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UNIVERSITY OF IDAHO
Moscow, Idaho

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Measurable cardinals and analytic games

by

Donald A. Martin (New York)

Introduction. A subset P of ω^ω is *determinate* if, in the sense of [5] the game $G_\omega(P)$ is determined. The assumption that every projective set is determinate implies that every projective set is Lebesgue measurable (see [6]) and leads to a complete solution to the problem of reduction and separation principles in the classical and effective projective hierarchies [1], [4]. Because if these and other consequences it would be pleasant to have a proof that every projective set is determinate. The best available result is that every $F_{\sigma\delta}$ is determinate [2]. It is not provable in Zermelo–Fraenkel set theory that every analytic ($\boldsymbol{\Sigma}_1^1$) set is determinate [5]. (1)

We assume the existence of a measurable cardinal and prove that every analytic set is determinate. Our proof is fairly simple and makes a very direct use of the large cardinal assumption (we present it in terms of a Ramsey cardinal) and the fact that open games are determined. We believe that larger cardinals will yield a generalization of our proof to all projective sets. The assumption that measurable cardinals exist is known not to imply even that all $\boldsymbol{\Delta}_2^1$ sets are determinate. (This follows from [1], [4] and work of Silver.)

§ 1. Definitions. (For more information on the analytical hierarchy see [7], [8]; on infinite games see [5]; on large cardinals see [10], [11].)

Let ω be the set of all natural numbers. If $f\colon \omega \to A$, the function $\bar{f}$ is defined by setting $\bar{f}(n)$ equal to the sequence $\langle f(0), f(1), \ldots, f(n-1)\rangle$ of the first n values of f. Let Seq be the set of all finite sequences of natural numbers. Let $n \to k_n$ be some enumeration of Seq with the property that k_n has length $\leqslant n$. The *Kleene–Brouwer ordering* of Seq is defined by

$$\bar{f}(m) < \bar{g}(n) \leftrightarrow \begin{cases} \bar{f}(m) \text{ is a proper extension of } \bar{g}(n)\,, \\ \text{or at the least } p \text{ for which } f(p) \neq g(p)\,, \\ f(p) < g(p)\,. \end{cases}$$

(1) Harvey Friedman (unpublished) has shown that the determinateness of Borel sets cannot be proved in Zermelo set theory. Whether it can be proved in Zermelo-Fraenkel set theory remains open.

Let $R(i,j,k)$ be a relation in Seq^3 and let f and g map ω into ω. A sequence $\bar{k}(n)$ is *secured* with respect to f, g, and R if

$$(\exists m \leqslant n) R\big(\bar{f}(m), \bar{g}(m), \bar{k}(m)\big) .$$

A fundamental fact is that $(h)(\exists n) R\big(\bar{f}(n), \bar{g}(n), \bar{h}(n)\big)$ holds if and only if the Kleene–Brouwer ordering of the sequences unsecured with respect to f, g, and R is a well-ordering.

Let $\varkappa$ be an uncountable cardinal number. Let $\varkappa^{[n]}$ be the set of all subsets of $\varkappa$ of cardinality n. Let $\mathcal{F}$ be a set such that, if $F \in \mathcal{F}$, there is an $n < \omega$ such that $F\colon \varkappa^{[n]} \to \omega$. $X \subseteq \varkappa$ is a *homogeneous set* for $\mathcal{F}$ if, for each $F \in \mathcal{F}$ with $F\colon \varkappa^{[n]} \to \omega$ and elements a and b of $\varkappa^{[n]}$ which are subsets of X, $F(a) = F(b)$. If a is an ordinal number, $\varkappa \to (a)^{<\omega}$ means that, for every countable $\mathcal{F}$, there is a homogeneous set for $\mathcal{F}$ of order type a. $\varkappa$ is a *Ramsey cardinal* if $\varkappa \to (\varkappa)^{<\omega}$. Every measurable cardinal is Ramsey.

Let A and B be sets and let $C \subseteq A^\omega \times B^\omega$. The (Gale–Stewart) infinite game defined by A, B, and C is given as follows: Players I and II move alternately, choosing elements of A and B respectively at each turn. In this way functions $f\colon \omega \to A$ and $g\colon \omega \to B$ are produced. I wins if $\langle f, g\rangle \in C$. I has a winning strategy if there is a function which, given the first n plays of II, gives an $(n+1)$st play for I in such a manner that I wins whatever II plays. A game is *determined* if either I or II has a winning strategy. Let $A = B = \omega$. The game defined by A, B, and C is *analytic* (Σ_1^1) if there is a relation $R(i,j,k)$ in Seq^3 such that

$$C = \big\{\langle f, g\rangle\colon (\exists h)(n) R\big(\bar{f}(n), \bar{g}(n), \bar{h}(n)\big)\big\}$$

(i.e., if C is the projection of a closed set in $(\omega^\omega)^3$ under the product topology.) The game is *Borel* if some R satisfies the condition above and furthermore there is a countable ordinal a such that for no f and g is the Kleene–Brouwer ordering of sequences unsecured with respect to f, g, and $\mathrm{Seq}^3 - R$ a well-ordering of order type $> a$.

§ 2. The determinateness of analytic sets.

THEOREM. (a) *If* $(\exists\varkappa)[\varkappa \to (\omega_1)^{<\omega}]$ *every analytic game is determined.*

(b) *If* $(\exists\varkappa)(a < \omega_1)[\varkappa \to (a)^{<\omega}]$ *every Borel game is determined.*

Proof. Only (a) will be proved, as the prove of (b) is similar. Let $\varkappa \to (\omega_1)^{<\omega}$. Let $R \subseteq \mathrm{Seq}^3$. I and II, moving alternately, produce functions $f\colon \omega \to \omega$ and $g\colon \omega \to \omega$. II wins if

$$(h)(\exists n) R\big(\bar{f}(n), \bar{g}(n), \bar{h}(n)\big) .$$

Call this Game 1. We consider a second game (Game 2). I picks $f\colon \omega \to \omega$ and II picks not only $g\colon \omega \to \omega$ but also $G\colon \omega \to \varkappa$. (At stage n, II selects the ordered pair $\langle g(n), G(n)\rangle$.) Via the enumeration k_n, G induces a map

$G^*\colon \mathrm{Seq} \to \varkappa$. II wins Game 2 if $G^*(k) = 0$ for all k secured with respect to f, g, and R and G^* preserves the Kleene–Brouwer ordering on the unsecured sequences.

LEMMA 1. *Game 2 is determined.*

Proof. This is the Gale–Stewart result for open games [3]. If I has no winning strategy, II makes the least plays at each move such that I still has no winning strategy. Since II wins provided that he has not lost by some finite stage, this strategy wins for II. (There is a concealed use of the Axiom of Choice in this proof which can be eliminated.)

LEMMA 2. *If* II *has a winning strategy for Game* 2, II *has a winning strategy for Game* 1.

Proof. If II wins Game 2, the Kleene–Brouwer ordering on the unsecured sequences is a well-ordering. (The converse of Lemma 2 can be proved without assuming $\varkappa \to (\omega_1)^{<\omega}$ but only that $\varkappa$ is uncountable.)

LEMMA 3. *If* I *has a winning strategy for Game* 2, I *has a winning strategy for Game* 1.

Proof. Let $f(n) = f^{**}\big(\bar{g}(n), \bar{G}(n)\big)$ be a winning strategy for I for Game 2. Let $\bar{f}(n)$ and $\bar{g}(n)$ be any finite sequences. Let $k_{i_1}, \ldots, k_{i_m}$ for $i_j \leqslant n$ be the sequences unsecured with respect to f', g', and R for any f', g' agreeing with $\bar{f}(n)$, $\bar{g}(n)$ respectively. (Since k_j has length $\leqslant j$, its being secured depends only on $\bar{f}(j)$ and $\bar{g}(j)$.) Let $Q \in \varkappa^{[m]}$. There is a unique sequence $\bar{G}(n)$ such that $G(p) = 0$ if k_p is secured and G^* maps $\{k_{i_1}, \ldots, k_{i_m}\}$ into Q so as to preserve the Kleene–Brouwer ordering. We define

$$F_{\bar{f}(n),\bar{g}(n)}\colon \varkappa^{[m]} \to \omega \ \text{ by } \ F_{\bar{f}(n),\bar{g}(n)}(Q) = f^{**}\big(\bar{g}(n), \bar{G}(n)\big)$$

where $\bar{G}(n)$ is the sequence defined above.

Let $\mathcal{F} = \{F_{\bar{f}(n),\bar{g}(n)}\colon \bar{f}(n), \bar{g}(n) \in \mathrm{Seq}\}$. Let X be a homogeneous set for $\mathcal{F}$ order type ω_1. We define a strategy f^* for I for Game 1 inductively by

$$f^*\big(\bar{g}(n)\big) = F_{\bar{f}(n),\bar{g}(n)}(Q)$$

where $\bar{f}(n)$ is the result of applying f^* to the first n plays $g(p)$ and $Q \in \varkappa^{[m]}$ is any subset of X. If f^* is not a winning strategy, there is a play g such that, for the play f given by f^*, the Kleene–Brouwer ordering of the sequences unsecured with respect to f, g, and R is a well-ordering. Let G be such that $G(n) = 0$ for k_n secured and G^* maps the unsecured sequences in an order preserving manner into X. f is then the play according to f^{**} against g and G, and so we have a contradiction.

§ 3. Further results.

Combining our argument with the methods of [11], we can prove that, if $(\exists\varkappa)[\varkappa \to (\omega_1)^{<\omega}]$, then every Σ_1^1 game has a Δ_3^1 winning strategy. (We owe this observation to Solovay.)

For sets A and B, give $A^\omega \times B^\omega \times \omega^\omega$ the product topology, where A, B, and ω are given the discrete topology. Let $C \subseteq A^\omega \times B^\omega$ be the projection of a closed set in $A^\omega \times B^\omega \times \omega^\omega$. Our argument can be used to show that the game defined by A, B, and C is determined, on the assumption that a Ramsey cardinal larger than the cardinals of A and B exists.

In [5], using a theorem of Davis [2], Mycielski shows that, if every analytic game is determined, then every uncountable $\boldsymbol{CA}(\boldsymbol{\Pi}_1^1)$ set has a perfect subset. Our theorem thus gives a new and very different proof of a result of Solovay [12] and Mansfield: If $(\exists\varkappa)[\varkappa \to (\omega_1)^{<\omega}]$ then every uncountable $\boldsymbol{CA}$ set has a perfect subset.

By a theorem of Shoenfield [8], the assertion that all Borel games are determined relativizes to L, the universe of constructible sets. Hence we have that, if $(\exists\varkappa)(a < \omega_1)[\varkappa \to (a)^{<\omega}]$, then "all Borel games are determined" holds in L. We should note that Silver has shown that $(\exists\varkappa)(a < \omega_1)[\varkappa \to (a)^{<\omega}]$ relativizes to L.

§ 4. What games are determined? We believe that the best way of approaching this problem is to see what games one can prove to be determined using plausible large cardinal assumptions. Nevertheless, some guesses may prove useful.

Addison and Moschovakis [1] suggest that *definability* may be the crucial property which guarantees determinateness. Their "Axiom of Definable Determinateness" asserts that, if $A = B = \omega$ and C is ordinal definable from a member of ω^ω, then the game defined by A, B, and C is determined. A problem with this axiom is that there is little hope at present of proving it from large cardinal assumptions or of using anything like its full strength in deducing consequences (in both cases, because of the unmanageable "ordinal definable"). A weaker proposition suggested by Takeuti and by Solovay, which may not have these defects, is that the Axiom of Determinateness holds in the smallest transitive class containing all ordinals and all subsets of ω and satisfying the axioms of set theory.

Another approach is to consider games of arbitrary length. For any ordinal a, sets A, B, and $C \subseteq A^a \times B^a$ define a game of length $2a$ in the obvious way. (See § 7 of [5].) The proposition that all games of length 2 are determined is equivalent to the Axiom of Choice [5]. Give A and B the discrete topology and $A^a \times B^a$ the product topology. If C is open, the game determined by A, B, and C is an *open game of length* $2a$. The proposition that all open games of every length are determined is inconsistent. Even if $A = B = \omega$, the Axiom of Choice can be used to construct an undetermined open game of length ω_1. However, it apparently is possible that every open game of length $< \omega_1$ with $A = B = \omega$ is determined. Let P_n be the proposition that all open games of length $\omega \cdot n$ where $A = B = \omega$ are determined. It is easy to show that, for $n < \omega$, P_{n+1} is equivalent to the assertion that all Σ_n^1 sets are determinate. This suggests that length may well be a sufficient compensation for what we lose when we restrict ourselves to open games.

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THE ROCKEFELLER UNIVERSITY

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