Proof. Since $f(x)$ is BVG-ω on $[a, b]$, we can express the interval as the union of a countable family of closed sets F_i, $[a, b] = \sum_i F_i$, on each of which $f(x)$ is BV-ω. Consider the set F_i, where i is any positive integer. By Theorem 3.1, there is a function $g_i(x)$ in calss $\mathfrak{U}$ such that $g_i(x)$ is BV-ω on $[a, b]$ and $g_i(x) = f(x)$ for all $x \in F_i$. Denote by F'_i the set of points of F_i where the ω-derivative of $g_i(x)$ exists finitely. Then by Theorem 1.2, $|F_i - F'_i|_\omega = 0$. Let E_i denote the set of points of F'_i where the ω-density of $S - F_i$ is zero. Clearly F'_i and $S - F_i$ are ω-separated. So by Theorem 1.1, $|F'_i - E_i|_\omega = 0$. We have $F_i - E_i = (F_i - F'_i) + (F'_i - E)$. So $|F_i - E_i|_\omega = 0$. Let a be any point of E_i. Since $g_i(x) = f(x)$ on SF_i and the ω-derivative of $g_i(x)$ exists finitely at a, it follows that $(\mathrm{ap})f'_\omega(a)$ exists finitely. Since a is arbitrary, $(\mathrm{ap})f'_\omega(x)$ exists finitely at each point of E_i. Write $E = \sum_i E_i$. Then, at each point of E, $(\mathrm{ap})f'_\omega(x)$ exists finitely. Now $[a, b] - E \subseteq \sum_i (F_i - E_i)$. So $\omega^*([a, b] - E) \leqslant \sum_i \omega^*(F_i - E_i) = 0$. This proves the Theorem.

I am grateful to Dr. P. C. Bhakta for his kind help and suggestions in the preparation of the paper.

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DEPARTMENT OF MATHEMATICS,
KALYANI UNIVERSITY,
West Bengal, India

Reçu par la Rédaction le 12. 7. 1968

Structure spaces of lattices

by

J. R. Isbell and **J. T. Morse** (Cleveland, Ohio)

Introduction. This paper gives a simpler proof of the functoriality of the structure space of maximal l-ideals of an f-ring with unit. Like the previous proof [3], this one depends (when analyzed) on a different, visibly functorial construction that turns out to yield the maximal ideal space. Both constructions generalize to distributive lattices with base point. Hence the maximal ideal space of a unitary f-ring is determined by the underlying lattice. This was previously known for commutative semisimple unitary f-rings [6].

Kaplansky's original proof that the lattice of continuous functions $C(X)$ on a compact Hausdorff space X determines X [4], and its generalizations until now, have used *ad hoc* constructions to wring the space from the lattice. After an *ad hoc* beginning (the quickest), we exhibit the following natural structure. A based distributive lattice L has a T_0 space $\pi(L)$ of prime ideals containing 0. The present construction, and Kaplansky's, form the finest quotient space $\varkappa(L)$ in which the closure of every point is collapsed to a point. The more radical treatment of [3] yields a compact Hausdorff space $\beta(L)$, which, unlike π and $\varkappa$, is functorial for a category of lattice homomorphisms containing the unitary f-ring homomorphisms. Obvious mappings run $\pi(L) \to \varkappa(L) \to \beta(L)$. The easiest way to establish coincidence of $\varkappa(L)$ and $\beta(L)$ (and a space of prime ideals) is to find a subspace of $\pi(L)$ continuously cross-sectioning $\pi(L) \to \beta(L)$ and mapping surjectively to $\varkappa(L)$; that is what the maximal ideal space of a unitary f-ring does, and also the maximal ideal space of an abelian l-group with strong order unit [3]. A continuous cross-section is not enough (for a vector lattice). We find a sufficient additional condition, for abelian l-groups, to the effect that group elements positive at a point are non-negative on a $\beta(L)$-neighborhood.

1. Maximal ideal spaces. Let $\mathcal{M}(A)$ be the space of maximal l-ideals of a unitary f-ring. (It is compact Hausdorff [2]; compact by the usual maximality argument, Hausdorff by a simple argument due to Gillman [1] depending on the fact that for $M \in \mathcal{M}(A)$, A/M is totally ordered.) An l-ideal is *primary* if it is contained in a unique maximal l-ideal.

The only new results in this section are 1.2 and 1.5.

1.1. *If $h\colon A\to B$ is a unitary homomorphism of unitary f-rings and $M\in\mathcal{M}(B)$, then $h^{-1}(M)$ is primary.*

Proof. The image A' of A in B/M is totally ordered unitary. So of any two l-ideals in it, one contains the other; A' has a unique maximal l-ideal.

1.1 gives us a function $\mathcal{M}(h)\colon \mathcal{M}(B)\to\mathcal{M}(A)$, taking M to the maximal ideal containing $h^{-1}(M)$.

For $M\in\mathcal{M}(A)$, among the primary ideals contained in M (the largest is M, and) there is a smallest, $G(M)$ [2]. (Below we need its description from [2].)

1.2. *In $\mathcal{M}(A)$, if M_0 contains the intersection of $\{G(M_\lambda)\colon \lambda\in\Lambda\}$ then it contains the intersection of $\{M_\lambda\}$.*

Proof. If M_0 does not contain $\bigcap M_\lambda$, then there is t in $\bigcap M_\lambda$ such that $t>0 \pmod{M_0}$. Since A/M has no proper l-ideal, some multiple u of t exceeds $2 \pmod{M_0}$. The positive part $(u-1)^+=(1-u)^-$ is still not in M_0; but since $1-u>0 \pmod{M_\lambda}$, $(1-u)^-\in G(M_\lambda)$ ([2], 5.8).

1.3. THEOREM. *If $h\colon A\to B$ is a unitary homomorphism of unitary f-rings, then $\mathcal{M}(h)\colon\mathcal{M}(B)\to\mathcal{M}(A)$ is continuous. $\mathcal{M}$ is functorial.*

Proof. If a set $\{M_\lambda\}$ in $\mathcal{M}(B)$ has a limit point M_0, then $h^{-1}(M_0)$ contains $\bigcap h^{-1}(M_\lambda)\supset\bigcap G\big(\mathcal{M}(h)(M_\lambda)\big)$; by 1.2, $\mathcal{M}(h)(M_0)$ contains $\bigcap\mathcal{M}(h)(M_\lambda)$. Thus $\mathcal{M}(h)$ preserves limit points. For a composition hg,

$$\mathcal{M}(g)\mathcal{M}(h)(M)\supset g^{-1}\mathcal{M}(h)(M)\supset g^{-1}h^{-1}(M)=(hg)^{-1}(M)\,,$$

so it is $\mathcal{M}(hg)(M)$.

To show that the lattice structure of A determines $\mathcal{M}(A)$, first, it suffices to consider the based lattice $(A,\geqslant,0)$; for any other based lattice $(A,\geqslant,a)$ is isomorphic by a translation. (The introduction of 0 is a trivial departure from the basic argument of Kaplansky [4], intended rather for Section 2 than for the present problem.) Lattice ideals containing 0 will be called ***lz-ideals***; the ring l-ideals may be distinguished as ***lr-ideals***. Prime lz-ideals are defined in the lattice sense ($x\wedge y$ in I implies x or y in I), and so are prime lr-ideals J. Thus A/J is totally ordered, but may have proper zero divisors.

1.4. (Pierce) *Every prime lz-ideal of an f-ring contains a prime lr-ideal.*

Pierce stated the result: for a sublattice L consisting of non-negative non-zero elements, there is a homomorphism upon a totally ordered ring C taking L into $C-\{0\}$ [5].

In particular, a prime lz-ideal of a unitary f-ring A contains a germinal (lr-) ideal. It cannot contain two distinct germinal ideals $G(M_1)$, $G(M_2)$; for $G(M_1)$ is not contained in M_2, whence the image of $G(M_1)$ in A/M_2 is all of A/M_2, and $G(M_1)$ is cofinal modulo $G(M_2)$.

Every lr-ideal is an intersection of prime lr-ideals since the quotient (f-) ring is a subdirect product of totally ordered rings. The intersection of the prime lz-ideals containing germinal G is accordingly $G^-=\{x\colon x^+\in G\}$. The germinal ideals G_1, G_2 in two prime lz-ideals P_1, P_2 are the same if and only if $P_1\vee P_2$ is not all of A. For if $G_1\neq G_2$, $G_1^-\vee G_2^-$ is already all of A; if $G_1=G_2$, neither the image of P_1 nor the image of P_2 in A/G_1 is cofinal, so the image of $P_1\vee P_2$ is not. Thus the ideals G^- are the intersections of the equivalence classes of prime lz-ideals under the relation "$P_1\vee P_2\neq A$", and are determined by the lattice and 0.

One gets the correct topology on this set of intersections by defining G^- to be a limit point of $\{G_\lambda^-\}$ if $\bigcap G_\lambda^-$ is contained in some prime lz-ideal containing G_0^-. For if the maximal lr-ideal M_0 contains $\bigcap M_\lambda$, M_0^- contains $\bigcap M_\lambda^-\supset\bigcap G_\lambda^-$; if not, M_0 does not contain $\bigcap G_\lambda$ (by 1.2), so $\bigcap G_\lambda$ is cofinal modulo M_0 and not contained in a prime lz-ideal containing G_0.

1.5. THEOREM. *The lattice structure of a unitary f-ring A determines $\mathcal{M}(A)$.*

2. General lattices.

For any based distributive lattice L, the prime lz-ideals with the hull-kernel topology form a T_0 space $\pi(L)$ (topological, because the ideals are prime. Distributivity will be needed for $\beta(L)$; for $\pi(L)$, one may as well assume distributivity since in any case the distributive reflection of L would give the same space.) It is already clear from the proof of 1.5 that (there) $\pi(A)$ determined $\mathcal{M}(A)$; for the equivalence relation $P_1\vee P_2\neq A$ is non-disjointness of the closures in $\pi(A)$, and the topology is the quotient topology.

The construction generalizes as follows. Let the *K-classes* of prime ideals of a based lattice L be the equivalence classes for the smallest equivalence relation $\sim$ such that $P\sim Q$ when $P\subset Q$. A K-class c has a kernel ideal $k(c)$; define c_0 to be a limit point of $\{c_\lambda\}$ if $\bigcap k(c_\lambda)$ is contained in some member of c_0. Then the K-classes form a topological space $\varkappa(L)$, the quotient space of $\pi(L)$, by the finest partition into unions of closures of points. $\varkappa$ is functorial for the narrow category of homomorphisms upon cofinal subsets, as is π, and the quotient mappings $v\colon\pi(L)\to\varkappa(L)$ constitute a natural transformation.

The parallel construction of the first four pages of [3] generalizes as easily. The rest of this paper assumes knowledge of [3].

In distributive L with 0, the polar sets

$$J^\perp=\{x\colon \text{for all } j\in J\,,\ x\wedge j\leqslant 0\}$$

are lz-ideals. For left segments S and T, we have

$$(S^{\perp\perp} \cap T^{\perp\perp}) \wedge (S \cap T)^{\perp} \wedge S \wedge T \subset \{0\}^{-}$$

since $S \wedge T \subset S \cap T$. Thus

$$(S^{\perp\perp} \cap T^{\perp\perp}) \wedge (S \cap T)^{\perp} \wedge S \subset T^{\perp} \cap T^{\perp\perp} = \{0\}^{-},$$

so

$$(S^{\perp\perp} \cap T^{\perp\perp}) \wedge (S \cap T)^{\perp} \subset S^{\perp} \cap S^{\perp\perp} = \{0\}^{-},$$

whence $S^{\perp\perp} \cap T^{\perp\perp} = (S \cap T)^{\perp\perp}$. In particular, the distributive law

$$I \cap (J \cup K)^{\perp\perp} = \big((I \cap J) \cup (I \cap K)\big)^{\perp\perp}$$

holds for polar sets. It follows easily that the polar sets form a complete Boolean algebra. We conclude as in [3] that L has a *uniform structure space* $\beta(L)$, a compact Hausdorff space provided with a natural dense continuous mapping w: $\pi(L) \to \beta(L)$. Then w must be constant on closures of points, and it induces continuous t: $\varkappa(L) \to \beta(L)$ such that $tv = w$. β is functorial for a larger category of homomorphisms than π and $\varkappa$, for those h: $L \to L'$ such that $h(L)$ is not contained in the ideal join of two non-supplementary polar ideals (p. 67 of [3]). For example, on unitary f-rings, it suffices if h takes the value 1.

We say no more of general lattices. For an abelian l-group G, there appear to be two or more other structure spaces present; $\beta(G)$ exactly as defined in [3], and the T_0 space $\overline{\mu}(G)$ of prime l-group ideals. ($\mu(G)$ denotes the completely regular subspace of minimal prime ideals.) In this setting, of zero f-rings, 1.4 can be sharpened; there is a largest l-group ideal $r(I)$ contained in a prime lz-ideal I, and it is prime. Largest, because $2|x|$, $2|y|$ in I implies $|x|+|y|$ in I; prime, clearly. Without difficulty one sees that for J in $\overline{\mu}(G)$, $\varrho(J) = J^{-}$ is the smallest prime lz-ideal in $r^{-1}(J)$, and we have a retraction r: $\pi(G) \to \overline{\mu}(G)$ with coretraction ϱ. Since $\varrho(J)$ is smallest, v factors across r by s: $\overline{\mu}(G) \to \varkappa(G)$. s is still a quotient mapping, of the same description as v. Similarly (but for the simpler conclusion) the correspondences like r and ϱ between the two types of polar ideals are mutually inverse and identify the two uniform structure spaces of G. Let $\overline{u} = ts$: $\overline{\mu}(G) \to \beta(G)$.

If one can find a continuous cross-section f: $\beta(G) \to \overline{\mu}(G)$ ($\overline{u}f = 1$ on $\beta(G)$), then $f\big(\beta(G)\big)$ and $sf\big(\beta(G)\big)$ are, of course, homeomorphic with $\beta(G)$. Theorems 4 and 5 of [3] establish such cross-sections by the space of maximal l-group ideals, if G has a strong order unit, and by the space of maximal l-ring ideals, if G is an f-ring with a dominant element. In the former case it is trivial that $sf\big(\beta(G)\big)$ is all of $\varkappa(G)$; in the latter case the argument of 1.5 applies.

Recalling the nature of the uniform ideals $\mathfrak{J} \in \beta(G)$ and the mapping $\overline{u}$ (viz. $\overline{u}(P) \subset 2^{P}$), we can supplement this information. Call a cross-section f: $\beta(G) \to \overline{\mu}(G)$ *strong* if $x > 0$ $\big(\mathrm{mod}\, f(\mathfrak{J})\big)$ implies that for some $J \in \mathfrak{J}$, $x \geqslant 0$ (mod J).

2.1. *For an abelian l-group G, if $\overline{u}$ has a strong continuous cross-section then t: $\varkappa(G) \to \beta(G)$ is a homeomorphism.*

Proof. Given a strong cross-section f, any prime lz-ideal P must be bounded above modulo $fw(P)$, for if P had elements $p > x$ $\big(\mathrm{mod}\, fw(P)\big)$ for arbitrary x, $p \geqslant x$ (mod J) and $J \subset P$ would yield $x \in P$. Hence if $w(P) = w(Q)$, both P and Q are bounded modulo $fw(P)$, and they are in the same K-class. So t is one-to-one. If there is a continuous cross-section of $\overline{u}$, t is homeomorphic.

These cross-section arguments apply over any subspace of $\beta(G)$ on which the cross-sections exist.

We conclude with four counterexamples.

2.2. *In 2.1, "continuous" cannot be omitted.*

Proof sketch. Let V be a lexicographically ordered real vector space on the basis $e_1 < e_2 < \ldots$ Let X consist of the non-negative rationals which have the form m or $m+n^{-1}$, m and n integral, with the natural topology and order. Among the V-valued functions on X note those f_i which have the value e_i outside the open $(i+1)^{-1}$ neighborhoods of integers and vanish inside; note that $f_{i+1} > cf_i$ for all scalar c. Let G consist of those locally constant V-valued functions on X which are finally equal to a finite linear combination of the f_i.

$\beta(G)$ is just the one-point compactification of X. For, if $f_i = g+h$, one of g and h is non-zero on all but a compact part of support of f_i; thus of two supplementary polar ideals, the common zeros of the members of one must form a bounded set. To construct a strong cross-section, take the ideals M_x of functions vanishing at x and M_∞ of functions finally zero. So t is one-to-one. But in $\varkappa(G)$, ∞ is not a limit point of the integers; the kernel of the K-class at any integer contains each f_i, but each member of the K-class at ∞ is finally bounded.

2.3. *The restriction $s|\mu(G)$ need not be a quotient map.*

Proof. Take the same G, and note that the minimal prime ideal at any point x of $X \cup \{\infty\}$ is unique and is M_x. f_1 is in every minimal prime ideal except those at $m+\frac{1}{2}$ and at ∞; in particular, the (inverse) set of all $m+\frac{1}{2}$ is closed in $\mu(G)$, but its image is not.

2.4. *In 2.1, "strong" cannot be omitted.*

The details are much as in 2.2. Use the same V, let $X = [0,1]$, and take the functions $f = \sum f_i(x)e_i$ with f_i real-valued continuous and constant on $[i^{-1}, 1]$. There are no supplementary pairs of polar ideals, and $\beta(G)$ is a single point; but there are two K-classes, living at 0 and elsewhere.

2.5. *The Kaplansky space $\varkappa(G)$ of an abelian l-group G need not be a T_0 space.*

Do 2.4 on two halves of a circle.

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CASE WESTERN RESERVE UNIVERSITY
CLEVELAND STATE UNIVERSITY

Reçu par la Rédaction le 27. 7. 1968

Lawvere's elementary theories and polyadic and cylindric algebras

by

Aubert Daigneault (Montreal)

À la mémoire de Léon Leblanc
mon regretté ami et collègue

Contents

Introduction *

In his short papers [8], [9], [10] and some talks, Lawvere has presented a new approach to the problem of the algebraization of first order Logic in which elementary theories become categories. It is the purpose of this paper to describe the exact relationship that the new approach bears to the older one constituted by the theory of polyadic and cylindric algebras. We hope thus to call attention to Lawvere's important contribution to Algebraic Logic. (Throughout the paper, we shall mean by "polyadic algebra", locally finite polyadic algebra with equality and a fixed infinite set of variables. "Cylindric algebra" has a similarly restricted meaning).

(*) This paper is an amended version of a paper read at the conference on the Construction of Models for Axiomatic Systems in Warsaw, August 26–September 1, 1968.