On weak cardinal powers in generic extensions

by

F. R. Drake (Leeds, England)

We shall work in Zermelo–Fraenkel set theory with the axiom of
choice (ZFC), and identify cardinals with initial ordinals. If \( \sigma, \tau \), are
 cardinals, the weak power \( \sigma^\tau \) is defined to be

\[
\bigcup_{\beta < \tau} \sigma^\beta,
\]

where \( \beta \) ranges through cardinals less than \( \tau \), and \( \sigma^\beta \) is the usual cardinal
power (the cardinal of all functions from \( \beta \) into \( \sigma \)).

In [1], J. Derrick and the author studied the extensions obtained
by adding generic subsets of a regular cardinal \( \kappa \) to a model of ZFC, where
\( \kappa > \omega \). In order to prove that all cardinals were preserved in the extension,
the assumption was made that for cardinals \( \beta < \kappa \), in the original model
\( \mathcal{P}^\beta \preceq \kappa \). This is, of course, the assumption that \( \mathcal{P} = \kappa \)
in the original model. (This cardinal arose in that work since it is just the cardinal of
the set of conditions used in defining “generic” in the extensions studied.)

The question of the preservation of cardinals in the absence of this
condition is the subject of this paper. As a corollary of the main result,
it will be seen that if \( \mathcal{P} \) is a cardinal greater than \( \kappa \) in the original model,
then in the extension this cardinal is not preserved but collapses and
becomes similar to \( \kappa \). Thus e.g. if \( \kappa \) is \( \kappa^+ \), then the continuum hypothesis
will always hold in the extension.

Notation. We let \( \mathcal{M} \) be a transitive \( \varepsilon \)-model of ZFC, and \( \kappa \) an
uncountable cardinal in \( \mathcal{M} \). We use the following notion of forcing (\(^*\)):
a condition is a partial function in \( \mathcal{M} \) from \( \kappa \) into \( (0,1] \), with cardinality
in \( \mathcal{M} \) less than \( \kappa \). If \( p, q \) are conditions, then \( p \) extends \( q \) if \( p \supseteq q \). Let \( \mathcal{P} \)
be the set of all conditions (\( \mathcal{P} \in \mathcal{M} \)). A subset \( X \) of \( \mathcal{P} \) is dense if every
condition has an extension in \( X \). Conditions \( p, q \) are compatible if there
is a condition extending both, and a set of (pairwise) compatible con-
ditions is generic over \( \mathcal{M} \) for this notion of forcing if it meets every dense

\(^*\) This paper was read at a meeting on the Foundations of Mathematics in Warsaw,
August 26–September 2, 1968.

\(^*\) The elegant formulation of a notion of forcing, as used here, has been seen by

Fundamenta Mathematicae, T. LXVI 15
subset of \( P \) which is in \( M \). In the usual way we can prove that if \( G \) is generic over \( M \), then \( M[G] \), the extension of \( M \) by \( G \), is again a model of ZFC, which contains a new subset of \( u \) (not in \( M \)) (see [1] or [2] for details).

If \( M \) is countable, then the existence of a generic set of conditions is easily proved, as in Cohen's work (see [2]). Other assumptions than countability of \( M \) can also lead to proofs of the existence of generic sets; but one corollary of the result to be proved is a condition for the non-existence of sets of conditions generic over \( M \).

If \( x \) is an infinite cardinal, we write \( cf x \) (\( \text{cofinality of } x \) in \( M \)) for the least ordinal \( \lambda \) such that there is a function in \( M \): \( f: \lambda \to \alpha \), such that \( \bigcup f(\alpha) = x \) (\( \alpha \) is regular iff \( cf x = \alpha \), otherwise singular; \( cf x \) is always a regular cardinal of \( M \)).

Assuming these preliminaries and that \( G \) is generic over \( M \) for this notion of forcing, we state the main theorem:

**Theorem 1.** Let \( \alpha < \kappa \) be a cardinal in \( M \), and let \( \lambda = 2^\kappa \) in \( M \) (i.e. \( \lambda \) is an initial ordinal of \( M \)). Then, in \( M[G] \), there is a function mapping \( \lambda \) one-to-one into \( cf \kappa \).

**Proof.** Let \( G \) be generic over \( M \), and \( \alpha \) be a subset of \( \alpha \) which is in \( M \). Let \( p: \alpha \to \omega \) be an increasing function in \( M \) onto regular cardinals less than \( \alpha \), if \( x \) is singular; or if \( cf \alpha = \alpha \), let \( p(\beta) = \alpha \cdot \beta \) for \( \beta < \alpha \) (1). If \( p \) is a condition, we examine \( \beta \in cf \kappa \) of \( \beta \) and \( \beta + 1 \) for \( \beta < cf \kappa \).

For ordinal \( \beta \), define the \( \alpha \)-cut of \( p \) at \( \beta \), \( p^\alpha(\beta) \), as the partial function from \( \alpha \) into \( [0, 1] \) given by:

\[ p^\alpha(\gamma) = \gamma < \alpha \text{ if } p(\beta + \gamma) \text{ is defined, and if so then } p^\alpha(\gamma) = \beta(\beta + \gamma) \number{2} \]

- Now, for \( \alpha \in \alpha \in \alpha \), we say that \( p \) is \( \alpha \)-like at \( \beta \) if \( p^\alpha \) is defined on the whole of \( \alpha \), and for \( \gamma < \alpha \), \( p^\alpha(\gamma) = 1 \) if \( \gamma < \alpha \); i.e. \( p^\alpha(\gamma) \) is the characteristic function of \( \alpha \).

We now look at \( \alpha \)-cuts of \( p \) between \( \beta \) and \( \beta + 1 \) for \( \beta < cf \kappa \).

If \( x \) is regular, then each such interval is of length \( \alpha \) and we are interested in the \( \alpha \)-cut of \( p \) at \( \beta \); but if \( \alpha = \kappa \), then for each \( \beta \) such that \( \beta \) is singular, there will be \( \beta + 1 \) such cuts between \( \beta \) and \( \beta + 1 \). We look at all cuts of \( p \) at \( \beta + 1 \) for \( \beta + 2 \) such that \( \beta + 1 \) and \( \beta + 2 \), and say, for \( \beta < cf \kappa \), that \( p \) is good for \( x \) at \( \beta \) if there is an ordinal \( \gamma \) such that:

- \( \beta + 1 \) is \( \gamma \)-like at \( \beta + 1 \)
- \( \beta + 1 \) and \( \beta + 2 \) are similar to \( \kappa \) in the extension \( M[G] \), all cardinals outside this range are preserved.

**Lemma.** For \( x \in M \), \( x \in M \), the set of conditions such that there is a \( \beta < cf \kappa \) at which \( p \) is good for \( x \) is dense (and in \( M \)).

**Proof:** Given any condition \( p \), we have to show how to extend \( p \) to a condition which is good for \( x \) at some \( \beta < cf \kappa \). Since \( p \) is a condition, card \( p < \kappa \). We take the case of \( x \) regular first. If \( x \) is regular, then \( \text{dom}(p) \) must be bounded below \( x \); say \( \text{dom}(p) \subset x : \xi < x \) for some \( \xi < \kappa \).

Then extend \( p \) to \( q \) by defining \( q \) on all \( \gamma \),

\[ a \cdot \xi < \gamma < a \cdot (\xi + 1) \]

so that \( q \) is \( x \)-like at \( a \cdot \xi \). Then \( q \) is an extension of \( p \) which is good for \( x \) at \( \xi \).

If \( \kappa \) is singular, then \( \text{dom}(p) \) may not be bounded below \( x \); but for some \( \beta < cf \kappa \), \( p(\beta) > \text{card}(p, a) \). So the domain of \( p \) is bounded below \( \beta + 1 \), i.e. there is a \( \gamma \) with \( \beta(\beta + 1) < \gamma < \beta + 1 \) such that \( p \) is undefined for all \( \xi < \beta \).

\[ \beta(\beta + 1) < \gamma < \beta + 1 \]

Any such \( \xi \) is of the form \( \xi = \beta(\beta + 1) + \alpha \cdot \xi + \xi' \) for \( \xi' < \alpha \); we extend \( p \) to \( q \) by defining

\[ q(\xi) = \begin{cases} 1 & \text{if } \xi' \in \alpha, \\ 0 & \text{if } \xi' \notin \alpha \end{cases} \]

and \( q \) is then an extension of \( p \) which is good for \( x \) at \( \beta \).

Thus the lemma is proved, and the theorem follows: take an enumeration \( (\xi_\alpha)_{\alpha < \alpha} \) in \( M \) of all subsets of \( \alpha \) in \( M \), and since \( G \) enters every dense set, we can define, for \( \beta < \lambda \),

\[ f(\beta) = \mu(\beta(\beta + 1) < \gamma < \beta + 1) \]

Then \( f \) is a \( 1 \)-1 function from \( \lambda \) into \( \alpha \), and \( f \) is in \( M[G] \).

**Corollary 2.** If \( \kappa \) is regular in \( M \), then all cardinals \( \alpha \) with \( \kappa < \alpha < \beta \), are similar to \( \kappa \) in the extension \( M[G] \); all cardinals outside this range are preserved.

**Proof.** By theorem 1, \( 2^{\alpha} \) collapses to \( \kappa \) in the extension; but \( 2^{\alpha} \) is then the union of \( \kappa \) ordinals similar to \( \kappa \) and so (since the axiom of choice will hold in the extension), \( 2^{\alpha} \) also collapses. Since the union of a set of compatible conditions of \( M \), of cardinality in \( M \) less than \( \kappa \), is again a condition, and also the set \( F \) of all conditions has cardinality \( 2^{\alpha} \) in \( M \), it follows by the usual methods that cardinals greater than \( 2^{\alpha} \) or less than \( \alpha \), or equal to \( \alpha \) are preserved in the extension (see e.g. [2]).

**Corollary 3.** If \( \kappa \) is singular in \( M \), then all cardinals \( \alpha \) with \( \kappa < \alpha < \beta \), are similar to \( \kappa \) in the extension \( M[G] \); all cardinals less than or equal to \( \alpha \) are preserved.

(*) We use \( \alpha \cdot \beta \) and \( \alpha + \beta \) for ordinal multiplication and addition throughout.

(*) We use \( \mu(x) \) for the least ordinal \( \beta \) such that...

(*) This result, that \( 2^{\kappa} \) collapses to \( \kappa \) when \( \kappa \) is singular, was obtained independently by Yves and Paris.
Proof. Exactly as above, $2^\kappa$ collapses to $\text{cf} \kappa$ in the extension. In this case we can only say that the union of a set of $M$ of compatible conditions of cardinality less than $\text{cf} \kappa$ is again a condition; so cardinals less than or equal to $\text{cf} \kappa$ will be preserved.

Note. In this case, the set of conditions $P$ will have cardinality in $M$, $\kappa^+$; this may be greater than $2^\kappa$, and we do not know whether this also collapses to $\text{cf} \kappa$.

Corollary 4. If the real cardinal of $2^\kappa$ of $M$ is greater than the real cardinal of $\text{cf} \kappa$ of $M$, then there is no set generic over $M$ for this notion of forcing.

Proof. By Corollaries 2 and 3.

In the case of $\kappa$ singular, another notion of forcing is immediately suggested, which turns out to be simpler to deal with than the notion above: namely, to take as conditions, those partial functions in $M$ from $\kappa$ into $[0,1]$, whose domain is bounded by an ordinal less than $\kappa$. (Clearly, this coincides with the previous notion for regular $\kappa$.) Assuming $G$ is generic over $M$ for this second notion, we can prove:

Theorem 5. If $\alpha$ is an ordinal with $\text{cf} \kappa < \alpha \leq 2^\kappa$, then in the extension $M[G]$, $\alpha$ is similar to $\text{cf} \kappa$; all cardinals outside this range are preserved.

Proof. The proof that for a cardinal $\alpha < \kappa$ of $M$, $2^\alpha$ collapses to $\text{cf} \kappa$ in $M[G]$, can be taken over from Theorem 1 without change (though it is essentially simpler in this case); and the proof that cardinals less than or equal to $\text{cf} \kappa$ are preserved is as in Corollary 3.

To see that cardinals greater than $2^\kappa$ are now preserved we simply note that the set $P^*$ of conditions in the new sense has cardinality $2^\kappa$ in $M$.

Added in proof. Since presenting this paper, the author has been informed, that some of these results were known previously: in particular the case $\kappa = \aleph_1$ was known to Vojtěch. Also, Lee has pointed out that the question noted after Corollary 3 can be answered in the Zermelo-Fraenkel theory.

References


Rejs par la Edition is 5. 8. 1964

Modified Vietoris theorems for homotopy
by
J. Dugundji* (Los Angeles, Calif.)

1. Introduction. Smale's Vietoris theorem for homotopy [9] and its various generalizations ([8], [10]) impose local connectivity conditions on the fibres of the given map $p : X \to Y$; in this paper we obtain versions that depend on the manner that the fibres of $p$ are embedded in $X$ rather than on their actual structure.

In the first part § 2 we study a condition, called $\text{PC}_n$, on the embedding of a set $A$ in a space $X$; in particular (2.4) suitable conditions on $A$ itself are sufficient (but not necessary) for $A$ to be $\text{PC}_n$. In § 3, 4, upper semi-continuous decompositions of a space $X$ into $\text{PC}_n$ subsets having a paracompact decomposition space $Y$ are characterized; under certain assumptions (4.4–4.7), for example, when $Y$ is metrizable, then $X$ must have strong local properties. The Vietoris-type theorems for $p : X \to Y$ are given in § 5; the general result (5.1) can be improved considerably if either $Y$ is dominated by a polytope (5.2) or if $Y$ has suitable local properties. Some applications are given in § 6.

2. Proximally $\nu$-connected sets. In writing homotopy groups, the base point will be omitted unless explicitly noted. Let $A \subset B$; for $\nu \geq 1$ we denote by $\pi_\nu(A/B)$ the image of $\pi_\nu(A)$ in $\pi_\nu(B)$ under the homomorphism induced by the inclusion map; $\pi_\nu(A/B) = 0$ will denote that any two points of $A$ can be joined by a path in $B$.

2.1. Definition. Let $X$ be a Hausdorff space. The set $A \subset X$ is called proximally $\nu$-connected in $X$ (written: $\text{PC}_\nu$) if for each neighborhood $U(A)$ of $A$ in $X$ there is a neighborhood $V(A) \subset U$ of $A$ in $X$ such that $\pi_\nu(V|U) = 0$. The set $A \subset X$ is $\text{PC}_\nu$ if it is $k$-$\text{PC}_\nu$ for all $0 \leq k \leq \nu$; and $A \subset X$ is $\text{PC}_\nu$ if it is $\text{PC}_\nu$ for every $\nu \geq 0$.

This notion reduces to that of $\text{LO}^\nu([0,1], [0,1])$ whenever $A$ is a single point, in that $\pi_\nu$ is $\text{PC}_\nu$ if and only if $X$ is $\text{LO}^\nu$ at $a_0$. No $0$-$\text{PC}_\nu$ set can be embedded into two disjoint open subsets so, in particular, a closed $0$-$\text{PC}_\nu$ subset of a normal $X$ is necessarily connected. Other than this, even the