On functions of generalized bounded ω-variation

by

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1. Introduction. Let $\omega(x)$ be non-decreasing on the closed interval $[a, b]$. Outside the interval it is defined by $\omega(x) = \omega(a)$ for $x < a$ and $\omega(x) = \omega(b)$ for $x > b$. Let S denote the set of points of continuity of $\omega(x)$ and $D = [a, b] - S$. Let $\mathfrak{U}$ [5] denote the class of functions $f(x)$ defined as follows: $f(x)$ is defined on $[a, b]\cdot S$ and is continuous at each point of $[a, b]\cdot S$ relative to the set S. If $x_0 \in D$, $f(x)$ tends to a (finite) limit as x tends to x_0+ and x_0- over the points of the set S. These limits are denoted by $f(x_0+)$ and $f(x_0-)$ respectively. When $x < a$, $f(x) = f(a+)$ and $f(x) = f(b-)$ when $x > b$. $f(x)$ may or may not be defined at the points of D.

Let the function $f(x)$ be defined on $[a, b]$ and be in class $\mathfrak{U}$. $f(x)$ is absolutely continuous [5] relative to ω, AC-ω, on $E \subseteq [a, b]$ if given any $\varepsilon > 0$ there is a $\delta > 0$ such that for every set of pairwise disjoint open intervals (x_i, x_i') on $[a, b]$ with $x_i, x_i' \in E$, we have

$$\sum_i |f(x_i'+) - f(x_i-)| < \varepsilon$$

whenever $\sum_i \{\omega(x_i'+) - \omega(x_i-)\} < \delta$. $f(x)$ is a generalized absolutely continuous function [5] relative to ω, ACG-ω on $[a, b]$ if the interval can be expressed as the union of a countable family of closed sets on each of which $f(x)$ is AC-ω. Any set of points $a \leqslant x_0 < x_1 < x_2 < \ldots < x_n \leqslant b$ with $\omega(x_{i-1}) < \omega(x_i)$ $(i = 1, 2, \ldots, n)$ is an ω-subdivision of $[a, b]$. $f(x)$ is of bounded variation ([1], [2]) relative to ω, BV-ω on $[a, b]$ if the least upper bound $V_\omega(f; a, b)$ of the sums $\sum_{i=1}^{n} |f(x_i+) - f(x_{i-1}-)|$ is finite for all possible ω-subdivisions $x_0, x_1, x_2, \ldots, x_n$ of $[a, b]$.

Let A be a subset of $[a, b]$, x be any point and

$$v = [x, x+h] \quad (h > 0,\ x+h \in S).$$

Define $d(x, h)$ as follows:

$$d(x, h) = \omega^*(Av)/|v|_\omega \text{ if } |v|_\omega \neq 0 \quad \text{and} \quad d(x, h) = 0 \text{ if } |v|_\omega = 0,$$

where $\omega^*(E)$ denotes the outer ω-measure [5] and $|E|_\omega$ denotes the ω-measure of the set E. The $\lim_{h\to 0} d(x, h)$ $(x+h \in S)$, whenever it exists, is the right ω-density [3] of A at x. Similarly the left ω-density is defined. If the left and right ω-densities of A at x are equal, the common value is the ω-density of A at x.

Two sets A and B are ω-separated [3] if for every $\varepsilon > 0$ there are open sets $G_1 \supseteq A$ and $G_2 \supseteq B$ such that $|G_1 G_2|_\omega < \varepsilon$. A function $f(x)$ defined on $[a, b]$ possesses the property (N_ω) [4] on $E \subseteq [a, b]$ if $mf(e) = 0$ for every set $e \subseteq E$ with $|e|_\omega = 0$.

We now introduce the following definitions.

Definition 1.1. Let $f(x)$ be defined on $[a, b]$ and be in class $\mathfrak{U}$. $f(x)$ is of *bounded variation relative to* ω, BV-ω, *on* $E \subseteq [a, b]$ if the least upper bound $V_\omega(f; E)$ of the sums

$$\sum_{i=1}^{n} |f(x_i+) - f(x_{i-1}-)|$$

is finite for all possible ω-subdivisions $x_0, x_1, x_2, \ldots, x_n$ of $[a, b]$ with $x_i \in E$ $(i = 0, 1, 2, \ldots, n)$. $f(x)$ is of generalized bounded variation relative to ω, BVG-ω, on $[a, b]$ if the interval can be expressed as the union of a countable family of closed sets on each of which $f(x)$ is BV-ω.

Definition 1.2. Let $f(x)$ be defined on $[a, b]$ and be in class $\mathfrak{U}$ and let $x \in [a, b]$. For any point ξ $(\neq x)$ in S we define $\psi(x, \xi)$ by the relations:

$$\psi(x, \xi) = \begin{cases} \dfrac{f(\xi) - f(x-)}{\omega(\xi) - \omega(x-)}, & \xi > x,\ \omega(\xi) - \omega(x-) \neq 0, \\ \dfrac{f(\xi) - f(x+)}{\omega(\xi) - \omega(x+)}, & \xi < x,\ \omega(\xi) - \omega(x+) \neq 0, \\ 0, & \omega(\xi) - \omega(x\pm) = 0. \end{cases}$$

If $\psi(x, \xi)$ tends to a limit as ξ tends to x over S except for a subset of S of ω-density zero at x, then the limit is the *approximate ω-derivative of* $f(x)$ *at* x and is denoted by $(\mathrm{ap}) f'_\omega(x)$. (If $\psi(x, \xi)$ tends to a limit as ξ tends to x over S, then this limit is the ω-derivative [5] of $f(x)$ at x. It is denoted by $f'_\omega(x)$.)

In the present paper we show that

(i) if $f(x)$ is ACG-ω on $[a, b]$, then it possesses the property (N_ω) on $[a, b]$,

(ii) if $f(x)$ is BVG-ω on $[a, b]$ and possesses the property (N_ω) on $[a, b]$, then it is ACG-ω on $[a, b]$ and

(iii) if $f(x)$ is BVG-ω on $[a, b]$, then $(\mathrm{ap}) f'_\omega(x)$ exists finitely at ω-almost all points of $[a, b]$.

We require the following known results.

Theorem 1.1 ([3], Th. 3.2). *Let A and B two subsets of $[a, b]$. If they are ω-separated, then at ω-almost all points of one set the ω-density of the other is zero.*

Theorem 1.2. ([3], Th. 6.2). *If $f(x)$ is BV-ω on $[a, b]$, then $f'_\omega(x)$ exists finitely at ω-almost all points of $[a, b]$.*

Theorem 1.3. ([4], Th. 3.5). *If $f(x)$ is BV-ω on $[a, b]$ and possesses the property (N_ω) on $[a, b]$, then it is AC-ω on $[a, b]$.*

Theorem 1.4. ([4]. Th. 3.2). *If $f(x)$ is AC-ω on $[a, b]$, then it possesses the property (N_ω) on $[a, b]$.*

2. Preliminary lemmas. Let F be a closed subset of $[a, b]$ and $[a_0, b_0]$ be the smallest closed interval containing the set F. Let $f(x)$ be BV-ω on F.

Lemma 2.1. *If $\{(a_i, b_i)\}$ be any set of pairwise disjoint open intervals on $[a_0, b_0]$ with $a_i, b_i \in F$ and increasing end-points $(a_1 < a_2 < a_3 < \ldots)$ such that $\omega(b_i) < \omega(a_{i+1})$ $(i = 1, 2, \ldots)$, then each of the following is finite:*

$$\text{(i)}\ \sum_i |f(a_i+) - f(b_i+)|, \qquad \text{(ii)}\ \sum_i |f(a_i-) - f(b_i-)|,$$

$$\text{(iii)}\ \sum_i |f(a_i+) - f(b_i-)|, \qquad \text{(iv)}\ \sum_i |f(a_i-) - f(b_i+)|.$$

Proof: Since $f(x)$ is BV-ω on F, $f(x\pm)$ is bounded on F (cf. [1], Th. 2). So there is a positive constant K such that $|f(x\pm)| \leqslant K$ for all $x \in F$. Let n be any positive integer. We have

$$|f(a_1+) - f(b_1+)| \leqslant 2K,$$

$$|f(a_2+) - f(b_2+)| \leqslant |f(b_1-) - f(a_2+)| + |f(b_1-) - f(b_2+)|,$$

$$|f(a_3+) - f(b_3+)| \leqslant |f(b_2-) - f(a_3+)| + |f(b_2-) - f(b_3+)|,$$

$$|f(a_4+) - f(b_4+)| \leqslant |f(b_3-) - f(a_4+)| + |f(b_3-) - f(b_4+)|,$$

. . . .

Since $a_0 \leqslant b_1 < a_2 < b_3 < a_4 < \ldots \leqslant b_0$, $a_0 < b_2 < a_3 < b_4 < a_5 < \ldots \leqslant b_0$ and $a_0 \leqslant b_1 < b_2 < b_3 < \ldots < b_n \leqslant b_0$ are ω-subdivisions of $[a_0, b_0]$, we obtain

$$\sum_{i=1}^{n} |f(a_i+) - f(b_i+)| \leqslant 3V_\omega(f; F) + 2K.$$

Since n is arbitrary, result (i) follows.

The proofs of (ii), (iii) and (iv) are analogous and each of the quantities is $\leqslant 3V_\omega(f;F)+2K$.

Lemma 2.2. *Let $\{(a_i, b_i); i \in I\}$ be any set of pairwise disjoint open intervals on $[a_0, b_0]$ with $a_i, b_i \in F$ and increasing end-points, where $I = \{1, 2, \ldots, n\}$ or $I = \{1, 2, 3, \ldots\}$. If $\omega(a_i) < \omega(b_{i+1}) (i \in I)$, then each of the following is finite:*

$$\text{(i)} \sum_{i\in I} |f(a_i+)-f(b_i+)|, \qquad \text{(ii)} \sum_{i\in I} |f(a_i-)-f(b_i-)|,$$

$$\text{(iii)} \sum_{i\in I} |f(a_i+)-f(b_i-)|, \qquad \text{(iv)} \sum_{i\in I} |f(a_i-)-f(b_i+)|.$$

Proof: We divide the set I into three parts A, B, C such that $A = \{1, 4, 7, \ldots\}$, $B = \{2, 5, 8, \ldots\}$ and $C = \{3, 6, 9, \ldots\}$. Then each of the sets of intervals $\{(a_i, b_i); i \in A\}$, $\{(a_i, b_i); i \in B\}$ and $\{(a_i, b_i); i \in C\}$ satisfies the conditions of Lemma 2.1. We have

$$\sum_{i\in I} |fa_i+)-f(b_i+)| = \sum_{i\in A} |f(a_i+)-f(b_i+)| + \sum_{i\in B} + \sum_{i\in C}.$$

From Lemma 2.1, it follows that

$$\sum_{i\in I} |f(a_i+)-f(b_i+)| \leqslant 3[3V_\omega(f;F)+2K].$$

The proofs of (ii), (iii) and (iv) are analogous and each of the quantities is $\leqslant 3[3V_\omega(f;F)+2K]$.

Lemma 2.3. *If $f(x)$ is AC-ω on a set $E \subseteq [a, b]\cdot S$, then given any $\varepsilon > 0$ there is a $\delta > 0$ such that for every set of pairwise disjoint open intervals (a_i, b_i) on $[a, b]$,*

$$\sum_i (u_i - l_i) < \varepsilon \quad \textit{whenever} \quad \sum_i \{\omega(b_i-) - \omega(a_i+)\} < \delta,$$

where u_i and l_i are respectively the supremum and the infimum of $f(x)$ on $[a_i, b_i]\cdot E$ ($\neq$ void) and the summation is taken over all those i for which $[a_i, b_i]\cdot E$ is non-void.

Proof. Since $f(x)$ is AC-ω on E, it is BV-ω on E (cf. [1], Th. 5) and therefore bounded on E (cf. [1], Th. 2). Choose $\varepsilon > 0$ arbitrarily. Then there is a $\delta > 0$ such that for any set of pairwise disjoint open intervals (x_i, x_i') on $[a, b]$ with $x_i, x_i' \in E$,

$$\sum_i |f(x_i')-f(x_i)| < \varepsilon \quad \text{whenever} \quad \sum_i \{\omega(x_i') - \omega(x_i)\} < \delta.$$

Let $\{(a_i, b_i)\}$ be any set of pairwise disjoint open intervals on $[a, b]$ with $\sum_i \{\omega(b_i-) - \omega(a_i+)\} < \delta$. There are points ξ_i, η_i on $[a_i, b_i]\cdot E$ ($\neq$ void) such that

$$|f(\xi_i) - u_i| < \varepsilon/2^{i+2} \quad \text{and} \quad |f(\eta_i) - l_i| < \varepsilon/2^{i+2} \quad (i = 1, 2, \ldots).$$

Denote by α_i, β_i the maximum and the minimum of ξ_i and η_i. Then $\alpha_i, \beta_i \in E$ and $\sum_i \{\omega(\beta_i) - \omega(\alpha_i)\} \leqslant \sum_i \{\omega(b_i-) - \omega(a_i+)\} < \delta$. Therefore

$$\sum_i (u_i - l_i) \leqslant \sum_i \{|f(\xi_i) - f(\eta_i)| + |f(\xi_i) - u_i| + |f(\eta_i) - l_i|\}$$
$$\leqslant \sum_i |f(\alpha_i) - f(\beta_i)| + \varepsilon\cdot \sum_i 1/2^{i+1} < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

3. Results on BVG-ω functions.

Theorem 3.1. *Let $f(x)$ be defined on $[a, b]$ and be in class $\mathfrak{U}$. If it is BV-ω on a closed set $F \subseteq [a, b]$, then there is a function $g(x)$ in $\mathfrak{U}$ such that $g(x)$ is BV-ω on $[a, b]$ and $g(x) = f(x)$ on F.*

Proof. Let $[a_0, b_0]$ be the smallest closed interval containing the set F. Then the set $G = [a_0, b_0] - F$ is open. Let $G = \sum_i (\alpha_i, \beta_i)$ where the intervals (α_i, β_i) are pairwise disjoint. We define the function $g(x)$ as follows:

$$g(x) = \begin{cases} f(x) & \text{for all } x \in F, \\ f(\alpha_i+) + \dfrac{\omega(x) - \omega(\alpha_i+)}{\omega(\beta_i-) - \omega(\alpha_i+)} \{f(\beta_i-) - f(\alpha_i+)\} & \\ \qquad \text{for } \alpha_i < x < \beta_i \text{ if } \omega(\beta_i-) \neq \omega(\alpha_i+), & \\ f(\alpha_i+) + \dfrac{x - \alpha_i}{\beta_i - \alpha_i} \{f(\beta_i-) - f(\alpha_i+)\} & \\ \qquad \text{for } \alpha_i < x < \beta_i \text{ if } \omega(\beta_i-) = \omega(\alpha_i+), & \\ f(a_0-) & \text{for } x < a_0, \\ f(b_0+) & \text{for } x > b_0. \end{cases}$$

Then clearly $g(x)$ belongs to the class $\mathfrak{U}$ and $g(x\pm) = f(x\pm)$ for all $x \in F$. Let $a_0 \leqslant x_0 < x_1 < x_2 < \ldots < x_n \leqslant b_0$ be any ω-subdivision of $[a_0, b_0]$ and let

$$V = \sum_{i=1}^{n} |g(x_i+) - g(x_{i-1}-)|.$$

If $x_0, x_1, x_2, \ldots, x_n \in F$, then we have

$$V = \sum_{i=1}^{n} |f(x_i+) - f(x_{i-1}-)| \leqslant V_\omega(f; F).$$

Let $x_{r-1} \in F$, $x_r, x_{r+1}, \ldots, x_{k-1} \in (\alpha_s, \beta_s)$, $x_k, x_{k+2} \in F$ and $x_{k+1} \in (\alpha_t, \beta_t)$ where $\omega(\beta_t-) = \omega(\alpha_t+)$. Then

$$\sum_{i=r}^{k+2} |g(x_i+)-g(x_{i-1}-)|$$
$$= |f(x_{r-1}-)-g(x_r+)| + \sum_{i=r+1}^{k-1} |g(x_i+)-g(x_{i-1}-)| +$$
$$+ |f(x_k+)-g(x_{k-1}-)| + |f(x_k-)-g(x_{k+1}+)| +$$
$$+ |g(x_{k+1}-)-f(x_{k+2}+)|$$
$$\leqslant |f(x_{r-1}-)-f(\alpha_s+)| + 4|f(\alpha_s+)-f(\beta_s-)| + |f(\alpha_s+)-f(x_k+)| +$$
$$+ |f(x_k-)-f(\alpha_t+)| + 2|f(\alpha_t+)-f(\beta_t-)| + |f(\alpha_t+)-f(x_{k+2}+)| .$$

These considerations show that

$$V \leqslant \sum_i |f(a_{i1}-)-f(a'_{i1}+)| + \sum_i |f(a_{i2}+)-f(a'_{i2}+)| +$$
$$+ 4\sum_i |f(a_{i3}+)-f(a'_{i3}-)| ,$$

where the sets $\{(a_{ij}, a'_{ij});\ i = 1, 2, \ldots\}$ $(j = 1, 2, 3)$ of intervals are all pairwise disjoint and satisfy the conditions of the Lemma 2.2. Therefore by the same lemma,

(1) $V \leqslant 18[3V_\omega(f; F)+2K]$, where $K = \sup\{|f(x\pm)|;\ x \in F\}$.

Thus in any case relation (1) holds. Since $x_0, x_1, \ldots, x_n$ is an arbitrary ω-subdivision of $[a_0, b_0]$ it follows that $g(x)$ is BV-ω on $[a_0, b_0]$. Since $g(x)$ is constant on $[a, a_0)$ and $(b_0, b]$ it is BV-ω on $[a, a_0]$ and $[b_0, b]$. Hence by Theorem 3 [1], $g(x)$ is BV-ω on $[a, b]$. This proves the Theorem.

THEOREM 3.2. *Let $f(x)$ be BV-ω on a closed set $F \subseteq [a, b]$. If $f(x)$ is constant on each open interval (α, β) complementary to F where $\omega(x)$ is constant and if $f(x)$ possesses the property (N_ω) on F, then $f(x)$ is AC-ω on F.*

Proof. Let $[a_0, b_0]$ be the smallest closed interval containing the set F and let $[a_0, b_0]-F = \sum_i (\alpha_i, \beta_i)$, where the intervals are pairwise disjoint. Let $g(x)$ be the function as defined in Theorem 3.1. Then $g(x)$ is constant on each $[\alpha_i, \beta_i]$ where $\omega(\beta_i-) = \omega(\alpha_i+)$. So $g(x)$ is AC-ω on each such interval. Also from the definition of $g(x)$ we see that $g(x)$ is AC-ω on each $[\alpha_i, \beta_i]$ where $\omega(\beta_i-) \neq \omega(\alpha_i+)$. Therefore by Theorem 1.4, $g(x)$ possesses the property (N_ω) on each $[\alpha_i, \beta_i]$. Since $g(x) = f(x)$ on F, $g(x)$ possesses the property (N_ω) on F. So $g(x)$ possesses the property (N_ω) on $[a, b]$. By Theorem 1.3, $g(x)$ is AC-ω on $[a, b]$. Hence $f(x)$ is AC-ω on F.

THEOREM 3.3. *If $f(x)$ is BVG-ω on $[a, b]$ and possesses the property (N_ω) on $[a, b]$, then it is ACG-ω on $[a, b]$.*

Proof. Since $f(x)$ is BVG-ω on $[a, b]$ we can express the interval as the union of a countable family of closed sets F_i, $[a, b] = \sum_i F_i$, on each of which $f(x)$ is BV-ω. Let (α, β) be any open interval on which $\omega(x)$ is constant. Then $f(x)$ is continuous on (α, β). Let $e = [c, d] \subseteq (\alpha, \beta)$. Then $f(e)$ is either an interval or consists of a single point. Now $|e|_\omega = 0$ and $f(x)$ possesses the property (N_ω) on $[a, b]$. So $mf(e) = 0$, which shows that $f(e)$ consists of a single point. This implies that $f(x)$ is constant on $[c, d]$ and therefore on (α, β). From Theorem 3.2, it follows that $f(x)$ is AC-ω on each F_i. Hence $f(x)$ is ACG-ω on $[a, b]$.

THEOREM 3.4. *If $f(x)$ is ACG-ω on $[a, b]$, then $f(x)$ possesses the property (N_ω).*

Proof. Let E be any subset of $[a, b]$ with $|E|_\omega = 0$. Then $E \subseteq [a, b]\cdot S$. Since $f(x)$ is ACG-ω on $[a, b]$ we can express the interval as the union of a countable family of closed sets F_i, $[a, b] = \sum_i F_i$, on each of which $f(x)$ is AC-ω. Write $E_i = EF_i$ $(i = 1, 2, 3, \ldots)$. Then $E = \sum_i E_i$ and $f(E) = \sum_i f(E_i)$. We show that $mf(E_i) = 0$ for each i which will give that $mf(E) = 0$ and the proof of theorem will be complete.

Let i be any positive integer. Consider the set E_i. Write $A = E_i(a, b)$. Choose $\varepsilon > 0$ arbitrarily. Since $f(x)$ is AC-ω on A, by Lemma 2.3, there is a $\delta > 0$ such that for any set of pairwise disjoint open intervals $\{(a_k, b_k)\}$ on $[a, b]$,

(2) $$\sum_k (u_k - l_k) < \varepsilon \quad \text{whenever} \quad \sum_k \{\omega(b_k-)-\omega(a_k+)\} < \delta ,$$

where u_k, l_k have meanings as in Lemma 2.3. Since $|A|_\omega = 0$ there is an open set $G \supseteq A$ and $G \subseteq (a, b)$ such that $|G|_\omega < \delta$. Let $G = \sum_k (a_k, b_k)$, where the intervals are pairwise disjoint. Then $\sum_k \{\omega(b_k-)-\omega(a_k+)\} < \delta$. Since $A = \sum_k [a_k, b_k]\cdot A$, we have $f(A) = \sum_k f([a_k, b_k]\cdot A)$. So,

(3) $$m^*f(A) \leqslant \sum_k m^*f([a_k, b_k]\cdot A) \leqslant \sum_k (u_k - l_k) < \varepsilon \quad \text{(using (2))} .$$

Since $\varepsilon > 0$ is arbitrary, from (3) we obtain $m^*f(A) = 0$. If $B = \{a, b\}$, then $E_i \subseteq A + B$ and $f(E_i) \subseteq f(A) + f(B)$. So $m^*f(E_i) \leqslant m^*f(A) + m^*f(B) = 0$.

THEOREM 3.5. *If $f(x)$ is BVG-ω on $[a, b]$, then $(\mathrm{ap})f'_\omega(x)$ exists finitely at ω-almost all points of $[a, b]$.*

Proof. Since $f(x)$ is BVG-ω on $[a, b]$, we can express the interval as the union of a countable family of closed sets F_i, $[a, b] = \sum_i F_i$, on each of which $f(x)$ is BV-ω. Consider the set F_i, where i is any positive integer. By Theorem 3.1, there is a function $g_i(x)$ in calss $\mathfrak{U}$ such that $g_i(x)$ is BV-ω on $[a, b]$ and $g_i(x) = f(x)$ for all $x \in F_i$. Denote by F'_i the set of points of F_i where the ω-derivative of $g_i(x)$ exists finitely. Then by Theorem 1.2, $|F_i - F'_i|_\omega = 0$. Let E_i denote the set of points of F'_i where the ω-density of $S - F_i$ is zero. Clearly F'_i and $S - F_i$ are ω-separated. So by Theorem 1.1, $|F'_i - E_i|_\omega = 0$. We have $F_i - E_i = (F_i - F'_i) + (F'_i - E)$. So $|F_i - E_i|_\omega = 0$. Let a be any point of E_i. Since $g_i(x) = f(x)$ on SF_i and the ω-derivative of $g_i(x)$ exists finitely at a, it follows that $(\mathrm{ap})f'_\omega(a)$ exists finitely. Since a is arbitrary, $(\mathrm{ap})f'_\omega(x)$ exists finitely at each point of E_i. Write $E = \sum_i E_i$. Then, at each point of E, $(\mathrm{ap})f'_\omega(x)$ exists finitely. Now $[a, b] - E \subseteq \sum_i (F_i - E_i)$. So $\omega^*([a, b] - E) \leqslant \sum_i \omega^*(F_i - E_i) = 0$. This proves the Theorem.

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References

[1] P. C. Bhakta, *On functions of bounded ω-variation*, Rev. Math. Univ. Parma (2) 6 (1965).

[2] — *On functions of bounded ω-variation* — II, J. Aust. Math. Soc., Vol. V, part 3, (1965), pp. 380–387.

[3] M. C. Chakrabarty, *Some results on ω-derivatives and* BV-ω *functions*, J. Aust. Math. Soc., Vol. IX, parts 3 and 4 (1969), pp. 345-360.

[4] — *Some results on* AC-ω *functions*, Fund. Math. 64 (1969), pp. 219–230.

[5] R. L. Jeffery, *Generalized integrals with respect to functions of bounded variation*, Canad. J. Math. 10 (1958), pp. 617–628.

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Structure spaces of lattices

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Introduction. This paper gives a simpler proof of the functoriality of the structure space of maximal l-ideals of an f-ring with unit. Like the previous proof [3], this one depends (when analyzed) on a different, visibly functorial construction that turns out to yield the maximal ideal space. Both constructions generalize to distributive lattices with base point. Hence the maximal ideal space of a unitary f-ring is determined by the underlying lattice. This was previously known for commutative semisimple unitary f-rings [6].

Kaplansky's original proof that the lattice of continuous functions $C(X)$ on a compact Hausdorff space X determines X [4], and its generalizations until now, have used *ad hoc* constructions to wring the space from the lattice. After an *ad hoc* beginning (the quickest), we exhibit the following natural structure. A based distributive lattice L has a T_0 space $\pi(L)$ of prime ideals containing 0. The present construction, and Kaplansky's, form the finest quotient space $\varkappa(L)$ in which the closure of every point is collapsed to a point. The more radical treatment of [3] yields a compact Hausdorff space $\beta(L)$, which, unlike π and $\varkappa$, is functorial for a category of lattice homomorphisms containing the unitary f-ring homomorphisms. Obvious mappings run $\pi(L) \to \varkappa(L) \to \beta(L)$. The easiest way to establish coincidence of $\varkappa(L)$ and $\beta(L)$ (and a space of prime ideals) is to find a subspace of $\pi(L)$ continuously cross-sectioning $\pi(L) \to \beta(L)$ and mapping surjectively to $\varkappa(L)$; that is what the maximal ideal space of a unitary f-ring does, and also the maximal ideal space of an abelian l-group with strong order unit [3]. A continuous cross-section is not enough (for a vector lattice). We find a sufficient additional condition, for abelian l-groups, to the effect that group elements positive at a point are non-negative on a $\beta(L)$-neighborhood.

1. Maximal ideal spaces. Let $\mathcal{M}(A)$ be the space of maximal l-ideals of a unitary f-ring. (It is compact Hausdorff [2]; compact by the usual maximality argument, Hausdorff by a simple argument due to Gillman [1] depending on the fact that for $M \in \mathcal{M}(A)$, A/M is totally