows simply from the equation $\varphi \langle \psi_1, \psi_2 \rangle = \langle \varphi\psi_1, \varphi\psi_2 \rangle$.

We can now state the next axiom:

(L2) For each $\chi\colon X \to Y$ and $\varphi\colon X \to L$ (in T) there exists $\exists_\chi[\varphi]\colon Y \to L$ such that for all $\psi\colon Y \to L$

$$\exists_\chi[\varphi] \vdash_Y \psi \quad \text{iff} \quad \varphi \vdash_X \chi\psi$$

where $\vdash_X$ is the order relation of $T(X, L)$ and similarly for Y.

It can be shown that there exists at most one such function $\exists$.

Finally here comes the last axiom:

(L3) Every object of $\mathcal{T}$ has a unique representation in the form $A^n \times L^k$ where n and k are finite non negative integers.

Perhaps the reason for stating (L3) last is that the important category Ens verifies all axioms (except being small!) and (L3). There $L = 2$, A is any non-void set, 1 is a singleton, V maps 1 on the element 1 of 2 and F maps it on 0. Moreover, if a map with 2 as its codomain is identified with the subset of its domain of which it is the characteristic function, $\exists_\chi[\varphi]$ is simply the image of φ under χ i.e. an element $y \in Y$ is in $\exists_\chi[\varphi]$ iff *there exists* $x \in \varphi$ such that $y = (x)\chi$.

Let X, Y and k be fixed. Each $\psi\colon X \times L^k \to Y$ determines a sequence $(\psi)\delta = \big((\psi)\delta_1, \ldots, (\psi)\delta_{2k}\big)$ where

$$(\psi)\delta_i = \langle 1_X, e_{ikX}^{\#} \rangle \psi\colon\ X \to Y .$$

If $e = e_{ik}$, we shall sometimes write δ_e instead of δ_i. Conversely, by (L1), to each sequence $\varphi = (\varphi_1, \ldots, \varphi_{2k})$ of maps $X \to Y$, there corresponds a map $(\varphi)\varepsilon\colon X \times L^k \to Y$ such that $(\psi)\delta_i = \varphi_i$ for all i $(1 \leqslant i \leqslant 2^k)$. If $Y = L$, this $(\varphi)\varepsilon$ can be expressed in terms of φ by an expression reminiscent of the disjunctive normal form of a propositional formula. For this we introduce some notations. If p is an element of a Boolean algebra we agree to write $p^1 = p$ and $p^0 = p'$. For $i = 1, \ldots, k$ we let p_i be the ith projection $X \times L^k \to L$ and for $e \in 2^{[1,k]}$ the basic conjunction corresponding to e is

$$\mathrm{Conj}(e) = p_1^{(1)e} \Lambda \ldots \Lambda p_k^{(k)e} .$$

(Eventually the p_j may denote other "propositional variables".)

THEOREM 0.3.

$$(1) \qquad (\varphi)\varepsilon = \bigvee \{\mathrm{Conj}(e_{ik}) \Lambda p_X \varphi_i |\ 1 \leqslant i \leqslant 2^k\}$$

where $p_X = \mathrm{pr}(X \times L^k \to X)$.

Proof. $(\varphi)\varepsilon$ is the unique $\psi\colon X\times L^k\to L$ such that $(\psi)\delta_r=\varphi_r$, $r=1,\ldots,2^k$. Therefore it suffices to check that the right member of (1) has this property i.e.

$$\langle 1_X, e^{\#}_{rkX}\rangle \bigvee \{\mathrm{Conj}(e_{ik})\Lambda p_X\varphi_i|\ 1\leqslant i\leqslant 2^k\} = \varphi_r\,.$$

This follows immediately upon remembering that multiplication on the left preserves Boolean operations since one easily computes that

$$\begin{aligned}\langle 1_X, e^{\#}_{rkX}\rangle p_j^{e_{ik}(j)} &= V_X \quad \text{if } e_{rk}(j)=e_{ik}(j)\\ &= F_X \quad \text{if } e_{rk}(j)\neq e_{ik}(j)\end{aligned}$$

and

$$\langle 1_X, e^{\#}_{rkX}\rangle p_X = 1_X\,. \qquad \text{Q.E.D.}$$

There is no disadvantage in assuming here that all theories have the same objects, the same A, and the same L, and we shall generally do this. A morphism of theories $W\colon T\to T'$ is a product preserving functor (hence is identity on objects and (v_{in}) $W=v'_{in}$ such that (V) $W=V'$, (F) $W=F'$ and which preserves quantification i.e.

$$(\exists_\chi[\varphi])W = \exists_{\chi W}[\varphi W]$$

where the notation is self explanatory. It follows in particular that, for $X\in|T|$, the restriction of W to $T(X,L)$ is a Boolean homomorphism into $T'(X,L)$. The elementary theories together with their morphisms form a category $\mathcal{C}$.

A model of a theory T is a product preserving functor $M\colon T\to \mathrm{Ens}$ such that $(A)M$ is a non empty set, $(L)M=2$, and if $\varphi\colon X\to L$ and $\chi\colon X\to Y$, $(\exists_\chi[\varphi])M$ is the characteristic function of the subset of $(Y)M$ which is the image under $(\chi)M$ of the subset of $(X)M$ of which $(\varphi)M$ is the characteristic function. [No harm is done but not much is gained by requiring further that $(1)M$ be the $1\in 2$ and that $(V)M$ be the function ("true") which maps 1 onto 1 and $(F)M$ the function ("false") which maps 1 onto 0. This is just standardisation of notations]. It follows that M preserves the coproducts $(\langle 1_X, V_X\rangle, \langle 1_X, F_X\rangle)$. This is true because of the requirement $(L)M=2$ and the validity of the equation $X\times 2=X+X$ in Ens.

The adequacy of any kind of axiomatisation of first order Logic must be established by some "completeness theorem". Lawvere has outlined his in [9]:

THEOREM 0.4. *For any theory T and $\varphi\colon 1\to L$ in T such that $\varphi\neq F$ there exists a model M_φ of T such that $(\varphi)M_\varphi=$ "true".*

For expediency's sake we shall use this theorem in our proof of the equivalence of $\mathcal{C}$ with the category $\mathcal{T}$ in Sections I–III. However, if its use were dispensed with there, the work of Section IV would reduce

Theorem 0.4 to the representation theorem for polyadic algebras and ultimately to the ordinary completeness theorem for first-order Logic.

The category $\mathcal{C}$ has arbitrary products. If $\{T_k|\ k\in K\}$ is a family of theories, say, all with the same objects, their product is a theory T with the same objects and such that for any X and Y, $T(X,Y)$ is the ordinary cartesian product $\prod_k T_k(X,Y)$ of sets. Composition of morphisms is done componentwise. A morphism of T is noted $\Pi\varphi_k$ and the projection $T\to T_k$ maps $\Pi\varphi_k$ onto φ_k. The completeness theorem can be reformulated as Theorem 0.5 below provided we add the axiom:

(L4) For all $n\geqslant 0$ and $f,g\colon A^n\to A$, if $f\neq g$ then $\langle f,g\rangle\exists_d V_a\neq V_{A^n}$ where $d=\langle 1_A, 1_A\rangle$.

This means that if f and g have the same interpretation in every model of T then $f=g$.

THEOREM 0.5. *A theory admits an imbedding into a direct product of theories each of which admits a faithful model.*

Polyadic and cylindric algebras. For accounts of the elements of the theory of polyadic and cylindric algebras one may consult Halmos [6], Henkin and Tarski [7] and Daigneault [3]. According to Galler [5] the concepts of polyadic and cylindric algebras in the restricted sense in which we understand them are equivalent and we shall use indifferently either term. We shall assume known the elements of the theory including such notions as those of constant, term, operation, predicate and equality. Some of our unpublished work with Léon LeBlanc simplifies substantially the treatment of Halmos [6] in what regard these notions. We shall be content here with recalling the definition of quantifier and of cylindric algebra. We assume that the set of (individual) variables $I=\{v_1,v_2,v_3,\ldots\}$ is denumerably infinite. A quantifier on a Boolean algebra B is a map $\exists$ of B into itself such that

(Q_1) $\exists(0)=0$,

(Q_2) $p\leqslant \exists(p)$,

(Q_3) $\exists(p\Lambda\exists q)=\exists(p)\Lambda\exists(q)$,

for all $p,q\in B$. A cylindric algebra is a Boolean algebra P together with, for each variable v_i, a quantifier $(\exists v_i)$ on P, and, for each pair (v_i,v_j) of variables, a distinguished element $E(v_i,v_j)$ of P such that

(C1) $(\exists v_i)(\exists v_j)=(\exists v_j)(\exists v_i)$,

(C2) $E(v_i,v_i)=1$,

(C3) $E(v_i,v_j)=(\exists v_k)[E(v_i,v_k)\Lambda E(v_j,v_k)]$,

(C4) $(\exists v_i)[p\Lambda E(v_i,v_k)]\Lambda(\exists v_i)[p'\Lambda E(v_i,v_k)]=0$

if $k \notin \{i, j\}$. The "diagonal elements" $E(v_i, v_j)$ constitute the "equality" of the algebra. The algebra P is locally finite if for all $p \in P$ there are only finitely many i such that $(\exists v_i)(p) \neq p$. A polyadic homomorphism $f: P_1 \to P_2$ is assumed to preserve equality. Polyadic algebras and homomorphisms form a category $\mathfrak{T}'$.

If X is a non empty set, the Boolean algebras of all functions $X^I \to 2$ depending on only finitely many elements of I becomes a polyadic algebra C_X with the quantifiers and equality defined by the equations:

$$[(\exists v_i)q](x) = \bigvee \{q(y) \mid y_j = x_j \quad \text{if} \quad j \neq i\}$$

and

$$[\mathrm{E}(v_i, v_j)](x) = 1 \quad \text{iff} \quad x_i = x_j \quad \text{where } x = (x_i) \in X^I .$$

The representation theorem for polyadic algebras, which takes lieu of "completeness theorem" for this approach to first order Logic is the following

THEOREM 0.6. *For any polyadic algebra P and any $q \in P$ such that $q \neq 0$, there exists a non empty set X and a 2-valued representation $f: P \to C_X$ such that $(q)f = 1$.*

If χ is an element of P with support $\{u_1, \ldots, u_n\}$, we shall write

$$\chi = \chi(u_1, \ldots, u_n) = \chi(u) .$$

This notation will tacitly be generalized to the case of terms and to cases where there are "propositional variables", in a sense to be presently defined, in addition to individual variables. Moreover we will sometimes write

$$\chi(u'_1, \ldots, u'_n) \quad \text{for} \quad S(u'_1/u_1, \ldots, u'_n/u_n)\chi .$$

An extension of a polyadic algebra P by k "propositional variables" is an extension $P_1 = P(p_1, \ldots, p_k)$ generated by P together with k closed elements $p_1, \ldots, p_k$ in such a way that for any polyadic homomorphism $f: P \to P_2$ and any map $e: \{p_1, \ldots, p_k\} \to P_2$ there exists a (necessarily unique) Boolean homomorphism $g: P_1 \to P_2$ such that the restriction of g to P is f and $(p_i)g = (p_i)e$ for $i = 1, \ldots, k$.

THEOREM 0.7. *For any P and k there exists an extension by k propositional variables $p_1, \ldots, p_k$ uniquely determined to within equivalence. Its generic element can be written uniquely as*

$$(*) \qquad \bigvee \{\mathrm{Conj}(e_{ik}) \Lambda q_i \mid 1 \leqslant i \leqslant 2^k\}$$

where $q_i \in P$. In order for an extension $P(p_1, \ldots, p_k)$ of P by adjunction of k closed elements to be an extension by the k propositional variables $p_1, \ldots, p_k$ it suffices (and is, of course, necessary) that for $i = 1, \ldots, 2^k$, there exists a Boolean homomorphism $g: P(p_1, \ldots, p_k) \to P$ which is identity on P and such that for $j = 1, \ldots, k$, $(p_j)g = (j)e_{ik}$.

Proof. Let Σ_k be a Boolean algebra freely generated by $p_1, \ldots, p_k$. It can be considered as a polyadic algebra (with equality) with trivial polyadic structure. It is easy to verify that the tensor product $P \otimes \Sigma_k$ as defined in Daigneault [2] has the required property. As a Boolean algebra, $P \otimes \Sigma_k$ is the free product of P and Σ_k. On the other hand it is easy to show that any extension of P by closed elements p_i which verifies the condition termed sufficient in the statement of the theorem must admit a unique representation $(*)$ for its generic element. This fact serves to establish the equivalence of any two such extensions. Q.E.D.

We shall retain throughout the notations Σ_k and $P \otimes \Sigma_k$ and shall denote by $S(\varphi_1/p_1, \ldots, \varphi_k/p_k)$ the homomorphism $P \otimes \Sigma_k \to P$ which is identity on P and maps p_i onto $\varphi_i \in P \otimes \Sigma_k$. Although not polyadic (unless the φ_j are closed), this operator commutes with $(\exists v_i)$ if the φ_j are independent of v_i, and therefore, it can be applied to operations of $P \otimes \Sigma_k$ through their reduction to monovalent predicates.

We will denote by Σ_∞ the union of the Σ_k and by $P \otimes \Sigma_\infty$ that of the $P \otimes \Sigma_k \cdot \Sigma_\infty$ is freely generated by $\{p_1, p_2, \ldots\}$.

If $q \in P \otimes \Sigma_k$, $t_1, \ldots, t_n$ are terms of $P \otimes \Sigma_k$ and $q_1, \ldots, q_k$ elements of $P \otimes \Sigma_k$, the result $S(t_1/v_1, \ldots, q_k/p_k)q$ of the simultaneous substitution of the t_i for the v_i and the q_j for the p_j in q is defined as

$$S(t_1/u_1, \ldots, t_n/u_n) S(q_1/p_1, \ldots, q_k/p_k) S(u_1/v_1, \ldots, u_n/v_n) q$$

where $u_1, \ldots, u_n$ are distinct (individual) variables not in a common support of the t_i, the q_j and q. The operator can be applied to terms of $P \otimes \Sigma_k$ also.

If $f: P_1 \to P_2$ is a polyadic homomorphism, it extends uniquely to a homomorphism $P_1 \otimes \Sigma_\infty \to P_2 \otimes \Sigma_\infty$ still denoted by f and which leaves the p_i fixed.

Let P be a polyadic algebra, k a positive integer and consider an extension $P \otimes \Sigma_k$ of P by k propositional variables $p_1, \ldots, p_k$. Each $\psi \in P \otimes \Sigma_k$ determines a sequence $(\psi)\delta = \big((\psi)\delta_1, \ldots, (\psi)\delta_2k\big)$ of elements of P where

$$(\psi)\delta_i = S\big((1)e_{ik}/p_1, \ldots, (k)e_{ik}/p_k\big)\psi .$$

Conversely each sequence $\varphi = (\varphi_1, \ldots, \varphi_2k)$ of elements of P determines a unique element $(\varphi)\varepsilon$ of $P \otimes \Sigma_k$ such that $(\varphi)\varepsilon\delta = \varphi$. We have

$$(2) \qquad (\varphi)\varepsilon = \bigvee \{\mathrm{Conj}(e_{ik}) \Lambda \varphi_i \mid 1 \leqslant i \leqslant 2^k\} .$$

That the right member of this equation has the property required of $(\varphi)\varepsilon$ can easily be checked directly. The uniqueness comes from the fact that any element of $P \otimes \Sigma_k$ can be written uniqnely in the form of the right member of (2) from which it is seen that if $\psi \neq 0$ then $(\psi)\delta_i \neq 0$ for at least one i. A similar (ε, δ)-correspondance can be established between n-operations of $P \otimes \Sigma_k$ and 2^k-sequences of n-operations of P using the corresponding monovalent $(n+1)$-predicates.

We will wish to quantify over propositional variables. The basic Boolean fact is the following.

THEOREM 0.8. *If P is a Boolean algebra, $P \otimes \Sigma_\infty$ is the free product (i.e. the coproduct) of P and Σ_∞ and H is a subset of $\{p_1, p_2, p_3, \ldots\}$ then a quantifier $(\exists H)$ is defined on $P \otimes \Sigma_\infty$ by the equation*

$$(\exists H)(\psi) = \bigvee \{S(\alpha)\psi \mid \alpha \in 2^H\}$$

where $S(\alpha)$ is the endomorphism of $P \otimes \Sigma_\infty$ into itself which extends the map into $P \otimes \Sigma_\infty$ which sends $p_i \in H$ onto $(p_i)\alpha$ and $p_i \notin H$ onto itself and which is identity on P.

Note that the set to which the supremum sign applies, is finite since ψ depends on only finitely many elements p_i. In case $H = \{p_1, \ldots, p_k\}$ the formula is

$$(\exists p_1, \ldots, p_k)(\psi) = \bigvee \{(\psi)\delta_i \mid 1 \leqslant i \leqslant 2^k\}.$$

In case P is a polyadic algebra, propositional quantification commutes with individual quantification and with the canonical extension to $P_1 \otimes \Sigma_\infty$ of a polyadic homomorphism $P_1 \to P_2$.

In any polyadic algebra P, to every n-term t_n there corresponds in an obvious way an m-term t_m for every $m > n$. Although closely related, t_n and t_m must be distinguished. The correspondance between t_n and t_m is the same as the one between a function of n variables and the same function considered as a function of m variables in which the $m-n$ last variables play a silent role. Every variable v_i can be considered as an n-term for $n \geqslant i$ which is denoted simply by v_i or, if necessary, by (v_i, n). A set D of n-terms of a polyadic algebra p is said to be *admissible* if

(i) if D contains an n-term, it contains the corresponding m-term for each $m > n$;

(ii) every variable is in D;

(iii) for every $k \geqslant 1$ and $m, n \geqslant 0$, every sequence $\varphi = (\varphi_1, \ldots, \varphi_{2^k})$ of n-terms in D, every sequence $(t_1, \ldots, t_n)$ of m-terms in D, every sequence $(q_1, \ldots, q_k)$ of elements of support $\{v_1, \ldots, v_m\}$ in P, the m-term of P

$$S(t_1/v_1, \ldots, t_n/v_n;\ q_1/p_1, \ldots, q_k/p_k)(\varphi\varepsilon) \text{ is in } D.$$

The set $(P)\,\mathrm{Max}$ of all n-terms of P is admissible. Every set of n-terms of P generates an admissible set of n-terms. The set $(P)\,\mathrm{Min}$ generated by the void set of n-terms or, equivalently, by the set of individual variables, will as a rule be different from $(P)\,\mathrm{Max}$. In the case of the algebra C_X, the n-ary operations Q on X corresponding to the n-terms in $(C_X)\,\mathrm{Min}$ have the property that $(x_1, \ldots, x_n)Q$ where $(x_1, \ldots, x_n) \in X^n$ is one of the x_i. This can be verified by "induction" on $(C_X)\,\mathrm{Min}$. As some operations on X fail to have this property as soon as X has more than one element, we have, under that hypothesis, $(C_X)\,\mathrm{Min} \nsubseteq (C_X)\,\mathrm{Max}$. Also, if D is admissible, the subset of D made up of the elements of D which are not constants is admissible.

We consider the category $\mathfrak{I}$ whose objects are pairs (P, D) where P is a polyadic algebra and D is an admissibles et of operations of P. A morphism $f\colon (P, D) \to (P', D')$ of such pairs is a polyadic homomorphism (preserving equality) such that for any $Q \in D$, the image Qf of Q under f is in D'. In Theorem 3.1 the categories $\mathfrak{I}$ and $\mathfrak{T}$ will be shown to be equivalent.

I. The functor $\theta^{-1}\colon \mathfrak{I} \to \mathfrak{T}$

We first let (P, D) be an object of $\mathfrak{I}$ and associate to it an elementary theory $T = (P, D)\theta^{-1}$. The objects of T are the "words" or formal expressions A^nL^k where n and k range over the non negative integers. The word A^0L^0 is also noted 1. The elements of $T(A^nL^k, L)$ are the elements q of $P \otimes \Sigma_k$ supported by $\{v_i \mid i \leqslant n\}$. More exactly they are the triples (q, n, k). The elements of $T(A^nL^k, A)$ are the n-terms (i.e. the $\{v_1, \ldots, v_n\}$-terms) of $P \otimes \Sigma_k$ of the form $(\varphi_1, \ldots, \varphi_{2^k})\varepsilon$ where $\varphi_i \in D$ for $i = 1, \ldots, 2^k$. More exactly again, they are the triples (φ, n, k). For any objects $X = A^nL^k$ and $Y = A^mL^h$, $T(X, Y)$ consists of the sequences

$$\varphi = (\varphi_1, \ldots, \varphi_m;\ \varphi^{(1)}, \ldots, \varphi^{(h)}) = (\varphi_i;\ \varphi^{(j)})$$

where $\varphi_i \in T(X, A)$ and $\varphi^{(j)} \in T(X, L)$. In particular $T(X, 1)$ consists of the single (empty and unnamed) map $X \to 1$. $T(1, A)$ is the set of constants of P in D and $T(1, L)$ is its set of closed elements. Composition of maps in T is defined as follows. Let $\psi\colon Y \to Z$ where $Z = A^sL^r$ and

$$\psi = (\psi_1, \ldots, \psi_s;\ \psi^{(1)}, \ldots, \psi^{(r)}).$$

Then $\varphi\psi = \chi = (\chi_1, \ldots, \chi_s;\ \chi^{(1)}, \ldots, \chi^{(r)})$.

$$\chi_i = S(\varphi_1/v_1, \ldots, \varphi_m/v_m;\ \varphi^{(1)}/p_1, \ldots, \varphi^{(h)}/p_h)\psi_i$$

and similarly for $\chi^{(i)}$ and $\psi^{(i)}$ in place of χ_i and ψ_i.

We have $\chi_i \in T(X, L)$ i.e. $(\chi_i)\delta_j \in D$ because D is admissible and the fact that $(\chi_i)\delta_j$ is

$$S(\varphi_1\delta_j/v_1, \ldots, \varphi_m\delta_j/v_m;\ \varphi^{(1)}\delta_j/p_1, \ldots, \varphi^{(h)}\delta_j/p_h)\psi_i$$

and $\psi_i = (\psi_i\delta_1, \ldots, \psi_i\delta_{2^k})\varepsilon$ with $\psi_i\delta_l \in D$ for all i, j, l.

We now verify Lawvere's axioms. Note first that each variable v_i considered as an n-term of $P\otimes\Sigma_k$, for $n\geqslant i$ and any k yields a map $A^nL^k\to A$. Similarly p_j yields maps $A^nL^k\to L$ for each n and all $k\geqslant j$. We leave out the verification that T is a category. To check (L0) and (L3) it suffices to see that $(A^mL^h; v_1, \ldots, v_m; p_1, \ldots, p_h)$ is a product. This means that if $X=A^nL^k$, $\varphi_i\colon X\to A$, $(i=1,\ldots,m)$ and $\varphi^{(j)}\colon X\to L$, $(j=1,\ldots,h)$, there is $\varphi\colon X\to A^mL^h$ such that $\varphi_i=\varphi v_i$ and $\varphi^{(j)}=\varphi p_j$. As for this, we must take $\varphi=(\varphi_i;\varphi^{(j)})$, we can write $A^nL^k=A^n\times L^k$.

The V of T is the 1 of P and the F of T is the 0 of P. For $X=A^n\times L^k$, $1_X=(v_i;p_j)$. We have $V_X=(1,n,k)$ where the 1 is that of P.

In order to verify axiom (L1) i.e. $(\langle 1_X, V_X\rangle, \langle 1_X, F_X\rangle, X\times L)$ is a coproduct it suffices to consider two maps $\varphi_1\colon X\to Y$ and $\varphi_2\colon X\to Y$ where Y is either A or L and to find $\psi\colon X\times L\to Y$ such that $\langle 1_X, V_X\rangle\psi=\varphi_1$ and $\langle 1_X, F_X\rangle\psi=\varphi_2$. Say we look at the case $Y=L$. We have

$$\langle 1_X, V_X\rangle=\langle v_1,\ldots,v_n; p_1,\ldots,p_k, 1\rangle$$

and

$$\langle 1_X, F_X\rangle=\langle v_1,\ldots,v_n; p_1,\ldots,p_k, 0\rangle$$

where we should write (v_i,n,k), (p_i,n,k), $(1,n,k)$, $(0,n,k)$ in place of v_i, p_i, 1, and 0. Therefore

$$\langle 1_X, V_X\rangle\psi=S(1/p_{k+1})\psi \quad\text{and}\quad \langle 1_X, F_X\rangle\psi=S(0/p_{k+1})\psi$$

and it suffices to set $\psi=(p_{k+1}\Lambda\varphi_1)\vee(p'_{k+1}\Lambda\varphi_2)$, which means that, in the ε notation for the algebra $P_k=P\otimes\Sigma_k$ and the variable p_{k+1}, $\psi=(\varphi_1,\varphi_2)\varepsilon$. The case $Y=A$ is similar using the reduction of n-ary operations to monovalent $(n+1)$-predicates in polyadic algebra.

Next we prepare the ground for the verification of (L2). First we note that the Boolean algebra structure of $T(X,L)$ where $X=A^nL^k$ coincides with that of the $\{v_1,\ldots,v_n\}$-compression of $P\otimes\Sigma_k$. To prove this let $q_1, q_2\in T(X,L)$ and let, say, Λ_Y denote the intersection in $T(Y,L)$ for any object Y while Λ will denote the intersection in $P\otimes\Sigma_\infty$. It is easy to see from the definition of Λ_L that $\Lambda_L=p_1\Lambda p_2$ and therefore by the definition of Λ_X and the definition of composition of maps in $(P)\theta^{-1}$,

$$q_1\Lambda_X q_2=\langle q_1,q_2\rangle\Lambda_L=q_1\Lambda q_2\,.$$

Let $g\colon P\otimes\Sigma_k\to C_Z$ be a 2-valued representation. If φ in $P\otimes\Sigma_k$ is supported by $\{v_1,\ldots,v_n\}$ we write $(\varphi)g_n$ for the corresponding predicate $Z^n\to 2$ ([1], p. 87). If $X=A^n\times L^k$ and $Y=A^m\times L^h$ are objects of $T=(P,D)\theta^{-1}$ and $\varphi=\langle\varphi_i;\varphi^{(j)}\rangle\colon X\to Y$ we obtain a function $\varphi^\#\colon Z^n\times 2^k\to Z^m\times 2^k$ thus. An element of $Z^n\times 2^k$ can be noted (z,e) where $z=(z_1,\ldots,z_n)\in Z^n$ and $e\in 2^{[1,k]}$. We set $(z,e)\varphi_i^\#=(z)[(\varphi)\delta_e]g_n$ and similarly for $\varphi^{(j)}$, and $\varphi^\#=\langle\varphi_i^\#;\varphi^{(j)\#}\rangle$.

Now we look at quantification. Let $\chi\colon X\to Y$, $\varphi\colon X\to L$, $\chi=\langle\chi_i;\chi^{(j)}\rangle$. We let $u=(u_1,\ldots,u_n)$ and $a=(a_1,\ldots,a_k)$ be sequences of new individual and propositional variables respectively and we set by definition

(1) $$\exists_\chi[\varphi]=\exists(u,a)\{\varphi(u,a)\Lambda\bigwedge_{i=1}^{m}\mathrm{E}(\chi_i(u,a),v_i)\Lambda\bigwedge_{j=1}^{h}[\chi^{(j)}(u,a)\leftrightarrow p_j]\}\,.$$

We must check that for all $\psi\colon Y\to L$

(2) $$\exists_\chi[\varphi]\vdash_Y \quad\text{iff}\quad \varphi\vdash_X\chi\psi$$

where $\vdash_X$, for instance is the order relation of the $\{v_1,\ldots,v_n\}$-compression of $P\otimes\Sigma_k$. By the representation theorem for polyadic algebras it suffices to do that in any 2-valued representation. φ, ψ and χ determine functions $\varphi^\#$, $\psi^\#$ and $\chi^\#$ as above. Denoting, for instance, by $\overline{\varphi^\#}$ the subset of $Z^n\times 2^k$ of which $\varphi^\#$ is the characteristic function we see that in $P\otimes\Sigma_k$, (2) reduces to the obvious set-theoretic equation

(3) $$\overline{(\varphi^\#)}\,\chi^\#\subset\overline{\psi^\#} \quad\text{iff}\quad \overline{\varphi^\#}\subset\overline{(\psi^\#)}\,\chi^{\#-1}\,.$$

Indeed, the idea behind definition (1) is that, thinking of $\chi\colon X\to Y$ as a function between sets and of φ as a subset of X, $\exists_\chi[\varphi]$ should be the image of φ under χ and therefore an element $(v_1,\ldots,v_m,p_1,\ldots,p_h)$ of Y should be in $\exists_\chi[\varphi]$ iff there exists an element $(u_1,\ldots,u_n,a_1,\ldots,a_k)$ of φ mapped onto (v_i,p_j) by χ. This terminates the definition of the functor θ^{-1} on objects of $\mathfrak{I}$. We still have to define θ^{-1} on morphisms.

Let $f\colon(P_1,D_1)\to(P_2,D_2)$ be a morphism of $\mathfrak{I}$ and let $T_i=(P_i,D_i)\theta^{-1}$, $i=1,2$ be the corresponding elementary theories. As noted before f extends canonically to a homomorphism $P_1\otimes\Sigma_\infty\to P_2\otimes\Sigma_\infty$ still denoted by f and maps a J-term t of $P_1\otimes\Sigma_\infty$ on a J-term $(t)f$ of $P_2\otimes\Sigma_\infty$ for any finite $J\subset I$. If $\varphi=\langle\varphi_i;\varphi^{(j)}\rangle\in T_1(X,Y)$ we set by definition

$$(\varphi)[(f)\theta^{-1}]=\langle(\varphi_i)f;(\varphi^{(j)})f\rangle\,.$$

It is immediate from the definitions that if $(f)\theta^{-1}\colon T_1\to T_2$ is defined to be identity on objects, $(f)\theta^{-1}$ is a morphism of $\mathcal{C}$ and $\theta^{-1}\colon\mathfrak{I}\to\mathcal{C}$ is a functor. The fact that $(f)\theta^{-1}$ preserves quantification uses the fact that f preserves it.

II. The functor $\theta\colon\mathcal{C}\to\mathfrak{I}$

Let T be an elementary theory. We shall define an object $(T)\theta=((T)\theta',(T)\theta'')$ of $\mathfrak{I}$. In T, multiplication on the left by $\check{v}_{in}=\langle v_{1n},\ldots,v_{in}\rangle$ determines a homomorphism of Boolean algebras

(1) $$T(A^i,L)\to T(A^n,L)$$

by virtue of Theorem 0.2. Since $\check{v}_{in}\check{v}_{ji} = \check{v}_{jn}$ $(j \leqslant i \leqslant n)$ we have there a linearly directed system of Boolean algebras and monomorphisms. It is indeed easy to see that the map (1) must be a monomorphism from the fact that $\check{v}_{in}$ is an epimorphism. We let the Boolean algebra $(T)\theta$ be the direct limit of this system. This is simply the union of these algebras upon identifying $T(A^n, L)$ with the subalgebra of $T(A^{n+1}, L)$ which is its image under the map induced by $\check{v}_{n,n+1}$. More precisely, an element of $(T)\theta'$ is an equivalence class $\overline{q}$ of morphisms $q\colon A^n \to L$, and $\overline{q} = \overline{q}_1$ where, say, $q_1\colon A^{n_1} \to L$ and $n_1 \geqslant n$, iff $q_1 = \check{v}_{nn_1}$.

We may similarly construct a larger algebra $(T)\theta'_k$, for each positive integer k, by replacing in the present discussion, A^n by $A^n \times L^k$. As we shall see later this leads to a polyadic algebra of the form $(T)\theta' \otimes \Sigma_k$.

Next we endow the Boolean algebra $(T)\theta'$ (or more generally, $(T(\theta'_k)$) with a polyadic structure. Let us begin with the question of substitutions. It is enough to consider finite substitutions $\alpha' \in I^I$. We denote by the same letter an element of I^I and the mapping of the set of positive integers that it determines so that $(v_i)\alpha' = v_{i\alpha'}$. Note that if $\alpha\colon [1, n] \to [1, m]$ is any function, it induces a morphism $\widetilde{\alpha}\colon A^m \to A^n$ defined by $\widetilde{\alpha} = \langle v_{(1\alpha)m}, \ldots, v_{(n\alpha)m}\rangle$. Now, for α' finite in I^I and $\overline{q} \in (T)\theta$, to define $S(\alpha')\overline{q}$ we select n and $q \in \overline{q}$ such that $q\colon A^n \to L$ and α' lives on $[1, n]$ i.e. $(i)\alpha' = i$ for all $i > n$, and we let $\alpha\colon [1, n] \to [1, n]$ have the same effect on $[1, n]$ as α'. We let, by definition, $S(\alpha')\overline{q}$ be the equivalence class of $\widetilde{\alpha}q$. An easy calculation shows that $S(\alpha')\overline{q}$ is well-defined and it follows from Theorem 0.2 that S is a homomorphism of the semigroup I^I into the semigroup of endomorphisms of $(T)\theta$.

The definition of the predicate "equality" E in $(T)\theta$ is based on the idea that in any set A the equality relation is the image under $d = \langle 1_A, 1_A\rangle$: $A \to A^2$ of the set A itself. Accordingly we define

$$E(v_1, v_2) = \overline{\exists_d V_A}$$

where now A is the distinguished object of T. The definition of E for other variables can now be given simply as

$$E(v_i, v_j) = S(v_i/v_1, v_j/v_2)E(v_1, v_2)\,.$$

Turning now to quantification we first note a simple logical fact. If q is an element of a polyadic algebra supported by $\{v_1, \ldots, v_n\}$ and $J = \{v_{n_1}, \ldots, v_{n_r}\}$ is a subset of $\{v_1, \ldots, v_n\}$, and $u_1, \ldots, u_n$ are other distinct variables, then

(2) $$(\exists J)q = \exists(u_1, \ldots, u_n)[q(u_1, \ldots, u_n)\Lambda \bigwedge \{E(u_i, v_i)|\ i \in K\}]$$

where $K = [1, n] - \{n_1, \ldots, n_r\}$. On the other hand if $q\colon A^n \to L$ is a morphism of T and we let $K = \{m_1, m_2, \ldots, m_{n-r}\}$ and $\alpha\colon [1, n-r] \to [1, n]$ be defined by $(i)\alpha = m_i$ we have $\widetilde{\alpha} = \langle v_{m_1,n}, \ldots, v_{m_{n-r},n}\rangle$ and $\exists_{\widetilde{\alpha}}[q]\colon A^{n-r} \to L$ must be thought of as the set of $(v_1, \ldots, v_{n-r})$ in A^{n-r} on which some element of q is mapped by $\widetilde{\alpha}$. This suggests exactly (2) except for a change of variables. For the expression (2) depends on $v_{m_1}, \ldots, v_{m_{n-r}}$ instead of $v_1, \ldots, v_{n-r}$. Therefore, if $q\colon A^n \to L$, we set by definition,

(3) $$(\exists J)\overline{q} = \overline{\widetilde{\alpha}\exists_{\widetilde{\alpha}}[q]}\,.$$

The direct verification that quantification is well-defined and that the axioms for cylindrical algebras C1–C4 are satisfied is arduous but if we allow ourselves to use Lawvere's completeness theorem they boil down to some easy set-theoretic computations. For instance, to show that $(\exists J)\overline{q}$ is well-defined let $q_1\colon A^{n'} \to L$, $n \leqslant n'$

$$\overline{q} = \overline{q}_1 \quad \text{i.e.} \quad q_1 = \check{v}_{nn'}q\,,$$

$$K' = [1, n'] - \{n_1, \ldots, n_r\} = \{m_1, \ldots, m_{n'-r}\}\,, \quad \alpha'\colon [1, n'-r] \to [1, n']$$

such that $(i)\alpha' = m_i$.

We must see that

(4) $$\check{v}_{nn'}\widetilde{\alpha}\exists_{\widetilde{\alpha}}[q] = \widetilde{\alpha}!\exists_{\widetilde{\alpha}'}[\check{v}_{nn'}q]\,.$$

As $\check{v}_{nn'}\widetilde{\alpha} = \widetilde{\alpha}'\check{v}_{n-r,n'-r}$ it suffices to see that

(5) $$\check{v}_{n-r,n'-r}\exists_{\widetilde{\alpha}}[q] = \exists_{\widetilde{\alpha}'}[\check{v}_{nn'}q]\,.$$

Since, as noted before (Theorem 0.5), it follows from Lawvere's completeness theorem, that T can be embedded in a direct product of theories each of which is a subcategory of Ens and in each of which $L = 2$, we may assume here that T is such a subcategory of Ens. Then replacing q by the subset of A of which it is the characteristic function we see that (5) means that

(6) $$\check{v}_{n-r,n'-r}{}^{-1}\widetilde{\alpha}(q) = \widetilde{\alpha}'\big(\check{v}_{nn'}{}^{-j}(q)\big)\,.$$

This equation is easily verified.

The verification of $E(v_i, v_i) = 1$ leads quickly to the obvious set-theoretic equation $\langle v_{in}, v_{in}\rangle^{-1}(\Delta) = A^n$ where $\Delta \subset A^2$ is the relation of equality on A. It can also be done fairly easily directly. For the verification of Q_2, Q_3, C3 and C4, we need the fact already noted that in a model M of T, $(\Lambda_L)M$ is the infimum in the Boolean algebra 2 so that when we pass from characteristic functions to sets A in those axioms is interpreted as intersection of sets. Therefore these verifications in the present context are close to what they are for cylindrical set algebras.

We will need the following theorem which discloses the reason for introducing $(T)\theta'_k$.

THEOREM 2.1. *The polyadic algebra $(T)\theta'_k$ has the form $(T)\theta'\otimes\Sigma_k$ where Σ_k is freely generated by $p_1, \ldots, p_k$ (p_i being the equivalence class of the ith projection $L^k\to L$).*

Proof: The statement of the theorem tacitly identifies $(T)\theta'$ with its image in $(T)\theta'_k$ under a polyadic monomorphism which maps $\bar{q}\in(T)\theta'$, where $q\colon A^n\to L$, onto the class of $\mathrm{pr}(A^n\times L^k\to A^n)q$. It follows from Theorem 0.3, that, after this identification, the algebra $(T)\theta'_k$ is an extension of $(T)\theta'$ by the elements $p_1, \ldots, p_k$. That the p_i are closed elements is immediate from the definition of quantification in $(T)\theta'_k$ where for $a\colon [1, n]\to[1, m]$, $\tilde{a}$ would now denote

$$\langle p_{A^m}v_{(1a)m}, \ldots, p_{A^m}v_{(na)m}, 1_{L^k}\rangle \quad \text{which is a map} \quad A^m\times L^k\to A^n\times L^k .$$

To prove the theorem it suffices (Theorem 0.7) to define, for each $i\in[1, 2^k]$, a polyadic homomorphism $\delta_i\colon (T)\theta'_k\to(T)\theta'$ such that for each $j=1,\ldots,k$ $(p_j)\delta_i=(j)e_{ik}$ and $(\bar{q})\delta_i=\bar{q}$ for $\bar{q}\in(T)\theta'$. The generic element of $(T)\theta'_k$ has the form $\bar{\psi}$ where $\psi\colon A^n\times L^k\to L$. We define δ_i by the equation $(\bar{\psi})\delta_i=\overline{(\psi)\delta_i}$ where the δ_i on the right has the meaning defined earlier. The required conditions can readily be verified. Q.E.D.

We observe that the (ε, δ)-formalism for the maps $A^n\times L^k\to L$ of T corresponds by θ'_k to the (ε, δ)-formalism for the extension $(T)\theta'_k$ of the polyadic algebra $(T)\theta'$ by the propositional variables $p_1, \ldots, p_k$.

We note that the Boolean homomorphism

$$S(\bar{\varphi}_1/p_1, \ldots, \bar{\varphi}_k/p_k)\colon\ (T)\theta'_k\to(T)\theta'$$

where $\varphi_i\colon A^n\to L$ and p_i has the same meaning as in the theorem, maps $\bar{\psi}\in(T)\theta'_k$ onto the equivalence class of $\langle 1_{A^n}, \varphi_1, \ldots, \varphi_k\rangle\psi$.

In order to define $(T)\theta''$, the distinguished set of terms of $(T)\theta'$, we associate to every morphism $t\colon A^n\to A$ of T and n-term $\bar{t}$ of $(T)\theta'$ and we let $(T)\theta''$ be the set of all such $\bar{t}$. First we have the following theorem in accordance with the well-known idea by which an n-ary operation determines an $(n+1)$-ary predicate monovalent in its last variable.

THEOREM 2.2. *The map which assigns to $t\colon A^n\to A$, $\varphi_t\colon A^{n+1}\to L$ defined by the equation*

$$\varphi_t=\langle\check{v}_{n,n+1}t, v_{n+1,n+1}\rangle\exists_d V_A$$

is an injection of $T(A^n, A)$ into the set of the morphisms $\varphi\colon A^{n+1}\to L$ which are monovalent in their last variable i.e.

$$\exists_{\check{v}_{n,n+1}}[\varphi]=V_{A^n}$$

and

$$\check{v}_{n+1,n+2}\varphi\Lambda\langle\check{v}_{n,n+2}v_{n+2,n+2}\rangle\varphi\leqslant\langle v_{n+1,n+2}v_{n+2,n+2}\rangle\exists_d V_A \quad in \quad T(A^{n+2}, L) .$$

We leave out the proof of the theorem which is in any case an immediate consequence of the completeness Theorem 0.5. That the above injection may fail to be surjective is the reason for which $\mathcal{C}$ is equivalent to $\mathcal{T}$ and not simply to $\mathcal{T}'$.

Similarly we have an injection of $T(A^n\times L^k, A)$ into the set of morphisms $t\colon A^{n+1}\times L^k\to A$ monovalent in their last individual variable. Such a monovalent morphism determines a monovalent $(n+1)$-ary predicate and hence an n-term of $(T)\theta'_k$. The n-term corresponding to the morphism φ_t we call $\bar{t}$. If E is the equality of $(T)\theta'$, $E(\bar{t}, v_{n+1})$ is the equivalence class of

$$\langle p_{A^{n+1}}\check{v}_{n,n+1}t, p_{A^{n+1}}v_{n+1,n+1}\rangle\exists_d V_A$$

where $p_{A^{n+1}}=\mathrm{pr}(A^{n+1}\times L^k\to A^{n+1})$. Equivalently $\bar{t}$ can be characterized by the condition that if $q\colon A^m\times L^k\to L$ and $n\leqslant m$, $S(\bar{t}/v_1)\bar{q}$ is the equivalence class of

$$\langle\langle p_{A^m}\check{v}_{nm}t, \hat{v}_{1m}\rangle, p_{L^k}\rangle q$$

where $p_{L^k}=\mathrm{pr}(A^m\times L^k\to L^k)$.

That the set $(T)\theta''$ is admissible follows from the fact that

(i) if $t\colon A^n\to A$ and $m>n$ the m-term determined by $\bar{t}$ is $\overline{\check{v}_{nm}t}$

(ii) $(v_i, n)=\overline{v_{in}}$; and

(iii) for every $k\geqslant 1$, $m, n\geqslant 0$, every sequence $\varphi=(\varphi_1, \ldots, \varphi_{2^k})$ of morphisms $A^n\to A$ in T, every sequence $(t_1, \ldots, t_n)$ of morphisms $A^m\to A$ in T, every sequence $(q_1, \ldots, q_k)$ of morphisms $A^m\to L$ in T, the m-term associated to

$$\langle t_1, \ldots, t_n; q_1, \ldots, q_k\rangle(\varphi_1, \ldots, \varphi_{2^k})\varepsilon\colon\ A^m\to A^n\times L^k\to A$$

is

$$S(\bar{t}_1/v_1, \ldots, \bar{t}_n/v_n;\ \bar{q}_1/p_1, \ldots, \bar{q}_k/p_k)(\bar{\varphi}_1, \ldots, \bar{\varphi}_{2k})\varepsilon .$$

That $\overline{(\varphi_1, \ldots, \varphi_{2^k})\varepsilon}=(\bar{\varphi}_1, \ldots, \bar{\varphi}_{2^k})\varepsilon$ follows from Theorem 2.1 and the remarks at the end of the last paragraph.

To define θ on morphisms let $W\colon T_1\to T_2$ be a morphism of $\mathcal{T}$. Then $(W)\theta\colon (T_1)\theta'\to(T_2)\theta'$ is defined by the equation $(\bar{q})[(W)\theta]=\overline{(q)W}$. That $(W)\theta$ is a well-defined polyadic homomorphism is a short and easy verification. Moreover if $t\colon A^n\to A$ then $(\bar{t})[(W)\theta]=\overline{(t)W}$ and therefore, $(W)\theta$ is a morphism of $\mathcal{T}$. If, similarly we define $(T)\theta''_k$ as $\{\bar{t}|\ t\colon A^n\times L^k\to A\}$ and $(T)\theta_k=((T)\theta'_k, (T)\theta''_k)$, the same equation can be used to complete the definition of θ_k as a functor $\mathcal{C}\to\mathcal{T}$.

III. Reciprocity of the constructions θ and θ^{-1}

We shall prove the following

THEOREM 3.1. *The functors $\theta\colon \mathcal{T}\to\mathcal{P}$ and $\theta^{-1}\colon \mathcal{P}\to\mathcal{T}$ are reciprocal equivalences.*

This means that we must (see, for instance, Mitchell [12], pp. 52 and 61) define for each theory T and object (P, D) of $\mathcal{P}$ isomorphisms

$$R_T\colon T\to(T)\theta\theta^{-1}$$

and

$$S_{(P,D)}\colon (P, D)\to(P, D)\theta^{-1}\theta$$

in such a way that for any morphisms $W\colon T\to T_1$ and $f\colon (P, D)\to(P_1, D_1)$ in $\mathcal{T}$ and $\mathcal{P}$ respectively the following two diagrams commute:

$$(1)\qquad \begin{array}{ccc} T & \xrightarrow{R_T} & (T)\theta\theta^{-1} \\ \downarrow{\scriptstyle W} & & \downarrow{\scriptstyle (W)\theta\theta^{-1}} \\ T_1 & \xrightarrow[R_{T_1}]{} & (T_1)\theta\theta^{-1} \end{array}$$

$$(2)\qquad \begin{array}{ccc} (P, D) & \xrightarrow{S_{(P,D)}} & (P, D)\theta^{-1}\theta \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle (f)\theta^{-1}\theta} \\ (P_1, D_1) & \xrightarrow[S_{(P_1,D_1)}]{} & (P_1, D_1)\theta^{-1}\theta \end{array}$$

Definition of R_T. There is no loss of generality in assuming that T and $(T)\theta\theta^{-1}$ have the same objects and that R_T leaves objects fixed. We define the effect of R_T on morphisms in stages. Beginning with morphisms $q\colon X\to L$, for an arbitrary $X = A^n\times L^k$, we define R_T so that it maps $T(X, L)$ bijectively onto $[(T)\theta\theta^{-1}]\ (X, L)$. Although this will turn out to be the case, we do not have at this stage to show that this bijection is an isomorphism of Boolean algebras. The definition is simple: $(q)R_T = \bar{q}$ or, more exactly $(\bar{q}, n, k)$. The verifications are immediate. Next we wish to define R_T on $T(X, A)$. Let $(t)R_T = \bar{t}$, or, more exactly $(\bar{t}, n, k)$. The fact that $\bar{t}$ is a morphism of $(T)\theta\theta^{-1}$ follows from the definition of θ^{-1} and of θ''. To complete the definition of R_T, let $\varphi = \langle\varphi_i; \varphi^{(j)}\rangle\colon A^n\times L^k\to A^m\times L^h$. By definition $(\varphi)R_T = \langle(\varphi_i)R_T; (\varphi^{(j)})R_T\rangle$. Therefore R_T maps the set of morphisms of T biuniquely onto the set of morphisms of $(T)\theta\theta^{-1}$ and preserves domains and codomains.

To show that R_T is an isomorphism of categories the only thing left is the verification that R_T preserves the composition of morphisms. We are easily reduced to the case of a pair of morphisms $\varphi\colon X\to Y$, $\psi\colon Y\to Z$ where Z is either A or L. In the latter case, that R_T preserves the composition means that

$$(\varphi\psi)R_T = S(\varphi_1 R_T/v_1, \ldots, \varphi_m R_T/v_m;\ \varphi^{(1)}R_T/p_1, \ldots, \varphi^{(h)}R_T/p_h)\,(\psi R_T)$$

where $Y = A^m\times L^h$. The left member is $\overline{\varphi\psi}$. The right member reduces to this after a computation based on the formulas established above.

Finally the commutativity of diagram (1) must be checked. For objects this is obvious. Let $\varphi\colon X\to Y$ in T, $\varphi = \langle\varphi_i, \varphi^{(j)}\rangle$. Then $[(\varphi)W]R_{T_1} = \langle\overline{\varphi_i W}; \overline{\varphi^{(j)}W}\rangle$ in $(T)\theta\theta^{-1}$. On the other hand $[(\varphi)R_T](W)\theta\theta^{-1}$ also easily reduces to that expression.

Definition of $S_{(P,D)}$. If $q\in P$ is supported by $\{v_1, \ldots, v_n\}$ we set $(q)S_{(P,D)} = \overline{(q, n, 0)}$. Remember that $(q, n, 0)$ is then a morphism of $(P)\theta^{-1}$. That $S_{(P,D)}$ is well-defined and bijective is easy to show. That $S_{(P,D)}$ preserves the Boolean structure follows from the fact already shown that $(P\theta^{-1})(A^n, L)$ has the same Boolean structure as the $\{v_1, \ldots, v_n\}$-compression of P. That $S_{(P,D)}$ preserves the transformation endomorphisms $S(a)$ and quantification must be computed. The reader will miss little if he does it, say, for $(\exists v_1)q(v_1, v_2)$. Equality is then preserved by virtue of the uniqueness of equality in a polyadic algebra. The fact that diagram (2) commutes is obvious.

The category $\mathcal{P}'$ of polyadic algebras can be imbedded as a full subcategory of $\mathcal{P}$ by either of two functors MIN and MAX, trivial on morphisms and otherwise defined by the equations

$$(P)\,\mathrm{MIN} = \big(P, (P)\,\mathrm{Min}\big)\quad\text{and}\quad (P)\,\mathrm{MAX} = \big(P, (P)\,\mathrm{Max}\big).$$

The following corollary is immediate from the definition of θ^{-1}.

COROLLARY 3.2. *MAX θ^{-1} is an isomorphism of $\mathcal{P}'$ onto the full subcategory of $\mathcal{T}$ whose objects T have the property of being definably complete i.e. the map $t\to\varphi_t$ of $T(A^n, A)$ into the set of morphisms $A^{n+1}\to L$ monovalent in their last variable is surjective.*

IV. The semantics functors on $\mathcal{P}$ and $\mathcal{T}$

To avoid set-theoretic difficulties we shall assume the existence of an inaccessible cardinal and hence of a Grothendieck universe. The categories $\mathcal{T}$, $\mathcal{P}$, Ens will henceforth be restricted to contain only objects in that universe. Following Lawvere [10], [11] we consider the category $\mathfrak{U}$ whose objects are all functors $U\colon \mathcal{C}\to\mathrm{Ens}$ where $\mathcal{C}$ is a small category although not necessarily in the above universe. (As defined, $\mathfrak{U}$ is a class. We could require that $\mathcal{C}$ lies in a larger Grothendieck universe if we assumed the existence of a second inaccessible cardinal. Then $\mathfrak{U}$ would again be a set.) If $U_i\colon \mathcal{C}_i\to\mathrm{Ens}$ for $i=1,2$ are objects in $\mathfrak{U}$, a morphism $U_1\to U_2$ is a functor $\mathcal{C}_1\to\mathcal{C}_2$ such that $U_1=(\mathcal{C}_1\to\mathcal{C}_2)\,U_2$.

We wish to show that the 2-valued representations of a polyadic algebra P correspond to the models of the associated theory $(P)\theta^{-1}$ no matter what set D of terms we distinguish in P. For convenience we will therefore henceforth denote by $\mathcal{T}$ the category of definably complete theories and write θ^{-1} instead of MAX θ^{-1}, and $\mathcal{S}$ instead of $\mathcal{S}'$. The idea is to eliminate the unnecessary duplications in our original $\mathcal{T}$ and $\mathcal{S}$. More precisely, what we will do is to define two contravariant "semantics" functors

$$\mathrm{Sem}\colon \mathcal{S}\to\mathfrak{U} \quad\text{and}\quad \mathrm{SEM}\colon \mathcal{T}\to\mathfrak{U}$$

and show that Sem and θ^{-1} SEM are naturally equivalent functors. The functor SEM is defined by Lawvere [8], [10]. It associates to a theory T, not only, as used to be the case in model theory, the set of all models of T, but the category $(T)\mathcal{M}$ of all models of T endowed with the underlying set functor U_T. More explicitely, the objects of $(T)\mathcal{M}$ are the models M of T, its morphisms are the natural transformations between models, and $(M)\,U_T=(A)M$. The action of SEM on morphisms is as follows: if $W\colon T_1\to T_2$ is a morphism of $\mathcal{T}$, $(W)\,\mathrm{SEM}\colon U_{T_2}\to U_{T_1}$ is the functor $(W)\,\mathrm{SEM}\colon (T_2)\mathcal{M}\to(T_1)\mathcal{M}$ which maps $M\in|(T_2)\mathcal{M}|$ onto $(W)M\colon T_1\to\mathrm{Ens}$.

Similarly, the functor Sem associates to the polyadic algebra P the functor $U_P\colon (P)\mathfrak{R}\to\mathrm{Ens}$ where $(P)\mathfrak{R}$ is the category of 2-valued representations f of P. More explicitely, the objects of $(P)\mathfrak{R}$ are the 2-valued representations $f\colon P\to C_X$ of P ($X\in|\mathrm{Ens}|$), $(f)\,U_P=X$ and a morphism $f_1\to f_2$, where $f_i\colon P\to C_{X_i}$ $(i=1,2)$ are objects of $(P)\mathfrak{R}$, is a polyadic monomorphism $g\colon [(P)f_1](\hat{X}_1)\to C_{X_2}$ such that $f_1g=f_2$. The domain of g is the subalgebra of C_{X_1} generated by $(P)f$, and all constants $\hat{x}_1$ of C_{X_1}, $\hat{x}_1$ being the constant associated to $x_1\in X_1$. If P is the polyadic algebra associated to an ordinary elementary theory and f_1, f_2 the representations associated to two models of that theory, then such a g corresponds to an elementary monomorphism between these models ([1], p. 124). $(g)\,U_P$ maps $x_1\in X_1$ on the element $x_2\in X_2$ such that $\hat{x}_2=(\hat{x}_1)g$.

A natural equivalence of θ^{-1} SEM onto Sem is a function φ which associates to every object P of $\mathcal{S}$ an isomorphism $\varphi_P\colon (P)\,\theta^{-1}\,\mathrm{SEM}\to (P)$ Sem in such a way that for every polyadic homomorphism $f\colon P_1\to P_2$ the following diagram commutes

$$(1)\qquad \begin{array}{ccc} (P_1)(\theta^{-1}\,\mathrm{SEM}) & \xrightarrow{\ \varphi_{P_1}\ } & (P_1)\,\mathrm{Sem} \\ \big\downarrow{\scriptstyle (f)(\theta^{-1}\,\mathrm{SEM})} & & \big\downarrow{\scriptstyle (f)\,\mathrm{Sem}} \\ (P_2)(\theta^{-1}\,\mathrm{SEM}) & \xrightarrow[\ \varphi_{P_2}\ ]{} & (P_2)\,\mathrm{Sem} \end{array}$$

([12], p. 59). Such a Φ_P is an isomorphism of categories $(P\theta^{-1})\mathcal{M}\to(P)\mathfrak{R}$ such that $\Phi_P U_P = U_{P\theta^{-1}}$.

We first define Φ_P^{-1} on objects $g\colon P\to C_X$. The model M of $T=(P)\,\theta^{-1}$ associated to g is defined by the conditions:

(i) $$(A^n\times L^k)M = X^n\times 2^k;$$

(ii) if $(q,n)\colon A^n\times L^k\to L$ in T, i.e. q is an element of $P\otimes\Sigma_k$ with support $\{v_1,\ldots,v_n\}$,

$$(x,e)[(q,n)M] = (x_1,\ldots,x_n)[S((1)e/p_1,\ldots,(k)e/p_k)\,q]\,g_n$$

where $(x,e)\in X\times 2^k$, and, for any element q_1 of P with support $\{v_1,\ldots,v_n\}$, $(q_1)g_n$ is the function $X^n\to 2$ determined by $(q_1)g$ as before;

(iii) a condition similar to (ii) for the morphisms $t\colon A^n\times L^k\to A$;

(iv) if generally, $\varphi=\langle\varphi_i;\varphi^{(j)}\rangle$,

$$(\varphi)M = \langle(\varphi_i)M;\,(\varphi^{(j)})M\rangle\,.$$

Next we define Φ_P on objects M of $(T)\mathcal{M}$ where $T=(P)\,\theta^{-1}$ as before. The representation g of P associated to M is defined by the condition:

if $q\in P$ has support $\{v_1,\ldots,v_n\}$, and $x\in X^I$ where $I=\{v_1,v_2,v_3,\ldots\}$

$$(x)[(q)_g] = (x_{v_1},\ldots,x_{v_n})[\overline{(q,n)}M]\,.$$

The correspondance thus established between M and g is strictly one-one. We leave out the straightforward verifications of this fact as well as that, M and g as defined above, are indeed a model of T and a representation of P respectively.

Finally we define Φ_P^{-1} on morphisms of $(P)\mathfrak{R}$. Let $g\colon f_1\to f_2$ where $f_i\colon P\to C_{X_i}$ $(i=1,2)$ be a morphism in $(P)\mathfrak{R}$. Let $M_i=(f_i)\Phi_P^{-1}$ and $t=(g)\Phi_P^{-1}$. To any object $Y=A^n\times L^k$ of T, t associates a map

$$t_Y\colon X_1^n\times 2^k\to X_2^n\times 2^k\,.$$

For $Y = L$, this map is 1_2. As the thing works componentwise, it suffices to look at the case $Y = A$. The function $t_A\colon X_1 \to X_2$ is $(g)\, U_P$ by definition.

We leave out the definition of Φ_P on morphisms of $(T)\mathcal{M}$ and, again, the remaining verifications that Φ_P^{-1} (or Φ_P) is one-one on morphisms and preserves their composition, and that the diagram (1) is commutative.

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The new interval topology on lattice products *

by

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Although the interval, ideal [3], and new interval [2] topologies all give the topology of the real line, only the latter two give the topology of the plane. That is, the ideal or new interval topology of the plane (ordered coordinate-wise) is equivalent to the product of the ideal or new interval topologies of the real line. It is reasonable to ask whether this property holds in the case of a finite product of chains or, more generally, a finite product of lattices.

Alo and Frink ([1], Theorem 2) proved that the ideal topology of a finite product of lattices is equivalent to the product of the ideal topologies of these lattices. However, for the new interval topology, they were forced to restrict the lattices to chains ([1], Theorem 9).

In this paper we show by means of a counterexample that the new interval topology on the finite product of lattices is not equal to the product of the new interval topology on the lattices. First, however, we prove two theorems that give conditions upon the individual lattices that insure the equivalence of the two topologies. These theorems, besides being of interest in themselves, give insight into the counterexample.

1. Definitions. The product order $\prod_{a\in A} L_a$ of an arbitrary number of lattices is defined coordinate-wise:

$$(a_a) \leqslant (b_a) \text{ if and only if } a_a \leqslant b_a \text{ for all } a\,.$$

$\prod_{a\in A} L_a$ is a lattice under this order:

$$(a_a)\vee(b_a) = (a_a\vee b_a) \quad \text{and} \quad (a_a)\wedge(b_a) = (a_a\wedge b_a)$$

where $\vee$ ($\wedge$) is the lattice supremum (infimum).

The interval topology is defined by taking as subbasic closed sets the closed rays $[a, +\infty) = \{x \mid x \geqslant a\}$ and $(-\infty, b] = \{x \mid x \leqslant b\}$. Clearly the intervals $[a, b] = \{x \mid a \leqslant x \leqslant b\}$ are closed in this topology.

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