Two complexes that are spines of the three ball

by

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Let $M^3$ be a 3-manifold with 2-sphere boundary. Using the usual terminology, we will call $M^3$ a **fake 3-ball** if $M^3$ has the homotopy groups of a 3-ball, and a **homology ball** if $M^3$ has the homology of a 3-ball. It is well known that there are homology 3-balls that are not 3-balls while the 3-dimensional Poincaré Conjecture (every fake 3-ball is a 3-ball) is unsolved.

A 3-manifold with non-empty boundary can be collapsed to a 2-complex $K$ in the interior of the manifold. Moreover, we may assume, without loss of generality, that $K$ is a normalized spine of the 3-manifold. That is, every one simplex in $K$ is a face of at most three two simplexes [2].

In this paper we develop a criterion on a normalized spine of a homology 3-ball that will ensure that the homology 3-ball is in fact a 3-ball.

Let $K$ be a finite geometric n-complex. The following notation will be used.

$F(K) = \{\sigma^{n-1} | \sigma^{n-1} \text{ is a face of exactly one } \sigma^n \in K \}$,

$S(K) = \{\sigma^{n-1} | \sigma^{n-1} \text{ is a face of more than two } \sigma^n \in K \}$.

That is, $F(K)$ is the union of all $(n-1)$-dimensional free faces of $K$, while $S(K)$ is the set of singular points of $K$.

Then we have:

**Lemma 1.** If $L = S(K) \cup F(K)$, then either $H_n(K, L)$ is non-trivial or $H_{n-1}(K, L)$ has a subgroup of order two.

**Proof.** Suppose that $H_n(K, L)$ is trivial and let $[\sigma_i] \ (i = 1, \ldots, q)$ be the set of all $n$-simplexes in $K$. Using the fact that every $(n-1)$-simplex in $K - L$ is a face of precisely two $n$-simplexes, an easy calculation will show that if $\sigma(\sum_{i=1}^q \sigma_i) = \sum_{i=1}^q \sum_{j=1}^q 2a_i \sigma_i \sigma_j^{-1}$ then $\sum_{i=1}^q \sigma_i^{-1}$ is in fact a non-trivial element of order two in $H_{n-1}(K, L)$. We will call a connected complex $K$ homologically trivial if $H_n(K) = 0$ for all $q > 1$.

**Lemma 2.** If $K$ is a homologically trivial n-complex with $n \geq 2$ and $L = S(K) \cup F(K)$, then either $H_{n-1}(L)$ is non-trivial or $H_{n-2}(L)$ has a subgroup of order two.
Proof. From the augmented homology sequence of the pair \((K, L)\)
and the hypothesis we find that \(H_q(K, L) = H_{q-1}(L)\) for \(q > 1\). The
previous lemma then yields the desired result.

Note that in case \(a = 2\) in the above lemma \(H_a(L)\) must be non-
trivial.

**Lemma 3.** If \(K\) is a homologically trivial connected 2-complex with \(S(K)\) homologically trivial, then \(K \times 0\).

**Proof.** From the previous lemma we must have \(F(K) \neq \emptyset\). Thus
we may start collapsing \(K\). Suppose that \(K \times K_a\) and we here reach an
impass in our collapsing. That is, \(K_a\) is a 2-complex with no free faces.
Clearly \(S(K_a) \subset S(K)\), so \(S(K_a)\) is homologically trivial, and from
the previous lemma \(F(K_a) \neq \emptyset\), a contradiction. Thus \(K \times 0\).

Let \(D_n\) be the polyhedron which is a disjoint copies of \(S^2\) joined by
line segments as in Figure 1.

![Fig. 1](image)

Further, let \(D\) be any polyhedron which is a disjoint union of copies
of various \(D_n\)'s. If \(K\) is a two-complex, let \(S_2(K)\) denote the set of all \(x\)
in \(S(K)\) with a neighborhood which is the product of an arc and a triode,
and let \(S_4(K)\) denote the set of all \(x\) in \(S(K)\) with a neighborhood which
is a cone over \(D\).

With this notation the following theorem may be given.

**Theorem 1.** Let \(K\) be a 2-complex which is the spine of a homology
three cell \(M^3\). If for each simple closed curve \(C\) in \(S(K)\), \(C \cap S_2(K) \neq \emptyset\),
then \(M^3\) is a 3-ball.

**Proof.** The idea is to expand \(K\) to a complex \(L \subset M^3\). Then since
\(M^3 \times K\) and \(L \times K\), we have by [2] that \(M^3 \times L, L\) will have
been constructed in such a way that \(L \times 0\). Thus \(M^3 \times 0\), and by [2] we have the desired
result.

First expand \(K\) at each point of \(S_2(K)\) by introducing a 3-ball into
each cone as indicated in Figure 2. The 3-complex so obtained is \(L\).

It is clear that \(L \times 0\). We now show \(L \times 0\). First we collapse \(L\) to
a two complex \(L'\) by collapsing out all the copies of \(S^3\) introduced in
the expansion of \(K\) to \(L\). The result of such collapses is indicated for a typical
case in Figure 3.

Note that these operations introduce no new singularities so that
\(S(L') \subset S(K)\). Moreover, note that by hypothesis these operations remove
a 1-cell from every simple closed curve in \(S(K)\). Thus \(S(L')\) is homotopically
trivial.

Since expansions and collapses are homotopy equivalences, and \(M^3\)
was homologically trivial, \(L'\) is homologically trivial. Thus by Lemma 3,
\(L' \times 0\) and the proof is complete.

![Fig. 2](image)

It is clear that the proof of the previous theorem depends only on
"breaking" the simple closed curves in \(S(K)\). This theorem was not
intended to be exhaustive. For example, the situation may arise where

![Fig. 3](image)

a simple closed curve in \(S(K)\) is embedded in \(K\) as indicated in Figure 4
(consider the house with two rooms for example). This simple closed
curve may be broken by "flattening" the disk \(D\) to a 3-ball and collapsing
the 3-ball across the 2-ball \(B^3\) as indicated in Figure 5.

![Fig. 4](image)

![Fig. 5](image)

Perhaps a modification of the method used by Casser in [1] could
be used to establish the following:

**Conjecture.** Every connected 3-manifold with non-empty boundary
has a spine \(K\) for which \(S(K) = S_2(K) \cup S_4(K)\).
Measurable cardinals and analytic games

by

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Introduction. A subset \( P \) of \( \omega^\omega \) is determine if, in the sense of [5] the game \( G_\omega(P) \) is determined. The assumption that every projective set is determine implies that every projective set is Lebesgue measurable (see [6]) and leads to a complete solution to the problem of reduction and separation principles in the classical and effective projective hierarchies [1], [4].

Because of these and other consequences it would be pleasant to have a proof that every projective set is determine. The best available result is that every \( F_\alpha \) is determine [2]. It is not provable in Zermelo–Fraenkel set theory that every analytic \( (\Delta^1_2) \) set is determine [5].

We assume the existence of a measurable cardinal and prove that every analytic set is determine. Our proof is fairly simple and makes a very direct use of the large cardinal assumption (we present it in terms of a Ramsey cardinal) and the fact that open games are determined. We believe that larger cardinals will yield a generalization of our proof to all projective sets. The assumption that measurable cardinals exist is known not to imply even that all \( \Delta^1_2 \) sets are determine. (This follows from [1], [4] and work of Silver.)

§ 1. Definitions. (For more information on the analytical hierarchy see [7], [8]; on infinite games see [5]; on large cardinals see [10], [11].)

Let \( \omega \) be the set of all natural numbers. If \( f: \omega \to \omega \), the function \( f \) is defined by setting \( f(n) \) equal to the sequence \( \langle f(0), f(1), \ldots, f(n-1) \rangle \) of the first \( n \) values of \( f \). Let \( \text{Seq} \) be the set of all finite sequences of natural numbers. Let \( n \to n_0 \) be some enumeration of \( \text{Seq} \) with the property that \( n_0 \) has length \( \leq n \). The Kleene–Brouwer ordering of \( \text{Seq} \) is defined by

\[
\begin{align*}
\langle f(m) \rangle & \text{ is a proper extension of } \langle g(n) \rangle, \\
\text{ if } (m) & < \langle g(n) \rangle \iff \text{ or at least } p \text{ for which } f(p) \neq g(p), \\
\text{ if } & f(p) < g(p).
\end{align*}
\]

(1) Harvey Friedman (unpublished) has shown that the determinateness of Borel sets cannot be proved in Zermelo set theory. Whether it can be proved in Zermelo-Fraenkel set theory remains open.