Remarks on analytic sets

by

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Let $I$ denote the unit interval, let $B, A, L$ be the $\sigma$-algebras on $I$ generated by open sets, analytic sets and Lebesgue measurable sets (or sets measurable w.r.t. any fixed nonatomic probability measure on $B$) respectively. Let $C$ be the class of all subsets of $I$ and $E$ be any $\sigma$-algebra such that

$$A \subset E \subset L.$$

Let $U$ be any analytic subset of $I \times I$ which is universal w.r.t. the analytic sets of $I$. As is well-known ([1], p. 368) such sets do exist. The purpose of this note is to prove

**Theorem 1.** $E$ is not countably generated.

**Theorem 2.** $U \notin C \times L$.

(Symbol $C \times L$ stands for the $\sigma$-algebra on $I \times I$ generated by sets of the form $X \times Y$ where $X \in C, Y \in L$).

Before proving Theorem 1, we shall make a remark. There is no general way of proving that a $\sigma$-algebra is not countably generated. The first method available in the literature is a simple cardinality argument which fails here because the cardinality of $E$ can be $\mathfrak{c}$. The second method is to exhibit a probability measure on $E$ giving zero mass to singletons and taking only two values zero and one. This also fails here, because probability measures on $E$ give rise to the corresponding probability measures on $B$.

**Proof of Theorem 1.** If $E$ has a countable generator say $\{A_n; n \geq 1\}$ then consider the Marczewski function on $I$ defined by

$$f(x) = \sum_{n \geq 1} \frac{2x_n(x)}{3^n}$$

with range, say, $X \subset I$. Let $B_X$ be the relativized Borel $\sigma$-algebra on $X$. Clearly $f$ is an isomorphism of $(I, E)$ onto $(X, B_X)$ if $B$ is a Borel subset
of \( I \) and \( B \subseteq X \), then the map \( f^{-1} \), restricted to \( B \), being Borel and one to one, we have, in view of (1), p. 397 that \( f^{-1}(B) \) is a Borel subset of \( I \). Since the Lebesgue measure \( \lambda \) on \( I, E \) is compact [2] and hence perfect [3] there is a Borel subset \( B \) of \( I \) with

\[
B \subseteq X \quad \text{and} \quad \lambda(f^{-1}B) = 1.
\]

Denoting by \( Y \) the set \( f^{-1}(B) \) and by \( E_Y \) the \( \sigma \)-algebra \( E \) restricted to \( Y \) and by \( f_Y \) the map \( f \) restricted to \( Y \), one observes that \( f_Y \) is a Borel isomorphism on \( (Y, E_Y) \) onto \( (B, B_B) \). As remarked above, \( B \) is a Borel subset of \( I \) and being clearly uncountable there is a non-Borel analytic set in \( E_Y \) whereas every set in \( B_B \) is Borel. This contradicts that \( f_Y \) is a Borel isomorphism. This proves Theorem 1.

The author is indebted to the referee for suggesting that our Theorem 2 answers a question of S. M. Ulam [4, page 19, lines 20–23].

Proof of Theorem 2. If \( U \in C \times L \) then obviously there exist countable number of rectangles \( \{E_n \times F_n : n \geq 1 \} \) such that \( U \) is in the \( \sigma \)-algebra generated by these rectangles. Define \( E \) to be the \( \sigma \)-algebra on \( I \) generated by \( \{F_n : n \geq 1 \} \). Clearly \( E \subseteq L \). Since \( U \in C \times E \) and \( U \) is universal w.r.t. the analytic subsets of \( I; A \subseteq E \). Since \( E \) is countably generated we have a contradiction to Theorem 1. This proves Theorem 2.

The author could not show that “if \( A \subseteq C \subseteq C \) then \( E \) is not countably generated”. Observe that if this is established then, \( U \notin C \times C \) which answers in the negative the following unsolved question of S. M. Ulam: “Is the product of discrete (class of all subsets) \( \sigma \)-algebras on \( I \), the discrete \( \sigma \)-algebra on the square?”

We conclude with observing that the following proposition, which is not difficult to prove, answers in the negative the above question when \( I \) is replaced by a set of cardinality greater than \( c \).

Let \( E \) be a \( \sigma \)-algebra on a set \( X \). The diagonal of \( X \times X \) belong to \( E \times E \) if and only if there is a countably generated \( \sigma \)-algebra \( D \subseteq E \) with singletons as atoms. Consequently if \( \text{card}(X) > c \) then whatever be \( E \), diagonal can not belong to \( E \times E \).

The only if part is essentially contained in an exercise in P. R. Halmos’s “Measure Theory”.

Acknowledgments: Thanks are to Drs. A. Maitra and J. K. Ghosh for the many useful discussions. Thanks are also to Professor C. Ryll-Nardzewski for suggesting many improvements in the original version of the paper.

References