Axiom systems for Lipschitz structures *

by

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1. Introduction. In 1954, Efremovich [1] suggested that the study of certain geometric properties of metric spaces could be advanced by studying the properties preserved by a Lipschitz function. Sandberg [7], in an attempt to carry out such a plan, generalized the notion of a Lipschitz function. Geraghty [4] gave a different generalization which was closer (in some sense) to the original concept of a Lipschitz function.

We give two axiom systems for a Lipschitz structure on a space and show that they are both natural and equivalent. Separation axioms are considered, yielding some conditions for which a Lipschitz structure is given by a metric. Finally, a "reasonable" topology is given for the space of real- (or complex-) valued functions which satisfy an "M-Lipschitz condition".

2. M-Lipschitz structures. Let (X, d) and (Y, e) be pseudometric spaces. A function $f\colon X\to Y$ is called a *local Lipschitz function* if there exist positive numbers K and δ such that $e\circ f_2(x_1, x_2)\leqslant Kd(x_1, x_2)$ whenever $d(x_1, x_2)<\delta$, where $f_2(x_1, x_2) = \big(f(x_1), f(x_2)\big)$. When d and e are pseudometrics on X such that Id: $(X, d)\to(X, e)$ is a local Lipschitz function, we denote it by $e\ll d$. Thus to say that $f\colon (X, d)\to(Y, e)$ satisfies a local Lipschitz condition can be denoted $e\circ f_2\ll d$. Finally, if $d\ll e$ and and $e\ll d$, we will say (following Efremovich) that d and e are strongly equivalent, and denote it by $d\approx e$ (¹).

2.1. Definition. Let X be a space. An M-Lipschitz structure for X is a non-empty collection D of pseudometrics satisfying the following conditions:

M1. If $d_1, d_2\in D$, then $d_1+d_2\in D$.

M2. If $d_1\in D$ and $d_2\leqslant d_1$, then $d_2\in D$.

M3. If $\min\{d, 1\}\in D$, then $d\in D$.

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(¹) Efremovich [2] actually defined two metrics to be strongly equivalent if a Lipschitz condition was satisfied both ways. Since the properties which he discussed are preserved by local Lipschitz mappings, we feel justified in usurping his definition for mappings which satisfy a *local* Lipschitz condition in both directions.

The pair (X, D) will be called an *M-Lipschitz space*.

We list a few easy consequences of the definition. As usual, $\vee$ denotes max and $\wedge$ denotes min.

2.2. If $d_1, d_2 \in D$, then $d_1 \vee d_2 \in D$.

2.3. For any two pseudometrics, $d_1 + d_2 \approx d_1 \vee d_2$.

2.4. If $d \in D$, then $nd \in D$ for any positive integer n.

2.5. If $d \in D$, then $rd \in D$ for any positive real r.

2.6. For any real $\delta < 0$, $d \wedge \delta \in D$ iff $d \in D$.

2.7. If $d \in D$ and $e \ll d$, then $e \in D$.

2.8. The intersection of any collection of M-Lipschitz structures on a space X is again an M-Lipschitz structure on X.

2.9. Given a non-void set of pseudometrics on a space X, there is a unique smallest M-Lipschitz structure on X containing the given set of pseudometrics.

We will refer to the smallest M-Lipschitz structure containing a given set of pseudometrics as the M-Lipschitz structure generated by that set of pseudometrics.

2.10. *Let X be a space, Δ a non-empty set of pseudometrics on X, and D the M-Lipschitz structure generated by Δ. Then $e \in D$ iff there exist $d_{a_1}, d_{a_2}, \ldots, d_{a_n} \in \Delta$ such that $e \ll \sum_{i=1}^{n} d_{a_i}$.*

Proof. Let $D' = \{e \mid e$ is a pseudometric on X for which there exist $d_{a_1}, d_{a_2}, \ldots, d_{a_n}$ such that $e \ll \sum_{i=1}^{n} d_{a_i}\}$. Clearly $D' \subseteq D$ by finite induction on M_1 and 2.6. Since $\Delta \subseteq D'$, we need only show that D' is an M-Lipschitz structure. This is routine.

2.11. *Let (X, d) be a pseudometric space and D the M-Lipschitz structure generated by d. Then $e \in D$ iff $e \ll d$.*

2.12. Definition. Let (X, D) and (Y, E) be M-Lipschitz spaces. A function $f: X \to Y$ will be called an *M-Lipschitz function* if given $e \in E$, there exists $d \in D$ such that $e \circ f_2 \ll d$.

We observe that the composition of two M-Lipschitz functions is again an M-Lipschitz function. This follows from the fact that the composition of two functions, each satisfying a local Lipschitz condition, yields a function which also satisfies a local Lipschitz condition. Thus if d_1, d_2, d_3 are all pseudometrics for X, such that $d_1 \ll d_2$ and $d_2 \ll d_3$, we see that $d_1 \ll d_3$.

2.13. Definition. Two M-Lipschitz spaces are *isomorphic* if there exists an M-Lipschitz function from one to the other which is one to one and onto with its inverse being an M-Lipschitz function.

2.14. *Let (X, d) and (Y, e) be pseudometric spaces with D and E the M-Lipschitz structures generated by d and e, respectively. Then a function $f: X \to Y$ is an M-Lipschitz function iff $e \circ f_2 \ll d$.*

Proof. Suppose f satisfies a local Lipschitz condition. Let $e' \in E$ be given. Then Id: $(Y, e) \to (Y, e')$ satisfies a local Lipschitz condition, and, by the remark above, $f(X, d) \to (Y, e')$ does also. That is, $e' \circ f_2 \ll d$.

Conversely, suppose f is an M-Lipschitz function. Then there exists $d' \in D$ such that $e \circ f_2 \ll d'$. But $d' \in D$ iff $d' \ll d$. Since $\ll$ is transitive, $e \circ f_2 \ll d$, and f satisfies a local Lipschitz condition.

2.15. A function $f: (X, D) \to (Y, E)$ is an M-Lipschitz function iff $e \circ f_2 \in D$ for every $e \in E$.

2.16. A function $f: (X, D) \to (Y, E)$ is an M-Lipschitz function iff $e \circ f_2 \in D$ for every e in a generating set for E.

2.17. If X is a space with pseudometrics d and e, then the corresponding M-Lipschitz structures are isomorphic under the identity iff $d \approx e$.

3. E-Lipschitz structures. Before we give the definition of an E-Lipschitz structure, it is necessary to establish some notation. Let X be a space and $S \subseteq X \times X$. For a function f defined on X, $f_2(S) = \{f_2(s_1, s_2) \mid (s_1, s_2) \in S\}$. If u_n, $n = 0, 1, 2, \ldots$ is a sequence of entourages, we shall write $\{u_n\}_{n=0}^{\infty}$, or more simply $\{u_n\}$ whenever there can be no confusion. Finally, $\{u_n\} < \{v_n\}$ will mean that $u_n \subseteq v_n$ for each integer $n \geqslant 0$.

3.1. Definition. Let X be a space. An *E-Lipschitz structure for X* is a filter U of sequences of entourages (partially ordered by $<$) with a basis U' satisfying

E1. If $\{v_n\} \in U'$, then $v_{n+1}^2 \subseteq v_n = v_n^{-1}$ for each n, and

E2. If $\{v_n\} \in U'$, then $\{v_{n+1}\} \in U'$.

The pair (X, U) will be called an *E-Lipschitz space*.

Comparing this definition with that of an M-Lipschitz structure, we see

a) Property E1 provides us with pseudometrics.

b) Property E2 implies that we are only interested in behaviour locally, and so corresponds to M3.

c) The "superset" property of a filter corresponds to M2, and the "intersection" property corresponds to M1.

3.2. Definition. Let (X, U) and (Y, V) be E-Lipschitz spaces. Then a function $f: X \to Y$ will be called an *E-Lipschitz function* if given $\{v_n\} \in V$, there exists $\{u_n\} \in U$ such that $\{f_2(u_n)\} < \{v_n\}$.

3.3. Let (X, U) and (Y, V) be E-Lipschitz spaces, with U' and V' the respectively bases guaranteed by the definition. The following are equivalent:

(i) The function f is an E-Lipschitz function.

(ii) Given $\{v_n\} \in V'$, there exists $\{u_n\} \in U'$ such that $\{f_2(u_n)\} < \{v_n\}$.

(iii) If $\{v_n\} \in V$, then $\{f_2^{-1}(v_n)\} \in U$.

(iv) If $\{v_n\} \in V'$, then $\{f_2^{-1}(v_n)\} \in U$.

The proof, being straightforward, is omitted.

3.4. Let X be a space and d a pseudometric on X. The *entourage sequence generated by* d is $\{u_n\}$, where $u_n = \{(x_1, x_2) \in X \times X|\ d(x_1, x_2) \leqslant 1/2^n\}$. The set of entourage sequences $\{(u_{n+k}\}_{n=0}^{\infty}|\ k = 0, 1, 2, \ldots\}$ satisfies the conditions for a basis of an E-Lipschitz structure, and the resulting E-Lipschitz structure will be called the *E-Lipschitz structure generated by d*. More generally, let Δ be a collection of pseudometrics on X. Then the set $\{\{_a u_{n+k}\}|\ d_a \in \Delta,\ k = 0, 1, 2, \ldots\}$ satisfies the conditions for a subbase of a filter (with the partial order $<$). The base which the subbase generates satisfies E1 and E2. Thus we call the resulting filter the *E-Lipschitz structure generated by Δ*.

3.5. *Let (X, d) and (Y, e) be pseudometric spaces, and U, V the E-Lipschitz structures generated by d, e respectively. Then a function $f\colon X \to Y$ is an E-Lipschitz function iff f satisfies a local Lipschitz condition.*

Proof. Suppose that $e \circ f_2 \ll d$. That is, there exist $\delta > 0$ and $k > 0$ such that $e \circ f_2(x_1, x_2) \leqslant k d(x_1, x_2)$ whenever $d(x_1, x_2) < \delta$. Without loss of generality, we assume that $k = 2^m$ and $\delta = 1/2^{n+m}$ for some pair of positive integers m and n. Then when $d(x_1, x_2) \leqslant 1/2^{m+n+k}$, we have $e \circ f_2(x_1, x_2) \leqslant 1/2^{n+k}$. That is, $f_2(u_{m+n+k}) \subseteq v_{n+k}$ for each positive integer k. Selecting an arbitrary $\{v_{l+k}\}_{k=0}^{\infty}$ from V', we see that $f_2(u_{n+m+l+k}) \subseteq v_{n+l+k} \subseteq v_{l+k}$ for every k. Thus f is an E-Lipschitz function by 3.3.

Conversely, assume that f is an E-Lipschitz function. That is, given m, there exists n such that $f_2(u_{m+k}) \subseteq v_{n+k}$ for each k. Now $(x_1, x_2) \in u_{m+k} - u_{m+k+1}$ implies that $1/2^{m+k+1} < d(x_1, x_2) \leqslant 1/2^{m+k}$. Then $e \circ f_2(x_1, x_2) \leqslant 1/2^{n+k} = 1/2^{m+k+1} \cdot 1/2^{n-m-1} \leqslant d(x_1, x_2)/2^{n-m-1}$. That is, if $d(x_1, x_2) \leqslant 1/2^m$, then $e \circ f_2(x_1, x_2) \leqslant 2^{m-n+1} d(x_1, x_2)$.

3.6. *Let X be a space with pseudometrics d and e. Let $\{u_n\}$ and $\{v_n\}$ be the entourage sequences generated by d and e, respectively. Then $d \ll e$ iff there exists k such that $\{v_{n+k}\} < \{u_n\}$.*

Proof. If $d \ll e$, then Id: $(X, V) \to (X, U)$ is an E-Lipschitz function, where V and U are the E-Lipschitz structures generated by e and d respectively. Since $\{\{u_{n+k}\}_{n=0}^{\infty}|\ k = 0, 1, \ldots\}$ forms a base for U, there exists a k such that $id_2(u_{n+k}) = u_{n+k} \subseteq v_n$ for all n.

Conversely, suppose such a k exists. We will show that Id: $(X, V) \to (X, U)$ is an E-Lipschitz function. It suffices by 3.3 to consider a sequence of the form $\{v_{n+l}\}$. By hypothesis, there is a k such that $u_{n+k+l} \subseteq v_{n+l}$ for every n. Thus $d \ll e$.

We recall the

3.7. Definition. Let $f\colon (X, d) \to (Y, e)$ be such that for any $\varepsilon > 0$, there is a $\delta > 0$ for which $e \circ f_2(x_1, x_2) \leqslant \varepsilon d(x_1, x_2)$ whenever $d(x_1, x_2) < \delta$. Then f will be said to *satisfy a lipschitz condition*, or to be a *lipschitz function*. Note that lipschitz should be read "little Lipschitz".

Little Lipschitz functions have their analogue in M-Lipschitz structures with the obvious definition. To discuss lipschitz functions in terms of entourage sequences, we use .

3.8. *Let X be a space with pseudometrics d and e. Let $\{u_n\}$ and $\{v_n\}$ be the entourage sequences generated by d and e, respectively. Then* Id: $(X, e) \to (X, d)$ *is a lipschitz function iff given any positive integer m, there exists a positive integer k such that $v_n \subseteq u_{m+n}$ for all $n \geqslant k$.*

Proof. If Id: $(X, e) \to (X, d)$ is a lipschitz function, then there exist positive integers i and j such that when $e(x, y) \leqslant 1/2^i$, then $d(x, y) \leqslant 1/2^j e(x, y)$. Thus when $(x, y) \in v_{n+i}$, we have $d(x, y) \leqslant 1/2^{i+n+j}$, and $(x, y) \in u_{i+n+j}$. That is, $v_n \subseteq u_{n+i}$ for all $n > j$.

Conversely, suppose that given m, there exists k such that $v_n \subseteq u_{m+n}$ for all $n > k$. That is, $e(x, y) \leqslant 1/2^n$ and $n > k$ implies that $d(x, y) \leqslant 1/2^{m+n}$. Choosing $(x, y) \in v_n - v_{n+1}$, with $n > k$, we have $e(x, y) \geqslant 1/2^{n+1}$. Hence $1/2^m e(x, y) \geqslant 1/2^{m+n+1} \geqslant 1/2\, d(x, y)$ when $e(x, y) < 1/2^k$, proving the result.

4. Equivalence of M-Lipschitz and E-Lipschitz structures.

In order to see the relation between M-Lipschitz structures and E-Lipschitz structures, we need some method of transferral from one to the other. We already have a method for obtaining an E-Lipschitz structure from an M-Lipschitz structure. To go the other way, we use the following:

4.1. Metrization Theorem ([3], p. 164). *For a space X, let $\{u_n\}$ be a sequence of entourages satisfying* E1. *Then there exists a pseudometric d for X with the property that*

$$\text{4.2}\qquad \{(x, y)|\ d(x, y) \leqslant 1/2^{n+1}\} \subseteq u_n \quad \text{and} \quad u_n \subseteq \{(x, y)|\ d(x, y) \leqslant 1/2^n\},$$

with $d(x, y) = 1$ if $(x, y) \notin u_0$.

Proof. For each n, set $A_{2^{-n}} = u_n$. If a is a dyadic rational,

$$a = \sum_{i=1}^{k} 2^{-n_i}, \quad n_i > n_{i+1}.$$

Then put

$$A_a = \Big\{(x, y)\colon (x, y) \in u_{n_1} \circ u_{n_2} \circ \ldots \circ u_{n_k} \text{ where } a = \sum_{i=1}^{k} 2^{-n_i}\Big\}.$$

This extends the definition of A_a to all dyadic rationals a.

Define

$$d(x,y)=\begin{cases}1 & \text{if } (x,y)\notin u_0,\\ \inf\{a\mid (x,y)\in A_a\} & \text{if } (x,y)\in u_0.\end{cases}$$

It is easily seen that d is in fact a pseudometric and satisfies 4.2.

The pseudometric constructed in the above manner will be referred to as the *pseudometric associated with an entourage sequence*, or the *pseudometric constructed from an entourage sequence*.

4.3. COROLLARY. *Let d be a pseudometric, $\{u_n\}$ its associated entourage sequence, and d' the pseudometric constructed from $\{u_n\}$. Then $d\approx d'$.*

Proof. Let $\{v_n\}$ be the entourage sequence constructed from d'. By the metrization theorem, $\{v_{n+1}\}<\{u_n\}<\{v_n\}$. Thus $d\ll d'\ll d$.

4.4. COROLLARY. *Let $\{u_n\}$ be an entourage sequence satisfying* E1 *and* E2, *d the pseudometric constructed from it, and $\{v_n\}$ the entourage sequence generated by d. Then $\{v_{n+1}\}<\{u_n\}<\{v_n\}$.*

We are now ready to prove the main theorem of this section. It should be noted that 4.3 and 4.4 form the keystone of the proof. Also, the sharpness of Gaal's proof of the metrization theorem is essential, since the proof in [6] (for instance) will not suffice.

4.5. THEOREM. *To each E-Lipschitz structure corresponds in a natural way a unique M-Lipschitz structure and conversely. This correspondence is reflexive. Moreover, the class of M-Lipschitz functions is precisely the class of E-Lipschitz functions.*

Proof. We first make some almost obvious observations. Let d and e be pseudometrics on X and $\{u_n\}$, $\{v_n\}$ their respective entourage sequences. Then $\{u_n\cap v_n\}$ is the entourage sequence generated by $d\vee e$. Similarly, if $\{u_n\}$ and $\{v_n\}$ are entourage sequences with d and e their respective pseudometrics constructed by the metrization theorem, then $d\vee e$ is the pseudometric constructed from $\{u_n\cap v_n\}$.

Now let U be an E-Lipschitz structure for a space X and U' the set of all entourage sequences of U satisfying E1 and E2. For each entourage sequence in U', we construct the associated pseudometric by means of the metrization theorem. Let D be the M-Lipschitz structure generated by the set of all such pseudometrics. Clearly D is unique.

Conversely, let D be an M-Lipschitz structure on X. Then the family of entourage sequences generated by the pseudometrics of D forms a base for an E-Lipschitz structure U. Once again, this construction yields a unique structure.

Let D be an M-Lipschitz structure, U the E-Lipschitz structure constructed from D, and D' the M-Lipschitz structure constructed from U. It follows from 4.3 that $D\subseteq D'$. To show $D'\subseteq D$, let $d'\in D'$.

Then by 2.3 and 2.10, $d'\ll d_1'\vee d_2'\vee\ldots\vee d_k'$ for some $d_1', d_2', \ldots, d_k'$, where each d_i' was constructed from an entourage sequence $\{{}_iu_n\}$ in U satisfying E1 and E2. Thus $d'\ll e'$, where e' was constructed from $\{\bigcap_{i=1}^{k}{}_iu_n\}$. Since the set of entourage sequences in U generated by pseudometrics of D forms a base for U, there is an entourage sequence $\{v_n\}\in U$ such that $\{v_n\}<\{\bigcap_{i=1}^{k}{}_iu_n\}$ and $v_n=\{(x,y)\mid d(x,y)\leqslant 1/2^n\}$ for some pseudometric d in D. Then we see that $e'\ll d$. By the transitivity, $d'\ll d$, and $d'\in D$ by 2.10.

To complete the proof of reflexivity, let U be an E-Lipschitz structure, D the M-Lipschitz structure constructed from U, and V the E-Lipschitz structure constructed from D. Let U' and V' be bases whose elements satisfy E1 and E2 for U, V, respectively. By 4.4, $\{u_n\}\in U'$ implies $\{u_n\}\in V$. Now let $\{v_n\}\in V'$. The paragraph above shows that the pseudometric d constructed from $\{v_n\}$ is in D. Then $d\ll\sum_{i=1}^{k}d_i$, where each d_i was constructed from an entourage sequence $\{{}_iu_n\}$ of U. Letting e be the pseudometric $d_1\vee d_2\vee\ldots\vee d_k$, we have e constructed from $\{\bigcap_{i=1}^{k}{}_iu_n\}$ and $\sum_{i=1}^{k}d_i\approx e$. Thus $d\ll e$, so $\{\bigcap_{i=1}^{k}{}_iu_n\}<\{v_n\}$, and $\{v_n\}\in U$.

Finally we show that the class of M-Lipschitz functions is identical with the class of E-Lipschitz functions. Let $f\colon X\to Y$ be a function, U and V E-Lipschitz structures for X and Y respectively, and D, E the M-Lipschitz structures constructed from U, V respectively. Now f is an E-Lipschitz function iff $\{f_2^{-1}(v_n)\}\in U$ for each $\{v_n\}$ in V satisfying E1 and E2. But f is an M-Lipschitz function iff $e\circ f_2\in D$ for each $e\in E$. Since $e\circ f_2$ is the pseudometric associated with $\{f_2^{-1}(v_n)\}$ when e is the pseudometric associated with $\{v_n\}$, the theorem follows.

5. Separation and metrization. It should be observed that either Lipschitz structure induces a uniformity in a natural way. The gage of the uniformity is usually "larger" than the M-Lipschitz structure. For example, let (X, d) be a compact metric space with a point of accumulation. Then the gage generated by d contains every pseudometric on X whose topology coincides with that of d. On the other hand, $\sqrt{d}$ is not in the M-Lipschitz structure generated by d.

Let (X, U) be an E-Lipschitz space. The structure is called *separated* if $\bigcap\{u_1\mid \{u_n\}\in U\}=\Delta$, the diagonal. This is clearly equivalent to saying that the pseudometrics in the corresponding M-Lipschitz structure separate the points of X. Thus a separated Lipschitz space induces a separated uniform space.

In the event that $\bigcap_{n=1}^{\infty} u_n = \Delta$ for some $\{u_n\} \in U$, we will call the structure strongly separated. It is easy to see that the M-Lipschitz structure for a strongly separated space is generated by a family of metrics. One might hope that a strongly separated structure which was generated by a countable collection of metrics would be generated by a single metric. This need not occur, even for a compact space.

5.1. Example. Let X be $[0,1]$. For each positive integer n, let $d_n(x,y) = |x-y|^{1/n}$. The M-Lipschitz structure D generated by $\{d_n|\ n = 1, 2, \ldots\}$ cannot be generated by a single metric.

Indeed, if $m < n$, then $d_m \ll d_n$. If D were generated by a single metric d, then we would have $d \ll d_{k_1} \vee d_{k_2} \vee \ldots \vee d_{k_n}$ for some finite set of indices. Letting $k = k_1 \vee k_2 \vee \ldots \vee k_n$, we have that $d \ll d_k$. If d generated D, by 2.11 we would have that $d_{k+1} \ll d$. But $|x-y|^{1/(k+1)} : |x-y|^{1/k} = |x-y|^{-1/k(k+1)} \to \infty$ as $|x-y| \to 0$. That is, $d_{k+1} \ll d_k$ is impossible, showing that D is not generated by a single metric.

If there is an entourage sequence $\{u_n\} \in U$ such that $\{\{u_{n+k}\}|\ k = 0, 1, 2, \ldots\}$ is a base for U, the structure will be called *simple*.

5.2. Proposition. *If X is a space with a simple E-Lipschitz structure U, then there is a pseudometric d such that U is the E-Lipschitz structure generated by d.*

Proof. Let $\{v_n\} \in U$ satisfy E1 and E2, and $\{v_n\} < \{u_n\}$, where $\{\{u_{n+k}\}_{n=0}^{\infty}|\ k = 0, 1, 2, \ldots\}$ is a base for U. Let d be the pseudometric constructed from $\{v_n\}$. Now $d_k = \inf\{2^k d, 1\}$ is the pseudometric constructed from $\{v_{n+k}\}$ and $d_k \approx d$. Since $\{\{v_{n+k}\}|k = 0, 1, 2, \ldots\}$ also form a base for U, d generates the E-Lipschitz structure.

5.3. Corollary. *If X is a space with a simple separated E-Lipschitz structure, then the E-Lipschitz structure is generated by a metric.*

If an M-Lipschitz space (X, D) is given, it is clear that $\approx$ is an equivalence relation on D. This equivalence relation induces a partial order in $\bar{D}$, the set of equivalence classes, by $\bar{d} \lessapprox \bar{e}$ iff $d \ll e$. It is easily shown that $\lessapprox$ is a well defined partial ordering. This order makes $\bar{D}$ a lattice in the usual way. We then have

5.4. Proposition. *The M-Lipschitz structure D on X is generated by a single pseudometric iff $\bar{D}$ has a maximal element.*

Proof. If $\bar{D}$ has a maximal element $\bar{d}$, then any $d \in \bar{d}$ will generate D. For let $e \in D$. Then $\bar{e} \lessapprox \bar{d}$, and so $e \ll d$. Conversely D is generated by a metric d means that any metric $e \in D$ must satisfy $e \ll d$. Thus $\bar{e} \lessapprox \bar{d}$ and $\bar{d}$ is maximal.

6. Constructions. Let $(X_\alpha, D_\alpha)_{\alpha \in \Gamma}$ be a family of M-Lipschitz spaces and for each α, let f_α: $Y \to X_\alpha$ be a function. The smallest M-Lipschitz structure D on Y such that each f_α is an M-Lipschitz function is called the *projective M-Lipschitz structure* determined by the family of f_α and X_α. It is fairly clear that the projective M-Lipschitz structure is the M-Lipschitz structure generated by $\{d_\alpha \circ f_{\alpha 2}|\ d_\alpha \in D_\alpha, \alpha \in \Gamma\}$. As examples of projective M-Lipschitz structures, we consider subspaces, product spaces, and projective limit spaces.

6.1. *Subspaces.* If (X, D) is an M-Lipschitz space and $Y \subseteq X$, we can consider the M-Lipschitz structure E on Y induced by Id: $Y \to X$. This structure is clearly generated by $E' = \{d|_Y|\ d \in D\}$. If $e \in E$, then $e \leqslant d|_Y$ for some $d \in D$. Since e has an extension e' to X given by $e'(x_1, x_2) = \inf\{d(x_1, y_1) + e(y_1, y_2) + d(y_2, x_2)|\ y_1, y_2 \in Y\}$, we see that $e' \leqslant d$ on X (²). Thus e (being equal to $e'|_Y$) is in E'.

6.2. *Product spaces.* Let $(X_\alpha, D_\alpha)_{\alpha \in \Gamma}$ be given. Then the projections p_β: $\prod_{\alpha \in \Gamma} X_\alpha \to X_\beta$ induce the *product M-Lipschitz structure.* If Γ is a finite set and each D_α is generated by a single pseudometric d_α, then the product M-Lipschitz structure is generated by the pseudometric $\sum_{\alpha \in \Gamma} d_\alpha$.

The product uniformity for a countable product of pseudometric uniformities is again a pseudometric uniformity. This is not true in general for M-Lipschitz structures, as one can verify by taking X_n to be $[0,1]$ and $d_n(x,y) = |x-y|^{1/n}$, $n = 1, 2, \ldots$ The "diagonal" space is the space described in 5.1.

6.3. Proposition. *Let Y be a space and $(X_\alpha, D_\alpha)_{\alpha \in \Gamma}$ a collection of M-Lipschitz spaces each having a map f_α of Y to X_α. Let (Y, E) be the projective Lipschitz structure induced by the (X_α, D_α). Then f: $(Y', E') \to (Y, E)$ is an M-Lipschitz map iff $f_\alpha \circ f$ is an M-Lipschitz map for each α.*

Proof. The "only if" part is trivial. Suppose then that $f_\alpha \circ f$: $Y' \to X_\alpha$ is an M-Lipschitz function for each α. Then $d_\alpha \circ (f_{\alpha 2} \circ f_2) \in E'$ for each $d_\alpha \in D_\alpha$ by 2.15. But $d_\beta \circ (f_{\alpha 2} \circ f_2) = (d_\alpha \circ f_{\alpha 2}) \circ f_2$. Since the pseudometrics $d_\alpha \circ f_{\alpha 2}$ generate the projective M-Lipschitz structure, f is an M-Lipschitz function by 2.16.

6.4. *Projective limit spaces.* Let (X_α, D_α), $\alpha \in \Gamma$ be a family of M-Lipschitz spaces. Let $\leqslant$ be a directed partial ordering on Γ such that for each pair of indices α, β with $\alpha \leqslant \beta$, there is a canonical M-Lipschitz mapping $f_{\alpha\beta}$: $X_\beta \to X_\alpha$. Further, the canonical mappings are to satisfy $f_{\alpha\beta} \circ f_{\beta\gamma} = f_{\alpha\gamma}$ and $f_{\alpha\alpha} = id$ on X_α. Then $\operatorname{limproj}\{(X_\alpha, D_\alpha)|\ \alpha \in \Gamma\} = \{x \in \prod_{\alpha \in \Gamma} X_\alpha|$ for $\alpha \leqslant \beta$, $f_{\alpha\beta} \circ p_\beta(x) = p_\alpha(x)\}$ where p_α is the projection map onto X_α. As an illustration, we show that each separated M-Lipschitz structure can be considered as the projective limit of M-Lipschitz structures generated by a single metric. Let (X, D) be an M-Lipschitz space. For each $d \in D$,

(²) This pseudometric is due to R. H. Bing.

let (X_d, d') denote the (metric) quotient space obtained by identifying points which are zero distance apart. D has the obvious partial order induced by the quasi-order $\leqslant$. The canonical map $f_{\alpha\beta}$ is the one induced by Id: $(X, d_\beta)\to(X, d_\alpha)$, where of course $d_\alpha \leqslant d_\beta$. Then the natural maps $(X, D)\to\prod_{d\in D}(X, d)\to\prod_{d\in D}(X_d, d')$ give the obvious M-Lipschitz isomorphism onto $\operatorname{limproj}(X_d, d')$.

For (X, D) an M-Lipschitz structure, define $\operatorname{Lip}(X, D)$ to be $[f\colon X\to R \mid f$ is a bounded, M-Lipschitz function$]$. (We use the customary metric for R.) For any pseudometric d on X, set $\operatorname{Lip}(X, d) = \{f\colon X\to R \mid |f(x)-f(y)| \leqslant kd(x, y)$ for some $k > 0\}$. $\operatorname{Lip}(X, d)$ is a Banach space [8]. In [2] it is shown that $d \leqslant e$ iff Id: $\operatorname{Lip}(X, d)\to\operatorname{Lip}(X, e)$ is a continuous imbedding. Hence $\{\operatorname{Lip}(X_d, d')\mid d \in D\}$ is an inductive family of Banach spaces. We topologize $\operatorname{Lip}(X, D)$ with the inductive limit topology.

As was mentioned before, from the fact that D is countably generated, we cannot conclude that D is generated by a single pseudometric. However, we do have

6.6. THEOREM. *Let D be countably generated. Then* $\operatorname{Lip}(X, D)$ *is a Frechet space iff D is generated by a single pseudometric.*

Proof. If D is generated by d, then $\operatorname{Lip}(X, D) \approx \operatorname{Lip}(X, d)$ by the identity map.

Now suppose $\operatorname{Lip}(X, D)$ is a Frechet space and D is generated by $\{d_i\}$, $i = 1, 2, \ldots$ We assume without loss of generality that $d_i \leqslant d_{i+1}$ for each i. The inductive limit topology is the strongest locally convex topology making all of the injection maps $\operatorname{Lip}(X, d_i)\to\operatorname{Lip}(X, D)$ continuous. By a theorem of Grothendieck [5], $\operatorname{Lip}(X, D)$ is isomorphic to $\operatorname{Lip}(X, d_n)$ for some n via the injection map. Then since $\operatorname{Lip}(X, d_n)\to\operatorname{Lip}(X, d_{n+k})$ must be an isomorphism for each k, $d_n \approx d_{n+k}$. Thus D is generated by d_n.

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Topologies uniquely determined by their continuous self map

by

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The major content of this paper is the search for topologies which are unique among the topologies for a given set with respect to their continuous self maps. Several classes of such spaces shall be given. Among them are the locally Euclidean, T_2-spaces; the separable metric, locally connected continua; all spaces of CW-complexes; and the non-discrete, cofinite spaces.

For notation let a pair (X, U) denote a topological space if X is the set of points in the space, and U is the collection of all *closed* subsets of the space. This variation from standard is a convenience to this study. If (X, U) is a topological space, let $C(U)$ denote the collection of all functions from X into X which are continuous with respect to U. A space (X, U) is *special* if and only if the only topology V on X such that $C(U) = C(V)$ is the topology $V = U$. If Q is a class of spaces, then a space (X, U) in Q is Q-*special* if and only if the only topology V on X such that (X, V) is in Q, and $C(V) = C(U)$ is the topology $V = U$. A space (X, U) is T_1-*special* if and only if it is Q-special when Q denotes the class of all T_1 spaces.

Now the problem may be described as the search for special spaces. The method will involve the study of spaces which are both maximal and minimal in the lattice of T_1-spaces with respect to their continuous self maps. Then conditions shall be given under which a T_1-special space is special. In the process a class of spaces which are absolutely minimal T_1-spaces with respect to their self maps shall be studied. There is a close relationship between this study and the study of S-admissibility [3]. This relationship shall be clearified, and several theorems on the construction of S-admissible classes shall be given.

Additional notation. If (X, U) and (Y, V) are spaces, let $C(U, V)$ denote the set of continuous functions from (X, U) into (Y, V). Let Y^X denote the set of all functions from X into Y. Let 2^X denote the set of all subsets of X. If f and g are two functions such that the composite