Extensions of uniform structures *

by

T. E. Gantner (Dayton, Ohio)

Let X be any completely regular space and let BX be any compactification of X. It is well known that BX has a unique admissible uniform structure $\mathcal{U}$, whereas X may have many distinct admissible uniform structures. Therefore there exists a unique admissible uniform structure $\mathcal{V}$ on X such that $\mathcal{V}$ is the relative uniform structure on X obtained from $\mathcal{U}$; we then call $\mathcal{U}$ an *exstension* of $\mathcal{V}$ to BX. In this paper we consider the following type of question: If X and Y are completely regular spaces and if X is a subspace of Y, then what conditions placed on X and Y will guarantee the existence of admissible extensions to Y of various admissible uniform structures on X? We show that if X is a closed P-embedded (see § 1) subset of a completely regular space Y, then every admissible uniform structure on X has an admissible extension to Y; and that if X is a closed C^*-embedded subset of Y, then every admissible precompact uniform structure on X has an admissible (even precompact) extension to Y. In the case of normal spaces, the latter assertion generalizes the classical Tietze–Urysohn Extension Theorem.

We also consider the class of γ-uniform structures which was defined by G. Aquaro in [1]. We first characterize γ-uniform structures in terms of normal covers and then we show that, under suitable conditions, they also have admissible extensions. As a result of this, a characterization of γ-collectionwise normal spaces is obtained that generalizes a theorem due to Aquaro [2].

It will be assumed that the reader is familiar with the approaches to uniform space theory by means of reflexive relations, or *entourages*, and by means of normal covers. It is well known that these two approaches to uniform space theory are equivalent (see for example [7], p. 149). For details concerning the theory of uniform spaces, we refer the reader to Kelley [11] and to Tukey [20]. We also refer the reader to [20] for the elementary properties of normal covers, normal sequences of covers, and star-refinements.

* This research was supported in part by National Science Foundation grant GP-04065.

Most of the results in this paper are from the author's doctoral dissertation which was written at Purdue University under the direction of Professor Robert L. Blair, to whom the author wishes to express his appreciation.

1. Preliminaries. In this section we give most of the basic definitions and notational conventions that will be used in this paper.

Let X be a set and let γ be an infinite cardinal number. By a *γ-separable pseudometric* on X we mean a pseudometric d on X such that the topological space $(X, \mathcal{T}_d)$ has a dense subset A of power at most γ, where $\mathcal{T}_d$ denotes the topology on X induced by d. A pair (X, d) will be called a *γ-separable pseudometric space* whenever d is a γ-separable pseudometric on the set X. Following Shapiro [17], we say that a subset S of a topological space X is *P-embedded* in X (resp. *P^γ-embedded* in X) if every continuous (resp. continuous γ-separable) pseudometric on S has a continuous (resp. continuous γ-separable) pseudometric extension to X. Shapiro [17] proved that a subset S of a topological space X is P-embedded in X if and only if S is P^γ-embedded in X for every infinite cardinal number γ.

Following Gillman and Jerison [6], we denote by $C(X)$ (resp. $C^*(X)$) the ring of all continuous (resp. bounded continuous) real-valued functions on the topological space X. A subset S of a topological space X is *C-embedded* in X (resp. *C^*-embedded* in X) if every $f \in C(S)$ (resp. $f \in C^*(S)$) has an extension in $C(X)$. We denote by $\wedge$ and $\vee$ the lattice operations in $C(X)$, and if $\alpha \in \boldsymbol{R}$, then we denote by $\boldsymbol{\alpha}$ the constant function in $C(X)$ with value α. If f is a real-valued function on a set X, then by the *pseudometric on X associated with f* we mean the pseudometric Ψ_f on X defined by the equality $\Psi_f(x, y) = |f(x)-f(y)|$. The following result will be needed:

1.1. PROPOSITION ([5], 2.2). *If X is a topological space and if $f \in C(X)$, then the pseudometric Ψ_f on X associated with f is a continuous $\aleph_0$-separable pseudometric on X.*

We recall that if X is a set and $\mathcal{S}$ is a collection of pseudometrics on X, then the collection of all subsets of $X \times X$ of the form

$$\{(x, y) \in X \times X:\ d(x, y) < \varepsilon\},$$

where $d \in \mathcal{S}$ and $\varepsilon > 0$, forms a subbase for a unique uniform structure $\mathcal{U}$ on X, called the *uniform structure on X generated by $\mathcal{S}$*; if $\mathcal{S} = \{d\}$, then the uniform structure on X generated by $\mathcal{S}$ is denoted by $\mathcal{U}_d$ and is called the *uniform structure on X generated by d*. If $\mathcal{A}$ is a collection of real-valued functions on a set X, then the uniform structure on X generated by $\{\Psi_f:\ f \in \mathcal{A}\}$ is called the *uniform structure on X generated by $\mathcal{A}$*.

The uniform topology on X induced by a uniform structure $\mathcal{U}$ on X is denoted by $T(\mathcal{U})$. If $(X, \mathcal{T})$ is a topological space, and if $\mathcal{U}$ is a uniform structure on X, then $\mathcal{U}$ is called *admissible* if $T(\mathcal{U}) = \mathcal{T}$. If $(X, \mathcal{U})$ is a uniform space and if $S \subset X$, then we denote the relative uniform structure on S by $\mathcal{U}|S \times S$. If X is a set and if $\mathcal{U}$ is a uniform structure on a subset S of X, then we say that a uniform structure $\mathcal{V}$ on X is an *extension* of $\mathcal{U}$ provided that $\mathcal{V}|S \times S = \mathcal{U}$.

We know that if $(X, \mathcal{T})$ is a completely regular space, then $\mathcal{U}_0(X)$, $\mathcal{C}(X)$, and $\mathcal{C}^*(X)$ are admissible uniform structures on X, where $\mathcal{U}_0(X)$ (resp. $\mathcal{C}(X)$, $\mathcal{C}^*(X)$) denotes the uniform structure on X generated by the collection of all continuous pseudometrics on X (resp. by $C(X)$, by $C^*(X)$). $\mathcal{U}_0(X)$ is called the *universal uniform structure* on X since it is the largest admissible uniform structure on X.

Let $\mathfrak{U} = (U_\alpha)_{\alpha \in I}$ and $\mathfrak{B} = (V_\beta)_{\beta \in J}$ be two covers of a set X. We denote by $\mathfrak{U} \wedge \mathfrak{B}$ the cover $(U_\alpha \cap V_\beta)_{(\alpha,\beta) \in I \times J}$. If $S \subset X$, then $\mathrm{st}(S, \mathfrak{U})$ denotes the union of all U_α, $\alpha \in I$, such that $U_\alpha \cap S \neq \emptyset$; if $S = \{x\}$, then we write $\mathrm{st}(x, \mathfrak{U})$ in place of $\mathrm{st}(\{x\}, \mathfrak{U})$. As usual, we say that $\mathfrak{U}$ is a *refinement* of $\mathfrak{B}$, written $\mathfrak{U} < \mathfrak{B}$, if, for each $\alpha \in I$, there exists $\beta \in J$ such that $U_\alpha \subset V_\beta$; and we say that $\mathfrak{U}$ is a *star-refinement* of $\mathfrak{B}$, written $\mathfrak{U} <^* \mathfrak{B}$, if $\big(\mathrm{st}(U_\alpha, \mathfrak{U})\big)_{\alpha \in I}$ is a refinement of $\mathfrak{B}$.

We will say that a collection μ of covers of a set X *generates a uniform structure* $\mathcal{U}$ on X if the collection

$$\{\textstyle\bigcup_{\alpha \in I}(V_\alpha \times V_\alpha):\ (V_\alpha)_{\alpha \in I} \in \mu\}$$

is a subbase for $\mathcal{U}$. If $(X, \mathcal{U})$ is a nonempty uniform space, then a cover $\mathfrak{B}$ of X is a *uniform* cover if there exists $V \in \mathcal{U}$ such that $\big(V(x)\big)_{x \in X} < \mathfrak{B}$. (Every cover of an empty uniform space is uniform.) It is easy to see that the collection μ of all uniform covers of a uniform space X satisfies:

(1) If $\mathfrak{B} \in \mu$ and if $\mathfrak{B} < \mathfrak{W}$, then $\mathfrak{W} \in \mu$.

(2) If $\mathfrak{B}, \mathfrak{W} \in \mu$, then $\mathfrak{B} \wedge \mathfrak{W} \in \mu$.

(3) If $\mathfrak{W} \in \mu$, there exists $\mathfrak{B} \in \mu$ such that $\mathfrak{B} <^* \mathfrak{W}$.

Moreover, if μ denotes the collection of all uniform covers of the uniform space $(X, \mathcal{U})$, then μ generates the uniform structure $\mathcal{U}$.

We assume the T_1-separation axiom in the definition of completely regular spaces, but not in the definitions of normal and collectionwise normal spaces. Following Aquaro [2], if X is a topological space and if γ is an infinite cardinal number, then X is a *γ-collectionwise normal* space if, for every discrete family $(F_\alpha)_{\alpha \in I}$ of closed subsets of X of power at most γ (i.e. $|I| \leqslant \gamma$), there exists a family $(U_\alpha)_{\alpha \in I}$ of pairwise disjoint open subsets of X such that $F_\alpha \subset U_\alpha$ for each $\alpha \in I$. Thus, a space X is collectionwise normal if and only if it is γ-collectionwise normal for every infinite cardinal number γ. Also, according to Kuratowski [12], a space X is normal if and only if it is $\aleph_0$-collectionwise normal.

2. γ-uniform structures and u_γ-embedding. Throughout this section we will use γ to denote an arbitrary infinite cardinal number. If X is a topological space, then we denote by $\mathfrak{U}_\gamma(X)$ the uniform structure on X that is generated by the collection of all continuous γ-separable pseudometrics on X.

Since γ-separability, like separability, is an hereditary property for pseudometric spaces, it follows that if X is a topological space, and if $S \subset X$, then we have the inclusion $\mathfrak{U}_\gamma(X)|S \times S \subset \mathfrak{U}_\gamma(S)$.

2.1. Lemma. *If $(X, \mathfrak{T})$ is a completely regular space, then $\mathfrak{U}_\gamma(X)$ is an admissible uniform structure on X.*

Proof. Clearly, $\mathfrak{U}_0(X) \supset \mathfrak{U}_\gamma(X)$. Moreover, for each $f \in C(X)$, the pseudometric Ψ_f on X associated with f is $\aleph_0$-separable (1.1), and is therefore γ-separable. Thus we have $\mathfrak{U}_\gamma(X) \supset \mathfrak{C}(X)$. But both $\mathfrak{U}_0(X)$ and $\mathfrak{C}(X)$ are admissible structures, and so $\mathfrak{T} = T(\mathfrak{U}_0(X)) \supset T(\mathfrak{U}_\gamma(X)) \supset T(\mathfrak{C}(X)) = \mathfrak{T}$. Therefore $\mathfrak{U}_\gamma(X)$ is admissible.

As a result of this lemma, if X is a completely regular space, then we have the following chain of admissible uniform structures on X:

$$\mathfrak{U}_0(X) \supset \ldots \supset \mathfrak{U}_\gamma(X) \supset \ldots \supset \mathfrak{U}_{\aleph_0}(X) \supset \mathfrak{C}(X) \supset \mathfrak{C}^*(X)\,.$$

G. Aquaro [1] calls an open cover $(V_\alpha)_{\alpha \in I}$ of a topological space X *reducible* if there exists a closed cover $(F_\alpha)_{\alpha \in I}$ of X such that F_α and $X \setminus V_\alpha$ are completely separated in X for each $\alpha \in I$ (i.e. for each $\alpha \in I$, there exists $f_\alpha \in C(X)$ such that $f_\alpha(F_\alpha) \subset \{1\}$ and $f_\alpha(X \setminus V_\alpha) \subset \{0\}$). Now let X be a topological space and let μ be the collection of all reducible locally finite open covers of X of power at most γ. Then, according to Aquaro [1], μ not only generates a unique uniform structure on X, but immediately gives a base for this unique structure, which he calls the *γ-uniform structure* on X. Our immediate goal is to provide two simpler descriptions of Aquaro's γ-uniform structures. For this we need the following result:

2.2. Proposition. *If X is a topological space, then the collection of all reducible locally finite open covers of X of power at most γ coincides with the collection of all locally finite normal open covers of X of power at most γ.*

This proposition follows immediately from the following unpublished characterization of normal covers due to H. L. Shapiro whose proof is included here for completeness.

2.3. Theorem (Shapiro [16]). *If $\mathfrak{U} = (U_\alpha)_{\alpha \in I}$ is an open cover of a topological space $(X, \mathfrak{T})$, then the following statements are equivalent*

(1) *$\mathfrak{U}$ is normal.*

(2) *There exists a locally finite cozero-set cover $\mathfrak{V} = (V_\alpha)_{\alpha \in I}$ of X such that $\mathrm{cl}V_\alpha$ is completely separated from $X \setminus U_\alpha$ for each $\alpha \in I$.*

Proof (Shapiro [16]). (1) implies (2). By hypothesis, there exists a normal sequence $(\mathfrak{U}_i)_{i \in N}$ of open covers of X such that $\mathfrak{U}_1 < \mathfrak{U}$. By [20], Chapter V, Theorem 7.4, there is a continuous pseudometric d on X associated with $(\mathfrak{U}_i)_{i \in N}$. If $\mathfrak{T}_d$ denotes the topology on X induced by d, then $\mathfrak{T}_d \subset \mathfrak{T}$ since d is continuous. Using the first part of the proof of ([17], Lemma 2.6), one obtains a locally finite open cover $\mathfrak{W} = (W_\alpha)_{\alpha \in I}$ of $(X, \mathfrak{T}_d)$ such that $W_\alpha \subset U_\alpha$ for each $\alpha \in I$. Since (X, d) is a pseudometric space, $(X, \mathfrak{T}_d)$ is normal. Therefore, by [13], Theorem 33.4, there exists an open cover $(V_\alpha)_{\alpha \in I}$ of $(X, \mathfrak{T}_d)$ such that, for each $\alpha \in I$, $\mathrm{cl}V_\alpha \subset W_\alpha$ relative to $\mathfrak{T}_d$. Thus, for each $\alpha \in I$, there exists a continuous map f_α, relative to $\mathfrak{T}_d$, from X into the unit interval such that $f_\alpha(x) = 1$ for $x \in \mathrm{cl}V_\alpha$ and $f_\alpha(x) = 0$ for $x \in X \setminus W_\alpha$. But each V_α, being an open set in the pseudometric space (X, d), is a cozero-set relative to $\mathfrak{T}_d$. Since $\mathfrak{T}_d \subset \mathfrak{T}$, each f_α is continuous relative to $\mathfrak{T}$, and each V_α is a cozero-set relative to $\mathfrak{T}$. Hence $(V_\alpha)_{\alpha \in I}$ is a cozero-set cover of $(X, \mathfrak{T})$ such that $\mathrm{cl}V_\alpha$ is completely separated from $X \setminus U_\alpha$ for each $\alpha \in I$.

(2) implies (1). This implication is well known (see, for example [14], Theorem 1.2).

2.4. Lemma. *Suppose that $(X, \mathfrak{T})$ is a topological space, that d is a continuous γ-separable pseudometric on X, and that $\varepsilon > 0$. Then there exists a locally finite normal open cover $\mathfrak{V} = (V_\alpha)_{\alpha \in I}$ of X of power at most γ such that*

$$(1) \qquad \bigcup_{\alpha \in I} (V_\alpha \times V_\alpha) \subset \{(x, y) \in X \times X\colon\ d(x, y) < \varepsilon\}\,.$$

Proof. Let $\mathfrak{T}_d$ be the topology on X induced by d, and consider the open cover $\mathfrak{S} = (S_d(x, \varepsilon/2))_{x \in X}$ of $(X, \mathfrak{T}_d)$, where $S_d(x, \varepsilon/2)$ is, of course, the open ball with center at x and radius $\varepsilon/2$. Since $(X, \mathfrak{T}_d)$ is paracompact and has a base of power at most γ, it follows that there exists a locally finite open cover $\mathfrak{V} = (V_\alpha)_{\alpha \in I}$ of $(X, \mathfrak{T}_d)$ of power at most γ such that $\mathfrak{V} < \mathfrak{S}$. But then, according to A. H. Stone [19] and the fact that Stone's theorem also holds for pseudometric spaces, $\mathfrak{V}$ is a normal open cover of $(X, \mathfrak{T}_d)$. Since $\mathfrak{T}_d \subset \mathfrak{T}$, $\mathfrak{V}$ is a locally finite normal open cover of $(X, \mathfrak{T})$ of power at most γ. Finally, since $\mathfrak{V} < \mathfrak{S}$, the inclusion (1) is immediate.

2.5. Lemma. *If $\mathfrak{V} = (V_\alpha)_{\alpha \in I}$ is a normal open cover of a topological space X such that $|I| \leqslant \gamma$, then there exists a continuous γ-separable pseudometric d on X such that*

$$(2) \qquad \{(x, y) \in X \times X\colon\ d(x, y) < 2^{-2}\} \subset \bigcup_{\alpha \in I} (V_\alpha \times V_\alpha)\,.$$

Proof. This is an easy consequence of [17], 2.6 and 2.4.

2.6. Theorem. *The uniform structure $\mathfrak{U}_\gamma(X)$ on a topological space X is precisely the γ-uniform structure on X. Moreover, the γ-uniform structure* *on X may also be described as the uniform structure on X generated by the collection of all normal open covers of X of power at most γ.*

Proof. Let $\mathcal{U}'_\gamma(X)$ denote the γ-uniform structure on X, and let $\mathcal{U}''_\gamma(X)$ denote the uniform structure on X generated by the collection of all normal open covers of X of power at most γ. By 2.4 and 2.2, $\mathcal{U}_\gamma(X) \subset \mathcal{U}'_\gamma(X)$; by 2.2, $\mathcal{U}'_\gamma(X) \subset \mathcal{U}''_\gamma(X)$; and by 2.5, $\mathcal{U}''_\gamma(X) \subset \mathcal{U}_\gamma(X)$. Therefore, $\mathcal{U}_\gamma(X) = \mathcal{U}'_\gamma(X) = \mathcal{U}''_\gamma(X)$, as desired.

2.7. Proposition. *Let X be a topological space and denote by μ the collection of all reducible locally finite open covers of X of power at most γ. Then every element of μ is a uniform cover of $(X, \mathcal{U}_\gamma(X))$, and every uniform cover of $(X, \mathcal{U}_\gamma(X))$ has a refinement in μ.*

Proof. We may assume that $X \neq \emptyset$. Suppose that $\mathfrak{V} \in \mu$, and choose, by [1], § 2, Lemma 8, $\mathfrak{W} \in \mu$ such that $\mathfrak{W} <^* \mathfrak{V}$. Let $\mathfrak{W} = (W_a)_{a \in I}$, and set $W = \bigcup_{a \in I} (W_a \times W_a)$. Then $W \in \mathcal{U}_\gamma(X)$, and, for each $x \in X$, $W(x) = \mathrm{st}(x, \mathfrak{W})$. Therefore, $(W(x))_{x \in X}$ refines $\mathfrak{V}$, and hence $\mathfrak{V}$ is a uniform cover of $(X, \mathcal{U}_\gamma(X))$.

Now suppose that $\mathfrak{V}$ is a uniform cover of $(X, \mathcal{U}_\gamma(X))$. Then there exists $V \in \mathcal{U}_\gamma(X)$ such that $(V(x))_{x \in X}$ refines $\mathfrak{V}$, and there exists $\mathfrak{W} = (W_a)_{a \in I}$ in μ such that $W = \bigcup_{a \in I} (W_a \times W_a) \subset V$. Therefore $(W(x))_{x \in X}$ refines $\mathfrak{V}$. But, again, $W(x) = \mathrm{st}(x, \mathfrak{W})$ for each $x \in X$, and so $\mathfrak{W} < (W(x))_{x \in X}$, whence $\mathfrak{W} < \mathfrak{V}$. This completes the proof.

2.8. Theorem. *Equip a topological space X with the uniform structure $\mathcal{U}_\gamma(X)$, and let S be a uniform subspace of X. Then every bounded uniformly continuous γ-separable pseudometric on S has a bounded uniformly continuous γ-separable pseudometric extension to X.*

Proof. In view of 2.7, this result follows immediately from [5], Theorem 3.6.

Now, suppose that X is a completely regular space, and that $S \subset X$. Then S is said to be *u_γ-embedded* in X if every admissible uniform structure on S generated by a collection of continuous γ-separable pseudometrics on S has an admissible extension to X.

2.9. Proposition. *Suppose that X is a completely regular space, that $S \subset X$, and that $\gamma \geqslant \gamma' \geqslant \aleph_0$. If S is u_γ-embedded in X, then S is $u_{\gamma'}$-embedded in X.*

2.10. Proposition. *If S is a subset of a completely regular space X, then the following statements are equivalent:*

(1) *S is P^γ-embedded in X.*

(2) $\mathcal{U}_\gamma(S) = \mathcal{U}_\gamma(X)|S \times S$.

(3) *$\mathcal{U}_\gamma(S)$ has an admissible extension to X.*

Proof. (1) implies (2). Clearly we need only show that the inclusion $\mathcal{U}_\gamma(S) \subset \mathcal{U}_\gamma(X)|S \times S$ holds, and for this we need only show that a subbase of $\mathcal{U}_\gamma(S)$ is contained in $\mathcal{U}_\gamma(X)|S \times S$. Thus, suppose that d is a continuous γ-separable pseudometric on S and that $\varepsilon > 0$. Then the set $V = \{(x, y) \in S \times S\colon d(x, y) < \varepsilon\}$ is a typical subbase element of $\mathcal{U}_\gamma(S)$. By (1), there exists a continuous γ-separable pseudometric r on X such that $r|S \times S = d$. But then

$$V = \{(x, y) \in X \times X\colon r(x, y) < \varepsilon\} \cap (S \times S)$$

is an element of $\mathcal{U}_\gamma(X)|S \times S$. Hence (2) holds.

(2) implies (3). This is immediate.

(3) implies (1). By (3), there is an admissible extension $\mathcal{V}$ of $\mathcal{U}_\gamma(S)$ to X. Therefore, $(S, \mathcal{U}_\gamma(S))$ is a uniform subspace of $(X, \mathcal{V})$. If d is a continuous γ-separable pseudometric on S, then, by [11], Theorem 6.11, d is uniformly continuous on $(S, \mathcal{U}_\gamma(S))$, so by [5], Theorem 3.4, d has a continuous γ-separable pseudometric extension to X. It follows that S is P^γ-embedded in X.

2.11. Corollary. *If S is a u_γ-embedded subset of a completely regular space X, then S is P^γ-embedded in X.*

We remark that the converse of 2.11 is not true in general (see 3.26). However, the converse of 2.11 is true for closed sets (see 2.14).

2.12. Lemma. *If S is a P^γ-embedded subset of a topological space X, then $\mathrm{cl}\, S$ is P^γ-embedded in X.*

Proof. Suppose that d is a continuous γ-separable pseudometric on $\mathrm{cl}\, S$. Then $d|S \times S$ is a continuous γ-separable pseudometric on S, and so there exists, by hypothesis, a continuous γ-separable pseudometric r on X such that $r|S \times S = d|S \times S$. But $S \times S$ is dense in $\mathrm{cl}\, S \times \mathrm{cl}\, S$, so by continuity, $d = r|\mathrm{cl}\, S \times \mathrm{cl}\, S$. Thus $\mathrm{cl}\, S$ is P^γ-embedded in X.

2.13. Theorem. *If S is a P^γ-embedded subset of a completely regular space X, then $\mathrm{cl}\, S$ is u_γ-embedded in X.*

Proof. Suppose that S is a P^γ-embedded subset of X, and set $F = \mathrm{cl}\, S$. Then, by 2.12, F is a closed P^γ-embedded subset of X. Let $\mathcal{U}$ be an admissible uniform structure on F generated by a collection $\mathcal{T}_0$ of continuous γ-separable pseudometrics on F, and let $\mathcal{T}$ be the collection of all uniformly continuous γ-separable pseudometrics on $(F, \mathcal{U})$. By [11], 6.11 and 6.15, $\mathcal{T}_0 \subset \mathcal{T}$ and $\mathcal{U}$ is generated by $\mathcal{T}$. Let $\mathcal{T}^*$ be the collection of all continuous γ-separable pseudometrics on X such that, for each $d \in \mathcal{T}^*$, we have $d|F \times F \in \mathcal{T}$. Since F is P^γ-embedded in X, we have $\{d|F \times F\colon d \in \mathcal{T}^*\} = \mathcal{T}$. Now let $\mathcal{V}$ be the uniform structure on X generated by $\mathcal{T}^*$. Then the uniform structure $\mathcal{V}|F \times F$ of F is generated by the collection $\{d|F \times F\colon d \in \mathcal{T}^*\} = \mathcal{T}$, and so $\mathcal{U} = \mathcal{V}|F \times F$. In order to complete the proof, we must show that $\mathcal{V}$ is admissible.

Let $\mathcal{T}$ be the topology on X. Since each element of $\mathcal{F}^*$ is continuous relative to $\mathcal{T}$, we have $T(\mathcal{V}) \subset \mathcal{T}$. We now show that $\mathcal{T} \subset T(\mathcal{V})$. Suppose that $G \in \mathcal{T}$ and that $x_0 \in G$. Since the cozero-sets in $(X, \mathcal{T})$ form a base for $\mathcal{T}$, we may assume that G is a cozero-set in $(X, \mathcal{T})$. We consider two cases.

If $x_0 \notin F$, then $G \cap (X \setminus F)$ is a neighborhood of x_0 in X, so there exists $f \in C(X)$ such that

$$0 \leqslant f \leqslant 1, \quad f(x_0) = 1, \quad \text{and} \quad f(F \cup (X \setminus G)) \subset \{0\}.$$

Then the pseudometric Ψ_f on X associated with f is a continuous $\aleph_0$-separable pseudometric on X (1.1), and if $(x, y) \in F \times F$, then $\Psi_f(x, y) = |f(x) - f(y)| = 0$, so Ψ_f is a continuous extension to X of the zero pseudometric $\mathbf{0}$ on F. Since $\mathbf{0}$ is obviously uniformly continuous with respect to $\mathcal{U}$ and is γ-separable, we have $\mathbf{0} \in \mathcal{F}$. Also, since $\gamma \geqslant \aleph_0$, it follows that Ψ_f is γ-separable, and consequently that $\Psi_f \in \mathcal{F}^*$. Then the set

$$W = \{x \in X\colon\ \Psi_f(x, x_0) < 1\}$$

is an element of $T(\mathcal{V})$, and we clearly have $x_0 \in W \subset G$, since $x \in X \setminus G$ implies $\Psi_f(x, x_0) = 1$.

Next we suppose that $x_0 \in F$. Since G is a cozero-set neighborhood of x_0 in X, there exists a zero-set neighborhood Z of x_0 in X such that $Z \subset G$. Then Z and $X \setminus G$ are disjoint zero-sets in X, so there is a function $g \in C(X)$ such that

$$0 \leqslant g \leqslant 1, \quad g(Z) = \{0\}, \quad \text{and} \quad g(X \setminus G) \subset \{1\}.$$

Since Z is a neighborhood of x_0 in X, and since $\mathcal{U}$ is admissible, then $Z \cap F$ is a neighborhood of x_0 in the uniform space $(F, \mathcal{U})$. Therefore, by a result of Weil [21] (see [8], I. 13), there exists a uniformly continuous real-valued function f on $(F, \mathcal{U})$ such that

$$0 \leqslant f \leqslant 1, \quad f(x_0) = 0, \quad \text{and} \quad f(F \setminus (F \cap Z)) \subset \{1\}.$$

Then the pseudometric Ψ_f on F associated with f is a uniformly continuous $\aleph_0$-separable (hence, γ-separable) pseudometric on $(F, \mathcal{U})$. It follows that $\Psi_f \in \mathcal{F}$, and so there exists $d \in \mathcal{F}^*$ such that $d|F \times F = \Psi_f$. We now define a function d_{x_0} from X into the real line $\boldsymbol{R}$ by setting

$$d_{x_0}(x) = d(x_0, x) \qquad (x \in X).$$

Then d_{x_0} is continuous, and, for each $x \in F$, we have

$$d_{x_0}(x) = d(x_0, x) = \Psi_f(x_0, x) = |f(x_0) - f(x)| = f(x),$$

so that $d_{x_0}|F = f$. Now we set

$$h = d_{x_0} \vee g.$$

Then h is a continuous function on X and $h(x) = f(x)$ for all $x \in F$. Therefore the pseudometric Ψ_h on X associated with h is a continuous $\aleph_0$-separable pseudometric on X (1.1) such that $\Psi_h|F \times F = \Psi_f$. Since $\gamma \geqslant \aleph_0$, it follows that Ψ_h is γ-separable and therefore $\Psi_h \in \mathcal{F}^*$. Finally, we set

$$W = \{x \in X\colon\ \Psi_h(x, x_0) < 1\}.$$

Then, obviously, $x_0 \in W$ and $W \in T(\mathcal{V})$. Moreover, if $x \in W$, then, since $h(x_0) = f(x_0) = 0$, we have $g(x) \leqslant h(x) = \Psi_h(x, x_0) < 1$, whence $x \in G$.

Therefore we have found, in either case, $W \in T(\mathcal{V})$ such that $x_0 \in W \subset G$. It follows that $G \in T(\mathcal{V})$ and so $\mathcal{T} \subset T(\mathcal{V})$. Hence $\mathcal{T} = T(\mathcal{V})$, and so $\mathcal{V}$ is admissible.

2.14. Corollary. *If F is a closed P^γ-embedded subset of a completely regular space X, then F is u_γ-embedded in X.*

Recall that a completely regular space is *realcompact* in case $(X, \mathcal{C}(X))$ is a complete uniform space (see [6], 15.14). We will now show that the converse of 2.14 is valid for realcompact subsets of a completely regular space.

2.15. Theorem. *If S is a realcompact subset of a completely regular space X, then the following statements are equivalent:*

(1) *S is u_γ-embedded in X.*

(2) *S is closed and P^γ-embedded in X.*

Proof. Suppose that S is u_γ-embedded in X. Then, by 2.11, S is P^γ-embedded in X. By 1.1, the uniform structure $\mathcal{C}(S)$ on S is generated by a collection of $\aleph_0$-separable (hence, γ-separable) continuous pseudometrics on S, and therefore, by (1), there is an admissible extension $\mathcal{U}$ of $\mathcal{C}(S)$ to X. But then, since S is realcompact, $(S, \mathcal{C}(S))$ is a complete uniform subspace of the (Hausdorff) uniform space $(X, \mathcal{U})$, and hence S is closed in X. The converse follows at once from 2.14.

2.16. Theorem. *Suppose that S is a u_γ-embedded subset of a completely regular space X. If $\mathcal{U}$ is an admissible uniform structure on S generated by a collection of continuous γ-separable pseudometrics on S, then there is an admissible extension of $\mathcal{U}$ to X that is generated by a collection of continuous γ-separable pseudometrics on X.*

Proof. We first note that, by 2.11, S is P^γ-embedded in X. Then, as the proof of 2.13 shows, every admissible uniform structure on $\operatorname{cl} S$ generated by a collection of continuous γ-separable pseudometrics on $\operatorname{cl} S$ has an admissible extension to X that is generated by a collection of continuous γ-separable pseudometrics on X. Therefore, we may assume that S is dense in X.

Now suppose that $\mathcal{U}$ is an admissible uniform structure on S generated by a collection $\mathcal{F}$ of continuous γ-separable pseudometrics on S. By virtue of [11], 6.15, we may assume that $\mathcal{I}$ is the collection of all uniformly continuous γ-separable pseudometrics on $(S, \mathfrak{U})$. Also, by hypothesis, there exists an admissible extension $\mathfrak{V}$ of $\mathfrak{U}$ to X, so that $(S, \mathfrak{U})$ is a uniform subspace of $(X, \mathfrak{V})$. Let $\mathcal{I}^*$ be the collection of all uniformly continuous γ-separable pseudometrics on $(X, \mathfrak{V})$. We claim that

$$\mathcal{I} = \{d|S\times S\colon\ d \in \mathcal{I}^*\}\,.$$

We clearly have $\{d|S\times S\colon\ d\in\mathcal{I}^*\}\subset\mathcal{I}$. Thus, suppose that $r\in\mathcal{I}$. Equipped with the product uniform structures, $S\times S$ is a uniform subspace of $X\times X$ and r is uniformly continuous on $S\times S$. Therefore there is a unique continuous map p from $\mathrm{cl}(S\times S) = X\times X$ into the nonnegative real numbers $\boldsymbol{R}^+$ that extends r (see [11], 6.26). It is easy to see, by continuity, that p must also be a pseudometric. Moreover, since r is γ-separable and S is dense in X, it follows that p is γ-separable. Therefore, $p\in\mathcal{I}^*$, and so $r = p|S\times S\in\{d|S\times S\colon\ d\in\mathcal{I}^*\}$.

Finally, suppose that $\mathcal{C}$ is the topology on X, and let $\mathfrak{W}$ denote the uniform structure on X generated by $\mathcal{I}^*$. Since $\mathcal{I} = \{d|S\times S\colon\ d\in\mathcal{I}^*\}$, it follows that $\mathfrak{W}|S\times S = \mathfrak{U}$. Moreover, since each element of $\mathcal{I}^*$ is a uniformly continuous pseudometric on $(X, \mathfrak{V})$, we have $\mathfrak{W}\subset\mathfrak{V}$, whence $T(\mathfrak{W})\subset T(\mathfrak{V}) = \mathcal{C}$. Thus, it remains to show that $\mathcal{C}\subset T(\mathfrak{W})$. Suppose that $G\in\mathcal{C}$ and let $x\in G$. Then, by a result of Weil [21], there exists a uniformly continuous real-valued function f on $(X, \mathfrak{V})$ such that

$$0\leqslant f\leqslant 1\,,\quad f(x) = 0\,,\quad \text{and}\quad f(X\setminus G)\subset\{1\}\,.$$

But then, the pseudometric Ψ_f on X associated with f is a uniformly continuous $\aleph_0$-separable (hence, γ-separable) pseudometric on $(X, \mathfrak{V})$, by 1.1, whence $\Psi_f\in\mathcal{I}^*$. Set $W = \{y\in X\colon\ \Psi_f(x, y) < 1\}$. Then $W\in T(\mathfrak{W})$ and $x\in W\subset G$, whence $\mathcal{C}\subset T(\mathfrak{W})$. Therefore $\mathfrak{W}$ is an admissible extension of $\mathfrak{U}$ to X that is generated by a collection of continuous γ-separable pseudometrics on X, and the proof is complete.

The following result is now obvious:

2.17. Proposition. *If X is a completely regular space and $A\subset B\subset X$, then the following statements are true:*

(1) *If A is u_γ-embedded in X, then A is u_γ-embedded in B.*

(2) *If A is u_γ-embedded in B, and if B is u_γ-embedded in X, then A is u_γ-embedded in X.*

In order to give a simple characterization of γ-collectionwise normal spaces, we need the following theorem:

2.18. Theorem (Katětov [10]). *The following statements are equivalent for a normal space X:*

(1) *X is γ-collectionwise normal.*

(2) *For every closed $S\subset X$, if $(F_\alpha)_{\alpha\in I}$* (resp. $(H_\alpha)_{\alpha\in I}$) *is a locally finite family of closed* (resp. *open*) *subsets of S such that $|I|\leqslant\gamma$ and $F_\alpha\subset H_\alpha$ for each $\alpha\in I$, then there exists a locally finite family $(G_\alpha)_{\alpha\in I}$ of open subsets of X such that $F_\alpha\subset G_\alpha\cap S\subset H_\alpha$ for each $\alpha\in I$.*

We refer the reader to [10], 3.1, for the proof of 2.18, and we note that, although Katětov originally stated this theorem for collectionwise normal spaces, his proof is valid for the case of γ-collectionwise normal spaces as stated above.

2.19. Theorem. *The following statements are equivalent for a topological space X:*

(1) *X is γ-collectionwise normal.*

(2) *Every closed subset of X is P^γ-embedded in X.*

Proof. (1) implies (2). This is immediate from 2.18 and [17], Theorem 2.7.

(2) implies (1). Let $(F_\alpha)_{\alpha\in I}$ be a discrete family of closed subsets of X such that $|I|\leqslant\gamma$, and set $F = \bigcup_{\alpha\in I} F_\alpha$. Then F is a closed subset of X. Let p be the discrete metric on I (i.e., $p(\alpha, \beta) = 1$ if $\alpha\neq\beta$), and equip I with the discrete topology $\mathcal{C}_p$ so that p is a continuous γ-separable metric on I. Moreover, let f be the map from F into I defined by

$$f(F_\alpha)\subset\{\alpha\}\quad \text{for all } \alpha\in I\,.$$

Then f is continuous and hence the pseudometric $r = p\circ(f\times f)$ is a continuous γ-separable pseudometric on F (see [5], 2.1). Consequently, by (2), there exists a continuous pseudometric d on X such that $d|F\times F = r$. Now, for each $\alpha\in I$, we set

$$U_\alpha = \bigcup_{x\in F_\alpha} S_d(x, \tfrac{1}{2})\,.$$

It is easily seen that $(U_\alpha)_{\alpha\in I}$ is a pairwise disjoint family of open subsets of X such that $F_\alpha\subset U_\alpha$ for each $\alpha\in I$. Therefore X is γ-collectionwise normal.

2.20. Theorem. *The following statements are equivalent for a completely regular space X:*

(1) *X is γ-collectionwise normal.*

(2) *Every closed subset of X is u_γ-embedded in X.*

(3) *For every closed subset F of X, $\mathfrak{U}_\gamma(F)$ has an admissible extension to X.*

(4) *For every closed subset F of X, $\mathfrak{U}_\gamma(F) = \mathfrak{U}_\gamma(X)|F\times F$.*

Proof. The equivalence of (1) and (4) was proved by Aquaro in [2]. The implications: (1) implies (2), (2) implies (3), and (3) implies (4), are immediate from, respectively, 2.19 and 2.14, the definitions, and 2.10.

3. Normal and collectionwise normal spaces. In section 2, we have seen that the γ-uniform structure is the natural structure to consider when talking about γ-collectionwise normal spaces. In this section, we show that collectionwise normal T_1-spaces can be characterized in terms of universal uniform structures, a result that can immediately be obtained from [2], and hence in terms of extensions of arbitrary admissible uniform structures. Then we discuss the relationships of the uniform structures $\mathfrak{U}_{\aleph_0}(X)$, $\mathfrak{C}(X)$, and $\mathfrak{C}^*(X)$ in a normal T_1-space X.

If S is a subset of a completely regular space X, then we say that S is *u_0-embedded* in X in case every admissible uniform structure on S has an admissible extension to X. The following result is obvious:

3.1. PROPOSITION. *If X is a completely regular space and $A \subset B \subset X$, then the following statements are true*:

(1) *If A is u_0-embedded in X, then A is u_0-embedded in B.*

(2) *If A is u_0-embedded in B, and if B is u_0-embedded in X, then A is u_0-embedded in X.*

3.2. PROPOSITION. *If S is a subset of a completely regular space X, then the following statements are equivalent*:

(1) *S is P-embedded in X.*

(2) $\mathfrak{U}_0(S) = \mathfrak{U}_0(X)|S \times S$.

(3) *$\mathfrak{U}_0(S)$ has an admissible extension to X.*

Proof. (1) implies (2). Let $\gamma = |X| + \aleph_0$. Then S is P^γ-embedded in X, by (1), and every continuous pseudometric on X (resp. on S) is γ-separable. Therefore $\mathfrak{U}_0(S) = \mathfrak{U}_\gamma(S)$ and $\mathfrak{U}_0(X) = \mathfrak{U}_\gamma(X)$, and so the implication follows from 2.10.

(2) implies (3). This is immediate.

(3) implies (1). Let $\mathfrak{V}$ be an admissible extension of $\mathfrak{U}_0(S)$ to X. Then $(S, \mathfrak{U}_0(S))$ is a uniform subspace of $(X, \mathfrak{V})$, and every continuous pseudometric on S is uniformly continuous on $(S, \mathfrak{U}_0(S))$. Therefore S is P-embedded in X, by [5], 3.5.

We note that the equivalence of (1) and (2) of 3.2 may be obtained from [17], Theorem 2.1, since the universal uniform structure on a completely regular space X is that structure on X generated by the collection of all normal covers of X (see [8], Theorem I.20).

3.3. COROLLARY. *If S is a u_0-embedded subset of a completely regular space X, then S is P-embedded in X.*

As the following example shows, the converse of 3.3 is not true in general.

3.4. EXAMPLE. Let X be any pseudocompact completely regular space such that $|X|$ is a nonmeasurable cardinal number and such that X is not almost compact (i.e. $|\beta X \setminus X| \geqslant 2$). A simple example of such a space is the topological sum of two copies of $W(\omega_1)$, where $W(\omega_1)$ denotes the ordered space of all countable ordinal numbers. Now, since X is pseudocompact, every continuous real-valued function on X is bounded, and so it follows that $vX = \beta X$, where βX denotes the Stone-Čech compactification of X, and vX denotes the Hewitt realcompactification of X (cf. [6]). Set $Y = \beta X$. Then X is dense and C-embedded in Y, and $|Y|$ is a nonmeasurable cardinal number. Consequently, by [17], 3.3, X is P-embedded in Y. On the other hand, since X is not almost compact, it follows that there are at least two distinct admissible (even admissible precompact) uniform structures $\mathfrak{U}_1$ and $\mathfrak{U}_2$ on X (see [6], Problem 15R). But Y is compact, so there is a unique admissible uniform structure $\mathfrak{V}$ on Y, and clearly $\mathfrak{V}$ cannot be an extension of both $\mathfrak{U}_1$ and $\mathfrak{U}_2$. Therefore X is not u_0-embedded in Y.

3.5. THEOREM. *If S is a P-embedded subset of a completely regular space X, then $\mathrm{cl}\,S$ is u_0-embedded in X.*

Proof. Let $\gamma = |\mathrm{cl}\,S| + \aleph_0$, and suppose that $\mathfrak{U}$ is an admissible uniform structure on $\mathrm{cl}\,S$. Then $\mathfrak{U}$ is generated by a collection $\mathcal{F}$ of continuous pseudometrics on $\mathrm{cl}\,S$. Clearly, for each $d \in \mathcal{F}$, d is γ-separable. Consequently, since S is P^γ-embedded in X, we may apply 2.13 to conclude that $\mathfrak{U}$ has an admissible extension to X. Therefore $\mathrm{cl}\,S$ is u_0-embedded in X.

3.6. COROLLARY. *If F is a closed P-embedded subset of a completely regular space X, then F is u_0-embedded in X.*

We recall that a completely regular space X is called *topologically complete* if there exists an admissible uniform structure $\mathfrak{U}$ on X such that $(X, \mathfrak{U})$ is a complete uniform space.

3.7. THEOREM. *If S is a topologically complete subset of a completely regular space X, then the following statements are equivalent*:

(1) *S is u_0-embedded in X.*

(2) *S is closed and P-embedded in X.*

The proof of this result is very similar to that of 2.15, and therefore will be omitted.

We now give several characterizations of collectionwise normal spaces. The following theorem, which easily follows from 2.19, was implicitly proved by C. H. Dowker in [4]:

3.8. THEOREM. *The following statements are equivalent for a topological space X*:

(1) *X is collectionwise normal.*

(2) *Every closed subset of X is P-embedded in X.*

Now, using 3.8, 3.6, and 3.2, we have:

3.9. THEOREM. *The following statements are equivalent for a completely regular space X:*

(1) *X is collectionwise normal.*

(2) *Every closed subset of X is u_0-embedded in X.*

(3) *For every closed subset F of X, $\mathfrak{U}_0(F)$ has an admissible extension to X.*

(4) *For every closed subset F of X, $\mathfrak{U}_0(F) = \mathfrak{U}_0(X)|F\times F$.*

If S is a subset of a completely regular space X, we say that S is *u-embedded* (resp. *u^*-embedded*) in X in case every admissible uniform structure on S generated by a collection of continuous (resp. bounded continuous) real-valued functions on S has an admissible extension to X.

It is well-known (cf. [15]) that an admissible uniform structure $\mathfrak{U}$ on a completely regular space X is generated by a collection of bounded continuous real-valued functions if and only if $\mathfrak{U}$ is precompact. Therefore a subset S of a completely regular space X is u^*-embedded in X if and only if every admissible precompact uniform structure on S has an admissible extension to X. The following result is obvious:

3.10. PROPOSITION. *If S is a pseudocompact subset of a completely regular space X, then S is u-embedded in X if and only if S is u^*-embedded in X.*

The proof of 2.10 is easily modified to obtain proofs of Propositions 3.11 and 3.13 below. To get the last implication (i.e., (3) implies (1)) of 3.11, one uses [5], 3.16, and to get the last implication of 3.13, one uses [9], Theorem 3. The first implication of 3.11 and all of 3.13 seem to be well-known (cf. [6]). Shapiro first noticed the implication "(3) implies (1)" of 3.11 (cf. [18]).

3.11. PROPOSITION. *If S is a subset of a completely regular space X, then the following statements are equivalent:*

(1) *S is C-embedded in X.*

(2) $\mathfrak{C}(S) = \mathfrak{C}(X)|\,S\times S$.

(3) $\mathfrak{C}(S)$ *has an admissible extension to X.*

3.12. COROLLARY. *Every u-embedded subset of a completely regular space X is C-embedded in X.*

3.13. PROPOSITION. *If S is a subset of a completely regular space X, then the following statements are equivalent:*

(1) *S is C^*-embedded in X.*

(2) $\mathfrak{C}^*(S) = \mathfrak{C}^*(X)|S\times S$.

(3) $\mathfrak{C}^*(S)$ *has an admissible extension to X.*

3.14. COROLLARY. *Every u^*-embedded subset of a completely regular space X is C^*-embedded in X.*

We now observe that the converse of neither 3.12 nor 3.14 is true in general, since Example 3.4 provides an example of a space X that is both C^*-embedded and C-embedded in βX, but is neither u^*-embedded nor u-embedded in βX.

3.15. LEMMA. *If S is a C-embedded* (resp. *C^*-embedded*) *subset of a topological space X, then* $\mathrm{cl}\,S$ *is C-embedded* (resp. *C^*-embedded*) *in X.*

Using this lemma, along with a suitable modification of the proof of 2.13 (i.e. use functions instead of pseudometrics), one obtains the following theorem:

3.16. THEOREM. *If S is a C-embedded* (resp. *C^*-embedded*) *subset of a completely regular space X, then* $\mathrm{cl}\,S$ *is u-embedded* (resp. *u^*-embedded*) *in X.*

Note that, in this theorem, the extension of an admissible uniform structure on $\mathrm{cl}\,S$ generated by a subcollection of $C(\mathrm{cl}\,S)$ can be chosen to be a uniform structure on X generated by an associated subcollection of $C(X)$. The details are left to the reader.

3.17. COROLLARY. *If F is a closed C-embedded* (resp. *C^*-embedded*) *subset of a completely regular space X, then F is u-embedded* (resp. *u^*-embedded*) *in X.*

3.18. THEOREM. *Let S be a realcompact subset of a completely regular space X. Then S is u-embedded in X if and only if S is closed and C-embedded in X.*

The proof of this result is similar to that of 2.15. There does not seem to be a nontrivial analogue of this result relative to u^*-embedded subsets.

3.19. THEOREM. *Suppose that X is a completely regular space and that S is a u-embedded subset of X. If $\mathfrak{U}$ is an admissible uniform structure on S generated by a collection of continuous real-valued functions on S, then there is an admissible extension of $\mathfrak{U}$ to X that is generated by a collection of continuous real-valued functions on X.*

The proof of this result is similar to that of 2.16. The following proposition is now obvious:

3.20. PROPOSITION. *If X is a completely regular space and $A\subset B\subset X$, then the following statements are true:*

(1) *If A is u-embedded in X, then A is u-embedded in B.*

(2) *If A is u-embedded in B, and if B is u-embedded in X, then A is u-embedded in X.*

For precompact uniform structures, we can obtain a result that is somewhat stronger that 3.19.

3.21. THEOREM. *Suppose that S is a subset of a completely regular space X and that $\mathcal{U}$ is an admissible precompact uniform structure on S. If $\mathcal{U}$ has an admissible extension to X, then $\mathcal{U}$ has an admissible precompact extension to X.*

Proof. Let $\mathcal{A}$ be the set of all $f \in C^*(S)$ that are uniformly continuous with respect to $\mathcal{U}$. Then $\mathcal{U}$ is generated by $\mathcal{A}$. Let $\mathcal{V}$ be an admissible extension of $\mathcal{U}$ to X, and let $\mathcal{B}$ be the set of all $g \in C^*(X)$ that are uniformly continuous with respect to $\mathcal{V}$. Then $\mathcal{B}$ generates a precompact uniform structure $\mathcal{W}$ on X such that $\mathcal{W} \subset \mathcal{V}$. We now show that $\mathcal{W}$ is admissible. Let $\mathcal{T}$ be the topology on X. Then, clearly, $T(\mathcal{W}) \subset T(\mathcal{V}) = \mathcal{T}$. Suppose that $G \in \mathcal{T}$ and that $x \in G$. Then, by a result due to Weil [21], there exists $f \in \mathcal{B}$ such that

$$0 \leqslant f \leqslant 1, \quad f(x) = 0, \quad \text{and} \quad f(X \setminus G) \subset \{1\}.$$

Therefore the set $W = \{y \in X\colon f(y) < 1\}$ is an element of $T(\mathcal{W})$, and we have $x \in W \subset G$. Hence $G \in T(\mathcal{W})$ and so $\mathcal{T} \subset T(\mathcal{W})$. It follows that $\mathcal{W}$ is admissible. Thus it remains to show that $\mathcal{W}|S \times S = \mathcal{U}$. Clearly, we have $\mathcal{W}|S \times S \subset \mathcal{V}|S \times S = \mathcal{U}$. To show that $\mathcal{U} \subset \mathcal{W}|S \times S$, it suffices to show that a subbase of $\mathcal{U}$ is contained in $\mathcal{W}|S \times S$. Thus, suppose that $h \in \mathcal{A}$, and $\varepsilon > 0$. Then consider the typical subbase element

$$U = \{(x, y) \in S \times S\colon |h(x) - h(y)| < \varepsilon\}$$

of $\mathcal{U}$. Since h is uniformly continuous with respect to $\mathcal{U}$, and since $(S, \mathcal{U})$ is a uniform subspace of $(X, \mathcal{V})$, it follows by [9], Theorem 3, that there exists a bounded uniformly continuous real-valued function g on $(X, \mathcal{V})$ such that $g|S = h$. Therefore $g \in \mathcal{B}$, and hence the set

$$V = \{(x, y) \in X \times X\colon |g(x) - g(y)| < \varepsilon\}$$

is an element of $\mathcal{W}$. But then, it is clear that

$$U = V \cap (S \times S) \in \mathcal{W}|S \times S,$$

whence $\mathcal{U} \subset \mathcal{W}|S \times S$. Therefore $\mathcal{W}$ is an admissible precompact extension of $\mathcal{U}$ to X.

The following proposition is now obvious:

3.22. PROPOSITION. *If X is a completely regular space and $A \subset B \subset X$, then the following statements are true:*

(1) *If A is u^*-embedded in X, then A is u^*-embedded in B.*

(2) *If A is u^*-embedded in B, and if B is u^*-embedded in X, then A is u^*-embedded in X.*

Remark. We recall (see [6], Problem 15D.1) that if $\mathcal{S}$ is a collection of continuous γ-separable pseudometrics on a topological space X that generates a uniform structure $\mathcal{U}$ on X, then $\mathcal{S}_0 = \{d \wedge 1\colon d \in \mathcal{S}\}$ also generates $\mathcal{U}$, and moreover, each element of $\mathcal{S}_0$ is still γ-separable. Therefore, assuming the generalized continuum hypothesis, we can prove a result similar to 3.21 for uniform structures generated by collections of continuous γ-separable pseudometrics, thus improving 2.16. To do this, one modifies the proof of 3.21 and uses [5], 3.11 in place of [9], Theorem 3. We note that [5], 3.11 is true for $\gamma = \aleph_0$ without assuming the generalized continuum hypothesis (see [5], 3.8). Therefore, for $\gamma = \aleph_0$, one gets:

3.23. THEOREM. *Suppose that S is a subset of a completely regular space X, and that $\mathcal{U}$ is an admissible uniform structure on S generated by a collection of continuous $\aleph_0$-separable pseudometrics on S. If $\mathcal{U}$ has an admissible extension to X, then $\mathcal{U}$ has an admissible extension to X that is generated by a collection of continuous $\aleph_0$-separable pseudometrics on X.*

The next result clarifies the relationship between the concepts of u-embedding and $u_{\aleph_0}$-embedding.

3.24. THEOREM. *If S is a subset of a completely regular space X, then the following statements are equivalent:*

(1) *S is $u_{\aleph_0}$-embedded in X.*

(2) *S is u-embedded in X.*

Proof. (1) implies (2). This is immediate by 1.1.

(2) implies (1). Let $\mathcal{T}$ be the topology on X, and let $\mathcal{U}$ be an admissible uniform structure on S generated by continuous $\aleph_0$-separable pseudometrics on S. Let $\mathcal{S}$ denote the collection of all uniformly continuous $\aleph_0$-separable pseudometrics on $(S, \mathcal{U})$, and let $\mathcal{A}$ denote the collection of all bounded uniformly continuous real-valued functions on $(S, \mathcal{U})$. Then $\{\Psi_f\colon f \in \mathcal{A}\} \subset \mathcal{S}$ and $\mathcal{S}$ generates $\mathcal{U}$. Let $\mathcal{U}^*$ be the uniform structure on S generated by $\mathcal{A}$. Clearly, $T(\mathcal{U}^*) \subset T(\mathcal{U}) = \mathcal{T}|S$. Using an argument similar to that in the last paragraph of the proof of 2.16, it is easily seen that $\mathcal{T}|S \subset T(\mathcal{U}^*)$. Therefore $\mathcal{U}^*$ is admissible on S. Since $\mathcal{U}^*$ is generated by bounded functions, it is precompact, and so, by (2) and 3.21, there is an admissible precompact uniform structure $\mathcal{V}^*$ on X such that $\mathcal{V}^*|S \times S = \mathcal{U}^*$. Moreover, $\mathcal{V}^*$ is generated by the collection $\mathcal{B}$ of all bounded uniformly continuous real-valued functions on $(X, \mathcal{V}^*)$. Next, we note that, by (2), 3.12, and [17], 4.7, S is $P^{\aleph_0}$-embedded in X. Finally, we let $\mathcal{S}^*$ denote the collection of all continuous $\aleph_0$-separable pseudometrics on X such that $d \in \mathcal{S}^*$ implies $d|S \times S \in \mathcal{S}$, and we let $\mathcal{V}$ denote the uniform structure on X generated by $\mathcal{S}^*$. Since $\mathcal{V}|S \times S$ is generated by $\{d|S \times S\colon d \in \mathcal{S}^*\} = \mathcal{S}$, we have $\mathcal{U} = \mathcal{V}|S \times S$. Moreover, since each $d \in \mathcal{S}^*$ is continuous on $(X, \mathcal{T})$, we have $T(\mathcal{V}) \subset \mathcal{T}$. It remains to show that $\mathcal{T} \subset T(\mathcal{V})$. But if $f \in \mathcal{B}$, then $f|S$ is uniformly continuous with respect to $\mathcal{U}^*$, and hence with respect to $\mathcal{U}$, since $\mathcal{U}^* \subset \mathcal{U}$. Therefore $\Psi_{f|S} \in \mathcal{S}$ and so $\Psi_f \in \mathcal{S}^*$ (1.1). It follows that $\mathcal{V}^* \subset \mathcal{V}$, and consequently $\mathcal{T} = T(\mathcal{V}^*)$

$\subset T(\mathfrak{V})$. Thus $\mathfrak{V}$ is an admissible extension of $\mathfrak{U}$ to X, so S is $u_{\aleph_0}$-embedded in X.

3.25. THEOREM. *For a completely regular space X, the following statements are equivalent:*

(1) *X is normal.*

(2) *Every closed subset of X is $u_{\aleph_0}$-embedded in X.*

(3) *Every closed subset of X is u-embedded in X.*

(4) *Every closed subset of X is u^*-embedded in X.*

(5) *For each closed $F \subset X$, $\mathfrak{U}_{\aleph_0}(F) = \mathfrak{U}_{\aleph_0}(X)|F \times F$.*

(6) *For each closed $F \subset X$, $\mathfrak{C}(F) = \mathfrak{C}(X)|F \times F$.*

(7) *For each closed $F \subset X$, $\mathfrak{C}^*(F) = \mathfrak{C}^*(X)|F \times F$.*

(8) *For each closed $F \subset X$, $\mathfrak{U}_{\aleph_0}(F)$ has an admissible extension to X.*

(9) *For each closed $F \subset X$, $\mathfrak{C}(F)$ has an admissible extension to X.*

(10) *For each closed $F \subset X$, $\mathfrak{C}^*(F)$ has an admissible extension to X.*

Proof. By a theorem of Kuratowski ([12], p. 260), it follows that X is normal if and only if X is $\aleph_0$-collectionwise normal. Therefore the equivalence of (1), (2), (5), and (8) follows from 2.20. Clearly (2) implies (3) and (3) implies (4). But (4) implies, by 3.14, that every closed subset of X is C^*-embedded in X, i.e. that X is normal. Finally, by using 3.11 and 3.13, the statements (6), (7), (9), and (10) are all equivalent to (1).

3.26. Remark. The following diagram now summarizes the implications that exist among the various notions of embedding that are studied in this paper:

$$\begin{array}{ccccccccc} u_0 & \Rightarrow & u_\gamma & \Rightarrow & u_{\aleph_0} & \Longleftrightarrow & u & \Rightarrow & u^* \\ \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\ P & \Rightarrow & P^\gamma & \Rightarrow & P^{\aleph_0} & \Longleftrightarrow & C & \Rightarrow & C^* \end{array}$$

(where γ is an infinite cardinal number greater that $\aleph_0$).

We note that 3.4 gives an example of a space X that is P-embedded in βX, but is not even u^*-embedded in βX. Thus none of the vertical implications can be reversed in general. Bing ([3], Example G) (see also [5], 2.6) gives an example of a normal T_1-space (hence $\aleph_0$-collectionwise normal) that is not $\aleph_1$-collectionwise normal. Thus, by virtue of 2.19 and 2.20, the implications "$u_\gamma \Rightarrow u_{\aleph_0}$" and "$P^\gamma \Rightarrow P^{\aleph_0}$" cannot be reversed in general. The equivalence of $u_{\aleph_0}$ and u is the content of 3.24, while the equivalence of C and $P^{\aleph_0}$ is the content of ([5], 2.4). Next, the space $\Lambda = \beta \boldsymbol{R} \setminus (\beta \boldsymbol{N} \setminus \boldsymbol{N})$ constructed by Katětov [9] (see also [6], Problem 6P), contains $\boldsymbol{N}$ as a closed C^*-embedded subset that is not C-embedded in Λ. In view of 3.17, $\boldsymbol{N}$ is therefore u^*-embedded in Λ, but is not u-embedded in Λ. It follows that the implications "$C \Rightarrow C^*$" and "$u \Rightarrow u^*$" cannot be reversed in general. We have been unable to show that the remaining two implications cannot be reversed (i.e. to construct, for $a > 0$, an $\aleph_a$-collectionwise normal space that is not $\aleph_{a+1}$-collectionwise normal).

Remark. It is possible to define absolutely u_0-embedded spaces, absolutely u^*-embedded spaces, etc., in the obvious manner; but these spaces are then none other than the almost compact spaces, i.e. those spaces differing from their Stone–Čech compactifications by at most a single point.

References

[1] G. Aquaro, *Ricovrimenti aperti a strutture uniformi sopra uno spazio topologico*, Ann. Math. Pura Appl., ser. 4, 47 (1959), pp. 319–390.

[2] — *Spazii collettivamente normali ed estensione di applicazioni continue*, Riv. Mat. Univ. Parma, ser. 2, 2 (1961), pp. 77–90.

[3] R. H. Bing, *Metrization of topological spaces*, Canad. J. Math. 3 (1951), pp. 175–186.

[4] C. H. Dowker, *On a theorem of Hanner*, Ark. Mat. 2 (1952), pp. 307–313.

[5] T. E. Gantner, *Extensions of uniformly continuous pseudometrics*, Trans. Amer. Math. Soc. 132 (1968), pp. 147–157.

[6] L. Gillman and M. Jerison, *Rings of Continuous Functions*, New York 1960.

[7] S. Ginsburg and J. R. Isbell, *Some operators on uniform spaces*, Trans. Amer. Math. Soc. 93 (1959), pp. 145–168.

[8] J. R. Isbell, *Uniform Spaces*, Providence 1964.

[9] M. Katětov, *On real-valued functions in topological spaces*, Fund. Math. 38 (1951), pp. 85–91 and 40 (1953), pp. 203–205.

[10] — *Extensions of locally finite covers*, Colloq. Math. 6 (1958), pp. 145–151. Russian).

[11] J. L. Kelley, *General Topology*, New York 1955.

[12] C. Kuratowski, *Sur le prolongement des fonctions continues et les transformations en polytopes*, Fund. Math., 24 (1935), pp. 259–268.

[13] S. Lefschetz, *Algebraic Topology*, New York 1942.

[14] K. Morita, *Paracompactness and product spaces*, Fund. Math. 50 (1962), pp. 223–236.

[15] S. B. Myers, *Functional uniformities*, Proc. Amer. Math. Soc. 2 (1951), pp. 153–158.

[16] H. L. Shapiro, *Extensions of Pseudometrics*, Thesis, Purdue University, 1965.

[17] — *Extensions of pseudometrics*, Canad. J. Math. 18 (1966), pp. 981–998.

[18] — *A note on extending uniformly continuous pseudometrics*, Bull. Soc. Math. Belgique 18 (1966), pp. 339–441.

[19] A. H. Stone, *Paracompactness and product spaces*, Bull. Amer. Math. Soc. 54 (1948), pp. 977–982.

[20] J. W. Tukey, *Convergence and Uniformity in Topology*, Princeton 1940.

[21] A. Weil, *Sur les espaces a structure uniforme et sur la topologie générale*, Act. Sci. Ind., no. 551, Paris 1937.

Reçu par la Rédaction le 23. 7. 1967