Some remarks on convex functions

by

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§ 1. All the sets in the following considerations are subsets of the set R of real numbers. The measure is always the Lebesgue measure. Symbols $m_i(A)$, $m_e(A)$ and $m(A)$ denote the inner Lebesgue measure, the outer Lebesgue measure and the measure of the set A, respectively.

DEFINITION 1. The set

$$\sum_{i=1}^{n} A_i \overset{\text{df}}{=} \Big\{x\colon x = \sum_{i=1}^{n} a_i\,,\ a_i \in A_i\Big\}$$

is called the *vector-sum* of the sets A_i, $i = 1, 2, \ldots, n$.

To simplify the notation we introduce the notation:

$$\sum_{1}^{n} A \overset{\text{df}}{=} \sum_{i=1}^{n} A_i\,,$$

where $A_i = A$ for $i = 1, 2, \ldots, n$.

DEFINITION 2. For a real number a and a set A we put

$$aA \overset{\text{df}}{=} \{x\colon x = aa\,,\ a \in A\}\,.$$

The obvious relations hold:

$$a\sum_{i=1}^{n} A_i = \sum_{i=1}^{n} (aA_i)\,, \tag{1}$$

$$a(\beta A) = (a\beta)A\,. \tag{2}$$

Moreover, we have for an arbitrary set A and a real number a

$$\begin{aligned} m_e(aA) &= |a|\, m_e(A)\,,\\ m_i(aA) &= |a|\, m_i(A)\,. \end{aligned} \tag{3}$$

In the sequel we shall use the following lemma.

LEMMA 1. *If $m_i(A) > 0$, then there exists a measurable set $B \subset A$ with a positive measure.*

DEFINITION 3. A real-valued function f defined on an interval $\Delta = (a, b)$ is called *convex in Jensen's sense* if for every $x, y \in \Delta$ the relation

$$f\left(\frac{x+y}{2}\right) \leqslant \frac{1}{2}\big(f(x)+f(y)\big)$$

holds.

In the sequel the symbol f always denotes a real-valued function defined on an interval $\Delta = (a, b)$ and convex in this interval. The convexity will always be understood in Jensen's sense.

DEFINITION 4. For an arbitrary set T we denote by $J(T)$ the set

$$J(T) \overset{\mathrm{df}}{=} \bigcup_{n=0}^{\infty} T_n ,$$

where

$$T_0 \overset{\mathrm{df}}{=} T , \qquad T_{n+1} \overset{\mathrm{df}}{=} \tfrac{1}{2}(T_n+T_n) .$$

It is easily seen (induction) that we can write the set $J(T)$ in the form

$$J(T) = \bigcup_{n=0}^{\infty}\left(\frac{1}{2^n}\sum_{1}^{2^n} T\right).$$

An important example of a convex function is an additive function, i.e. a function satisfying Cauchy's functional equation

$$f(x+y) = f(x)+f(y) .$$

Such a function is actually convex, since

$$f\left(\frac{x+y}{2}\right) = \frac{1}{2}\left(f\left(\frac{x+y}{2}\right)+f\left(\frac{x+y}{2}\right)\right) = \frac{1}{2}f\left(\frac{x+y}{2}+\frac{x+y}{2}\right)$$
$$= \frac{1}{2}f(x+y) = \frac{1}{2}\big(f(x)+f(y)\big) .$$

The general solution of Cauchy's equation is constructed with the aid of the Hamel basis of the set of real numbers (cf. G. Hamel [4], J. Aczél [1]). The solution is of the form

(4) $$f(x) = \sum r_a g(H_a) \quad \text{for} \quad x = \sum r_a H_a , \; r_a \in Q , \; H_a \in H ,$$

where H denotes a fixed Hamel basis, Q is the set of rational numbers, and g is an arbitrary function, $g\colon H \to R$.

§ 2. The fundamental problem from the theory of convex functions is to find the conditions which imply the continuity of such functions. It appears that even very weak hypotheses on a convex function guarantee its continuity (cf. for instance, Bernstein and Doetsch [2], Sierpiński [13], Ostrowski [11], Marcus [9], [10]). Recent results in this direction are

contained in papers by S. Kurepa [7], M. Kuczma [6] and M. R. Mehdi [8]. Kurepa's theorem reads as follows:

If a function f is defined and convex on an interval Δ and if f is bounded from above on a set $T \subset \Delta$ such that

$$m_i(T+T) > 0 ,$$

then f is continuous in Δ.

The condition $m_i(T+T) > 0$ cannot be replaced by the condition $m_e(T+T) > 0$.

A more general theorem is true [1].

THEOREM 1. *If a function f is defined and convex in an interval Δ, if f is bounded from above on a set $T \subset \Delta$, and if there exists a positive integer n such that*

$$m_i\Big(\sum_{1}^{n} T\Big) > 0 ,$$

then f is continuous in Δ.

Proof. Let us note that without loss of generality we can assume that $0 \in T$. In fact, if $0 \notin T$ then we can consider the interval

$$\Delta^* = \Delta - x_0 = (a-x_0, b-x_0) ,$$

where x_0 is an arbitrarily fixed point of the set T. The set $T^* = T - x_0$ has the property that $0 \in T$. The function

$$f^*(x) = f(x_0+x) , \qquad x \in \Delta^* ,$$

is defined on the interval Δ^*, convex and bounded from above on the set T^*. Therefore we can assume that $0 \in T$. Then we have the inclusion

$$T \subset T+T$$

since $t \in T$ and $0 \in T$ imply $t = t+0 \in T+T$.

Further, lest us note that if $f(t) \leqslant M$ for $t \in T$, then

$$f\left(\frac{t_1+t_2+\ldots+t_{2^j}}{2^j}\right) \leqslant M ,$$

where $t_i \in T$ for $i = 1, 2, \ldots, 2^j$ and j is an arbitrarily fixed positive integer. This follows easily by induction from the fact that the function f is convex.

The assumption $m_i(\sum_{1}^{n} T) > 0$ implies in virtue of Lemma 1 that there

[1] This theorem is essentially equivalent to the one proved by J. H. B. Kemperman [5] and S. Marcus [9], [10].

exists a measurable set $B\subset\sum_1^n T$ with a positive measure. Let us take a positive integer k such that $2^{k-1}<n\leqslant 2^k$. By (3),

$$m\left(\frac{1}{2^k}B\right)=\frac{1}{2^k}m(B)>0\,.$$

Let $x\in\frac{1}{2^k}B$. Then

$$x=\frac{t_1+t_2+\ldots+t_n}{2^k}=\frac{t_1+t_2+\ldots+t_n+0+0+\ldots+0}{2^k}\,,$$

where we have taken 2^k-n zero terms. According to the above remarks, $f(x)\leqslant M$. In virtue of Ostrowski's theorem (cf. [11]) the function f is continuous in Δ.

The above theorem is an improvement on the result of S. Kurepa. To show this we shall use the following lemma.

LEMMA 2 (cf. E. Borel [3]). *Let Z denote the set of such real numbers that the digit k, $0\leqslant k\leqslant N-1$ fixed, does not appear in their N-adic expansions. Then Z is measurable and $m(Z)=0$.*

It is easily seen that an arbitrary number in the N-adic system can be represented as a sum of $N-1$ terms such that their expansions consist of 0 or 1 only. In other words,

$$R=\sum_1^{N-1}T\,,$$

where

$$T\overset{\text{df}}{=}\left\{x\colon\, x=c+\sum_{i=1}^{\infty}\frac{a_i}{N^i},\ a_i\in\{0,1\}\right\}.$$

In view of Lemma 2, the sets T, $T+T$, ..., $\sum_1^{N-2}T$ are of measure zero. So if we use Theorem 1, then the boundedness of the function f on the set T is sufficient; however, the quoted theorem of Kurepa does not apply already for $N=4$.

§ 3. The purpose of the present paper is to establish the relation between the generalized Kurepa theorem and the following theorem of M. Kuczma (cf. [6]):

If a function f is defined and convex in a interval Δ and if f is bounded from above on a set $T\subset\Delta$ such that

$$m_i(J(T))>0\,,$$

then f is continuous in Δ.

Kuczma's theorem is more general than Theorem 1 (and thus also than Kurepa's theorem, which corresponds to the case $n=2$). In fact, since we have

$$m_i\Big(\sum_1^n T\Big)>0\,,$$

taking k such that $2^{k-1}<n\leqslant 2^k$, we obtain

$$\sum_1^n T\subset\sum_1^{2^k}T\,.$$

Thus

$$0<m_i\Big(\sum_1^n T\Big)\leqslant m_i\Big(\sum_1^{2^k}T\Big)\,.$$

By (3) and (1),

$$0<\frac{1}{2^k}m_i\Big(\sum_1^{2^k}T\Big)=m_i\Big(\frac{1}{2^k}\sum_2^{2^k}T\Big)=m_i(T_k)\leqslant m_i(J(T))\,.$$

Therefore the following implication holds:

$$\text{(5)}\qquad m_i\Big(\sum_1^n T\Big)>0\quad\text{implies}\quad m_i(J(T))>0\,.$$

It turns out that the infinite step in Kuczma's theorem is essential. Indeed, the converse implication to (5) does not hold. To show this, we shall construct a set T such that $m_i(\sum_1^n T)=0$ for every natural n, but $J(T)=R$.

Let $\{r_i\}$ denote the sequence of all rational numbers and H a fixed Hamel basis of the set of real numbers. Let us take

$$T\overset{\text{df}}{=}\bigcup_{i=1}^{\infty}(r_iH)\,.$$

$J(T)=R$ since for an arbitrary $x\in R$ we have

$$x=r_1h_1+r_2h_2+\ldots+r_nh_n=\frac{s_1}{2^j}h_1+\frac{s_2}{2^j}h_2+\ldots+\frac{s_n}{2^j}h_n\,,$$

where $r_i\in Q$, $h_i\in H$, $s_i=2^jr_i$ for $i=1,2,\ldots,n$, and j is so chosen that $2^{j-1}<n\leqslant 2^j$.

Then, after taking 2^j-n zero terms, we obtain

$$x=\frac{1}{2^j}(s_1h_1+s_2h_2+\ldots+s_nh_n)=\frac{1}{2^j}(s_1h_1+s_2h_2+\ldots+s_nh_n+0+0+\ldots+0)\,,$$

i.e.

$$x \in \frac{1}{2^j} \sum_1^{2^j} T = T_j \subset J(T).$$

Thus, $R \subset J(T)$. The converse inclusion is trivial.

Further, suppose that there exists an N such that $m_i(\sum_1^N T) > 0$. In virtue of Lemma 1 there exists a measurable set $B \subset \sum_1^N T$ of positive measure. According to H. Steinhaus's theorem [14], the set $B+B$ contains an interval $P = (\alpha, \beta)$. Of course, $P \subset \sum_1^{2N} T$, which means that every real number from the interval P has at most a $2N$-term Hamel representation. Let $x_0 \in (\alpha, \beta)$ and let

$$x_0 = r_1 h_1 + r_2 h_2 + \ldots + r_{2N} h_{2N}, \quad r_i \in Q, \; h_i \in H, \; i = 1, 2, \ldots, 2N.$$

(If such an element does not exist, then we take the element which has the longest Hamel expansion.)

Now, let us take an $h_0 \in H$ such that $h_0 \neq h_i$ for $i = 1, 2, \ldots, 2N$, and a rational number r_0 so small that

$$|r_0 h_0| < \min(|x_0 - \alpha|, |x_0 - \beta|).$$

Then

$$x_1 \stackrel{\text{df}}{=} x_0 + r_0 h_0 \in P$$

and it has a $(2N+1)$-term expansion.

This contradiction proves that for any natural n,

$$m_i\Big(\sum_1^n T\Big) = 0.$$

Now, taking an arbitrary convex function bounded from above on the set T we infer from Kuczma's theorem that it must be convex. This conclusion does not result from Theorem 1, because that theorem does not guarantee the continuity of a convex function bounded on the set T.

§ 4. The set $K(T) \stackrel{\text{df}}{=} \bigcup_{n=1}^{\infty} \sum_1^n T$ rather than $J(T)$ seems to be a more natural generalization of the sets occuring in Theorem 1. However, it turns out that it is impossible to replace in Theorem 1 a finite vector-sum by the set $K(T)$. Indeed, let us take the set

$$T \stackrel{\text{df}}{=} \{x\colon x = rh, \; r \in [-1, 1] \cap Q, \; h \in H\},$$

and a convex function f given by the formula

$$(6) \qquad f(x) = \sum r_a \quad \text{for} \quad x = \sum r_a H_a, \; r_a \in Q, \; H_a \in H.$$

This function is not continuous since $g(H_a) = 1$, whence

$$\frac{g(H_a)}{H_a} = \frac{1}{H_a} \neq \text{const},$$

while the constancy of this quotient is a necessary and sufficient condition of the continuity of the additive function (4) (cf. J. Aczél [1]). Function (6) is convex (§ 1) and bounded from above by 1 on the set T.

On the other hand, $K(T) = R$, because

(i) an arbitrary rational number r can be represented as a finite sum $r = r_1 + r_2 + \ldots + r_i$, where $r_m \in [-1, 1] \cap Q$ for $m = 1, 2, \ldots, i$;

(ii) an arbitrary real number x can be represented as the finite sum $x = r_1 h_1 + r_2 h_2 + \ldots + r_n h_n$, which can be written in the form:

$$x = (r_{11} + r_{12} + \ldots + r_{1i_1}) h_1 + (r_{21} + r_{22} + \ldots + r_{2i_2}) h_2 + \ldots + (r_{n1} + r_{n2} + \ldots + r_{ni_n}) h_n,$$

where the sums in brackets are decompositions of the rational numbers r_m as in (i).

Therefore, if $x \in R$ then

$$x \in \Big(\sum_1^{i_1} T + \sum_1^{i_2} T + \ldots + \sum_1^{i_n} T\Big) = \sum_1^{i_1 + i_2 + \ldots + i_n} T \subset K(T).$$

Thus, $R \subset K(T)$. The converse inclusion is trivial.

The above construction yields an example of a discontinuous and convex function bounded from above on a set T such that $K(T) = R$. Thus this kind of a generalization of the condition from Theorem 1 is not possible.

§ 5. M. R. Mehdi [8] has proved that if f is defined and convex in an interval Δ and if f is bounded from above on a second category Baire set $T \subset \Delta$, then f is continuous. This is an immediate consequence of S. Kurepa's theorem (2) since the conditions on T guarantee that the set $T+T$ contains an interval (cf. [12], p. 188).

(2) But only in the case of a single real variable. M. R. Mehdi's theorem is valid generally for real-valued functions defined on convex subsets of topological vector spaces.

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Reçu par la Rédaction le 2. 10. 1968