Remarks on analytic sets

by

B. V. Rao (Calcutta)

Let I denote the unit interval, let $\boldsymbol{B}, \boldsymbol{A}, \boldsymbol{L}$ be the σ-algebras on I generated by open sets, analytic sets and Lebesgue measurable sets (or sets measurable w.r.t. any fixed nonatomic probability measure on $\boldsymbol{B}$) respectively. Let $\boldsymbol{C}$ be the class of all subsets of I and $\boldsymbol{E}$ be any σ-algebra such that

$$\boldsymbol{A} \subset \boldsymbol{E} \subset \boldsymbol{L}.$$

Let U be any analytic subset of $I \times I$ which is universal w.r.t. the analytic sets of I. As is well-known ([1], p. 368) such sets do exist. The purpose of this note is to prove

THEOREM 1. *$\boldsymbol{E}$ is not countably generated.*

THEOREM 2. $U \notin \boldsymbol{C} \times \boldsymbol{L}$.

(Symbol $\boldsymbol{C} \times \boldsymbol{L}$ stands for the σ-algebra on $I \times I$ generated by sets of the form $X \times Y$ where $X \in \boldsymbol{C}$; $Y \in \boldsymbol{L}$).

Before proving Theorem 1, we shall make a remark. There is no general way of proving that a σ-algebra is not countably generated. The first method available in the literature is a simple cardinality argument which fails here because cardinality of $\boldsymbol{E}$ can be $\mathfrak{c}$. The second method is to exhibit a probability measure on $\boldsymbol{E}$ giving zero mass to singletons and taking only two values zero and one. This also fails here, because probability measures on $\boldsymbol{E}$ give rise to the corresponding probability measures on $\boldsymbol{B}$.

Proof of Theorem 1. If $\boldsymbol{E}$ has a countable generator say $\{A_n; n \geqslant 1\}$ then consider the Marczewski function on I defined by

$$f(x) = \sum \frac{2\chi_{A_i}(x)}{3^i}$$

with range, say, $X \subset I$. Let $\boldsymbol{B}_X$ be the relativized Borel σ-algebra on X. Clearly f is an isomorphism of $(I, \boldsymbol{E})$ onto $(X, \boldsymbol{B}_X)$. If B is a Borel subset of I and $B \subset X$, then the map f^{-1}, restricted to B, being Borel and one to one, we have, in view of ([1], p. 397) that $f^{-1}(B)$ is a Borel subset of I. Since the Lebesgue measure λ on $(I, \boldsymbol{E})$ is compact [2] and hence perfect [3] there is a Borel subset B of I with

$$B \subset X \quad \text{and} \quad \lambda(f^{-1}B) = 1 .$$

Denoting by Y the set $f^{-1}(B)$ and by $\boldsymbol{E}_Y$ the σ-algebra E restricted to Y and by f_1 the map f restricted to Y, one observes that f_1 is a Borel isomorphism on $(Y, \boldsymbol{E}_Y)$ onto $(B, \boldsymbol{B}_B)$. As remarked above, B is a Borel subset of I and being clearly uncountable there is a non-Borel analytic set in $\boldsymbol{E}_Y$ whereas every set in $\boldsymbol{B}_B$ is Borel. This contradicts that f_1 is a Borel isomorphism. This proves Theorem 1.

The author is indebted to the referee for suggesting that our Theorem 2 answers a question of S. M. Ulam [4, page 10, lines 20–23].

Proof of Theorem 2. If $U \in \boldsymbol{C} \times \boldsymbol{L}$ then obviously there exist countable number of rectangles $\{E_n \times F_n, n \geqslant 1\}$ such that U is in the σ-algebra generated by these rectangles. Define $\boldsymbol{E}$ to be the σ-algebra on I generated by $\{F_n;\ n \geqslant 1\}$. Clearly $\boldsymbol{E} \subset \boldsymbol{L}$. Since $U \in \boldsymbol{C} \times \boldsymbol{E}$ and U is universal w.r.t. the analytic subsets of I; $\boldsymbol{A} \subset \boldsymbol{E}$. Since $\boldsymbol{E}$ is countably generated we have a contradiction to Theorem 1. This proves Theorem 2.

The author could not show that "if $\boldsymbol{A} \subset \boldsymbol{E} \subset \boldsymbol{C}$ then $\boldsymbol{E}$ is not countably generated". Observe that if this is established then, $U \notin \boldsymbol{C} \times \boldsymbol{C}$ which answers in the negative the following unsettled question of S. M. Ulam: "Is the product of discrete (class of all subsets) σ-algebras on I; the discrete σ-algebra on the square?"

We conclude with observing that the following proposition, which is not difficult to prove, answers in the negative the above question when I is replaced by a set of cardinality greater than $\mathfrak{c}$.

Let $\boldsymbol{E}$ be a σ-algebra on a set X. The diagonal of $X \times X$ belong to $\boldsymbol{E} \times \boldsymbol{E}$ if and only if there is a countably generated σ-algebra $\boldsymbol{D} \subset \boldsymbol{E}$ with singletons as atoms. Consequently if $\operatorname{card}(X) > \mathfrak{c}$ then whatever be $\boldsymbol{E}$, diagonal can not belong to $\boldsymbol{E} \times \boldsymbol{E}$.

The only if part is essentially contained in an exercise in P. R. Halmos's "Measure Theory".

Acknowledgements: Thanks are to Drs. A. Maitra and J. K. Ghosh for the many useful discussions. Thanks are also to Professor C. Ryll–Nardzewski for suggesting many improvements in the original version of the paper.

Added in proof (October, 1969). i) Professor Jan Mycielski has kindly informed us that a weak form of Theorem 2 of this paper has been obtained by Dr. Richard Mansfield by using altogether different and difficult techniques.

ii) Regarding the problem of discrete σ-algebras see the author's paper "On discrete Borel spaces and projective sets" in Bull. Amer. Math. Soc. 75 (1969), pp. 614–617 and also a forthcoming paper of the author in Fund. Math.

References

[1] C. Kuratowski, *Topologie* I (édition troisième) Warszawa 1952.
[2] E. Marczewski, *On compact measures*, Fund. Math. 40 (1953), pp. (113–124).
[3] C. Ryll–Nardzewski, *On quasi compact measures*, Fund. Math. 40 (1953), pp. (125–130).
[4] S. M. Ulam, *A Collection of Mathematical Problems*, Interscience Publ., New York 1960.

INDIAN STATISTICAL INSTITUTE

Reçu par la Rédaction le 5. 8. 1968