Two complexes that are spines of the three ball

by

Paul F. Dierker (Moscow, Idaho)

Let M^3 be a 3-manifold with 2-sphere boundary. Using the usual terminology, we will call M^3 a *fake 3-ball* if M^3 has the homotopy groups of a 3-ball, and a *homology ball* if M^3 has the homology of a 3-ball. It is well known that there are homology 3-balls that are not 3-balls while the 3-dimensional Poincaré Conjecture (every fake 3-ball is a 3-ball) is unsolved.

A 3-manifold with non-empty boundary can be collapsed to a 2-complex K in the interior of the manifold. Moreover, we may assume, without loss of generality, that K is a normalized spine of the 3-manifold. That is, every one simplex in K is a face of at most three two simplexes [2]. In this paper we develop a criterion on a normalized spine of a homology 3-ball that will ensure that the homology 3-ball is in fact a 3-ball.

Let K denote a finite geometric n-complex. The following notation will be used.

$$F(K) = \{\sigma^{n-1} | \sigma^{n-1} \text{ is a face of exactly one } \sigma^n \in K\},$$

$$S(K) = \{\sigma^{n-1} | \sigma^{n-1} \text{ is a face of more than two } \sigma^n \in K\}.$$

That is, $F(K)$ is the union of all $(n-1)$-dimensional free faces of K, while $S(K)$ is the set of singular points of K.

Then we have:

LEMMA 1. *If $L = S(K) \cup F(K)$, then either $H_n(K, L)$ is non-trivial or $H_{n-1}(K, L)$ has a subgroup of order two.*

Proof. Suppose that $H_n(K, L)$ is trivial and let $\{\sigma_i^n | \ i = 1, \ldots, q\}$ be the set of all n-simplexes in K. Using the fact that every $(n-1)$-simplex in $K-L$ is a face of precisely two n-simplexes, an easy calculation will show that if $\partial\left(\sum_1^q \sigma_i^n\right) = \sum_{i\in I} 2a_i\sigma_i^{n-1}$ then $\sum_{i\in I} a_i\sigma_i^{n-1}$ is in fact a non-trivial element of order two in $H_{n-1}(K, L)$. We will call a connected complex K *homologically trivial* if $H_q(K) = 0$ for all $q \geqslant 1$.

LEMMA 2. *If K is a homologically trivial n-complex with $n \geqslant 2$ and $L = S(K) \cup F(K)$, then either $H_{n-1}(L)$ is non-trivial or $H_{n-2}(L)$ has a subgroup of order two.*

Proof. From the augmented homology sequence of the pair (K, L) and the hypothesis we find that $H_q(K, L) = H_{q-1}(L)$ for $q \geqslant 1$. The previous lemma then yields the desired result.

Note that in case $n = 2$ in the above lemma $H_1(L)$ must be non-trivial.

Lemma 3. *If K is a homologically trivial connected 2-complex with $S(K)$ homologically trivial, then $K \searrow 0$.*

Proof. From the previous lemma we must have $F(K) \neq \emptyset$. Thus we may start collapsing K. Suppose that $K \searrow K_n$ and we here reach an impass in our collapsing. That is, K_n is a 2-complex with no free faces. Clearly $S(K_n) \subset S(K)$, so $S(K_n)$ is homologically trivial, and from the previous lemma $F(K_n) \neq \emptyset$, a contradiction. Thus $K \searrow 0$.

Let D_n be the polyhedron which is n disjoint copies of S^1 joined by line segments as in Figure 1.

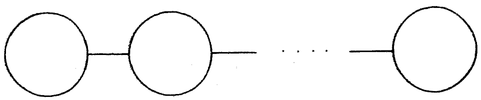

Fig. 1

Further, let D be any polyhedron which is a disjoint union of copies of various D_n's. If K is a two-complex, let $S_1(K)$ denote the set of all x in $S(K)$ with a neighborhood which is the product of an arc and a triode, and let $S_2(K)$ denote the set of all x in $S(K)$ with a neighborhood which is a cone over D.

With this notation the following theorem may be given.

Theorem 1. *Let K be a 2-complex which is the spine of a homology three cell M^3. If for each simple closed curve C in $S(K)$, $C \cap S_2(K) \neq \emptyset$, then M^3 is a 3-ball.*

Proof. The idea is to expand K to a complex $L \subset M^3$. Then since $M \searrow K$ and $L \searrow K$, we have by [2] that $M \searrow L$. L will have been constructed in such a way that $L \searrow 0$. Thus $M \searrow 0$, and by [2] we have the desired result.

First expand K at each point of $S_2(K)$ by introducing a 3-ball into each cone as indicated in Figure 2. The 3-complex so obtained is L.

It is clear that $L \searrow K$. We now show $L \searrow 0$. First we collapse L to a two complex L' by collapsing out all the copies of B^3 introduced in the expansion of K to L. The result of such collapses is indicated for a typical case in Figure 3.

Note that these operations introduce no new singularities so that $S(L') \subset S(K)$. Moreover, note that by hypothesis these operations remove

a 1-cell from every simple closed curve in $S(K)$. Thus $S(L')$ is homotopically trivial.

Since expansions and collapses are homotopy equivalences, and M^3 was homologically trivial, L' is homologically trivial. Thus by Lemma 3, $L' \searrow 0$ and the proof is complete.

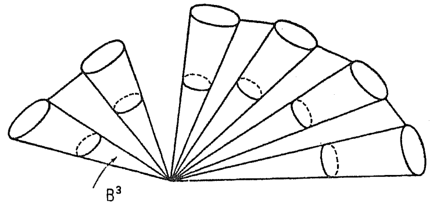

Fig. 2

It is clear that the proof of the previous theorem depends only on "breaking" the simple closed curves in $S(K)$. This theorem was not intended to be exhaustive. For example, the situation may arise where

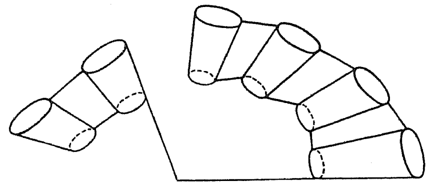

Fig. 3

a simple closed curve in $S(K)$ is embedded in K as indicated in Figure 4 (consider the house with two rooms for example). This simple closed curve may be broken by "fattening" the disk D to a 3-ball and collapsing the 3-ball across the 2-ball B^2 as indicated in Figure 5.

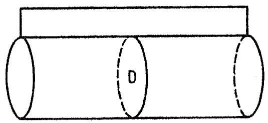

Fig. 4

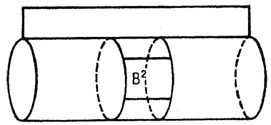

Fig. 5

Perhaps a modification of the method used by Casler in [1] could be used to establish the following:

Conjecture. *Every connected 3-manifold with non-empty boundary has a spine K for which $S(K) = S_1(K) \cup S_2(K)$.*

References

[1] B. G. Casler, *An imbedding theorem for connected 3-manifolds with boundary*, Proc. Amer. Math. Soc. 16 (1965), pp. 559–566.

[2] E. C. Zeeman, *Seminar on Combinatorial Topology* (mimeographed notes), Institute des Hautes Etudes Scientifiques, Paris 1963.

UNIVERSITY OF IDAHO
Moscow, Idaho

Reçu par la Rédaction le 19. 4. 1968

Measurable cardinals and analytic games

by

Donald A. Martin (New York)

Introduction. A subset P of ω^ω is *determinate* if, in the sense of [5] the game $G_\omega(P)$ is determined. The assumption that every projective set is determinate implies that every projective set is Lebesgue measurable (see [6]) and leads to a complete solution to the problem of reduction and separation principles in the classical and effective projective hierarchies [1], [4]. Because if these and other consequences it would be pleasant to have a proof that every projective set is determinate. The best available result is that every $F_{\sigma\delta}$ is determinate [2]. It is not provable in Zermelo–Fraenkel set theory that every analytic ($\boldsymbol{\Sigma}_1^1$) set is determinate [5]. (1)

We assume the existence of a measurable cardinal and prove that every analytic set is determinate. Our proof is fairly simple and makes a very direct use of the large cardinal assumption (we present it in terms of a Ramsey cardinal) and the fact that open games are determined. We believe that larger cardinals will yield a generalization of our proof to all projective sets. The assumption that measurable cardinals exist is known not to imply even that all $\boldsymbol{\Delta}_2^1$ sets are determinate. (This follows from [1], [4] and work of Silver.)

§ 1. Definitions. (For more information on the analytical hierarchy see [7], [8]; on infinite games see [5]; on large cardinals see [10], [11].)

Let ω be the set of all natural numbers. If $f\colon \omega \to A$, the function $\bar{f}$ is defined by setting $\bar{f}(n)$ equal to the sequence $\langle f(0), f(1), \ldots, f(n-1)\rangle$ of the first n values of f. Let Seq be the set of all finite sequences of natural numbers. Let $n \to k_n$ be some enumeration of Seq with the property that k_n has length $\leqslant n$. The *Kleene–Brouwer ordering* of Seq is defined by

$$\bar{f}(m) < \bar{g}(n) \leftrightarrow \begin{cases} \bar{f}(m) \text{ is a proper extension of } \bar{g}(n)\,, \\ \text{or at the least } p \text{ for which } f(p) \neq g(p)\,, \\ f(p) < g(p)\,. \end{cases}$$

(1) Harvey Friedman (unpublished) has shown that the determinateness of Borel sets cannot be proved in Zermelo set theory. Whether it can be proved in Zermelo-Fraenkel set theory remains open.