On weak cardinal powers in generic extensions *

by

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We shall work in Zermelo–Fraenkel set theory with the axiom of hoice (ZFC), and identify cardinals with initial ordinals. If σ, τ, are cardinals, the weak power $\sigma^{\underset{\smile}{\tau}}$ is defined to be

$$\bigcup_{\beta<\tau} \sigma^{\beta},$$

where β ranges through cardinals less than τ, and σ^{β} is the usual cardinal power (the cardinal of all functions from β into σ).

In [1], J. Derrick and the author studied the extensions obtained by adding generic subsets of a regular cardinal $\varkappa$ to a model of ZFC, where $\varkappa > \omega$. In order to prove that all cardinals were preserved in the extension, the assumption was made that for cardinals $\beta < \varkappa$, in the original model $2^{\beta} \leqslant \varkappa$. This is, of course, the assumption that $2^{\underset{\smile}{\varkappa}} = \varkappa$ in the original model. (This cardinal arose in that work since it is just the cardinal of the set of conditions used in defining "generic" in the extensions studied.)

The question of the preservation of cardinals in the absence of this condition is the subject of this paper. As a corollary of the main result, it will be seen that if $2^{\underset{\smile}{\varkappa}}$ is a cardinal greater than $\varkappa$ in the original model, then in the extension this cardinal is *not* preserved but collapses and becomes similar to $\dot{\varkappa}$. Thus e.g. if $\varkappa$ is $\aleph_1$, then the continuum hypothesis will always hold in the extension.

Notation. We let M be a transitive ε-model of ZFC, and $\varkappa$ an uncountable cardinal in M. We use the following notion of forcing (1): a *condition* is a partial function in M from $\varkappa$ into $\{0, 1\}$, with cardinality in M less than $\varkappa$. If p, q are conditions, then p *extends* q iff $p \supset q$. Let P be the set of all conditions ($P \in M$). A subset X of P is *dense* if every condition has an extension in X. Conditions p, q are *compatible* if there is a condition extending both, and a set of (pairwise) compatible conditions is *generic over M* for this notion of forcing if it meets every dense

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(1) The elegant formulation of a *notion of forcing*, as used here, has been seen by the author only in *Seminar notes* of J. Silver (1966–7) and of F. Rowbottom (1966–7).

subset of P which is in M. In the usual way we can prove that if G is generic over M, then $M[G]$, the extension of M by G, is again a model of ZFC, which contains a new subset of $\varkappa$ (not in M) (see [1] or [2] for details.).

If M is countable, then the existence of a generic set of conditions is easily proved, as in Cohen's work (see [2]). Other assumptions than countability of M can also lead to proofs of the existence of generic sets; but one corollary of the result to be proved is a condition for the non-existence of sets of conditions generic over M.

If $\varkappa$ is an infinite cardinal, we write cf $\varkappa$ (*confinality* of $\varkappa$ in M) for the least ordinal λ such that there is a function in M, $f\colon \lambda \to \varkappa$, such that $\bigcup_{a<\lambda} f(a) = \varkappa$. ($\varkappa$ is *regular* iff $\operatorname{cf}\varkappa = \varkappa$, otherwise singular; cf $\varkappa$ is always a regular cardinal of M).

Assuming these preliminaries and that G is generic over M for this notion of forcing, we state the main theorem:

THEOREM 1. *Let $a < \varkappa$ be a cardinal in M, and let $\lambda = 2^a$ in M (i.e. λ is an initial ordinal of M). Then, in $M[G]$, there is a function mapping λ one-to-one into* cf $\varkappa$.

Proof. Let G be generic over M, and x be a subset of a which is in M. Let $\varphi\colon \operatorname{cf}\varkappa \to \varkappa$ be an increasing function in M onto regular cardinals less than $\varkappa$, if $\varkappa$ is singular; or if $\operatorname{cf}\varkappa = \varkappa$, let $\varphi(\beta) = a\cdot\beta$ for $\beta < \varkappa$ (2). If p is a condition, we examine p between $\varphi(\beta)$ and $\varphi(\beta+1)$ for $\beta < \operatorname{cf}\varkappa$.

For ordinal β, define the *a-cut of p at β*, p^a_β, as the partial function from a into $\{0,1\}$ given by:

$p^a_\beta(\gamma)$ is defined for $\gamma < a$ if $p(\beta+\gamma)$ is defined, and if so then $p^a_\beta(\gamma) = p(\beta+\gamma)$.

Now, for $x \subset a$, we say that p is *x-like at β* if p^a_β is defined on the whole of a, and for $\gamma < a$, $p^a_\beta(\gamma) = 1$ iff $\gamma \in x$: i.e. p^a_β is the characteristic function of x.

We now look at a-cuts of p between $\varphi(\beta)$ and $\varphi(\beta+1)$ for $\beta < \operatorname{cf}\varkappa$. If $\varkappa$ is regular, then each such interval is of length a and we are interested in the a-cut of p at $\varphi(\beta)$; but if $\varkappa$ is singular, then for each β such that $\varphi(\beta) \geqslant a$, there will be $\varphi(\beta+1)$ such cuts between $\varphi(\beta)$ and $\varphi(\beta+1)$. We look at all cuts of p at $\varphi(\beta)+a\cdot\gamma$, where $\varphi(\beta)+a\cdot\gamma < \varphi(\beta+1)$, and say, for $\beta < \operatorname{cf}\varkappa$, that p is *good for x at β* if there is an ordinal γ such that:

(i) $\varphi(\beta)+a\cdot\gamma < \varphi(\beta+1)$,

(ii) for each $\xi \geqslant \gamma$, with $\varphi(\beta)+a\cdot\xi < \varphi(\beta+1)$, p is x-like at $\varphi(\beta)+a\cdot\xi$.

(If this is so, then p, from some point on, is a series of copies of the characteristic function of x, in the interval between $\varphi(\beta)$ and $\varphi(\beta+1)$.)

If $\varkappa$ is regular, we say that p is *good for x at β* if p is x-like at $\varphi(\beta)$.

(2) We use $a\cdot\beta$ and $a+\beta$ for ordinal multiplication and addition throughout.

LEMMA. *For x in M, $x \subset a$, the set of conditions such that there is a $\beta < \operatorname{cf}\varkappa$ at which p is good for x is dense (and in M).*

Proof: Given any condition p, we have to show how to extend p to a condition which is good for x at some $\beta < \operatorname{cf}\varkappa$. Since p is a condition, card $p < \varkappa$. We take the case of $\varkappa$ regular first. If $\varkappa$ is regular, then dom(p) must be bounded below $\varkappa$: say $\operatorname{dom}(p) \subset a\cdot\xi < \varkappa$ for some $\xi < \varkappa$.

Then extend p to q by defining q on all γ,

$$a\cdot\xi \leqslant \gamma < a\cdot(\xi+1),$$

so that q is x-like at $a\cdot\xi$. Then q is an extension of p which is good for x at ξ.

If $\varkappa$ is singular, then dom(p) may not be bounded below $\varkappa$; but for some $\beta < \operatorname{cf}(\varkappa)$, $\varphi(\beta) > \max(\operatorname{card} p, a)$. So the domain of p is bounded below $\varphi(\beta+1)$, i.e. there is a γ with $\varphi(\beta)+a\cdot\gamma < \varphi(\beta+1)$ such that p is undefined for all ζ,

$$\varphi(\beta)+a\cdot\gamma \leqslant \zeta < \varphi(\beta+1).$$

Any such ζ is of the form $\zeta = \varphi(\beta)+a\cdot\xi+\zeta'$ for $\zeta' < a$; we extend p to q by defining

$$q(\zeta) = \begin{cases} 1 & \text{if } \zeta' \in x, \\ 0 & \text{if } \zeta' \notin x, \end{cases}$$

and q is then an extension of p which is good for x at β.

Thus the lemma is proved, and the theorem follows: take an enumeration $(x_\delta)_{\delta<\lambda}$ in M of all subsets of a in M, and since G enters every dense set, we can define, for $\delta < \lambda$,

$$f(\delta) = \mu\beta(\exists p \in G(p \text{ is good for } x_\delta \text{ at } \beta)). \ (3)$$

Then f is a 1-1 function from λ into cf $\varkappa$, and f is in $M[G]$.

COROLLARY 2. *If $\varkappa$ is regular in M, then all ordinals a, with $\varkappa < a \leqslant 2^{\varkappa}$, are similar to $\varkappa$ in the extension $M[G]$; all cardinals outside this range are preserved.*

Proof. By theorem 1, 2^a collapses to $\varkappa$ in the extension; but $2^{\varkappa}$ is then the union of $\varkappa$ ordinals similar to $\varkappa$ and so (since the axiom of choice will hold in the extension), $2^{\varkappa}$ also collapses. Since the union of a set of compatible conditions of M, of cardinality in M less than $\varkappa$, is again a condition, and also the set P of all conditions has cardinality $2^{\varkappa}$ in M, it follows by the usual methods that cardinals greater than $2^{\varkappa}$ or less than or equal to $\varkappa$ are preserved in the extension (see e.g. [2]).

COROLLARY 3. *If $\varkappa$ is singular in M, then all ordinals a with $\operatorname{cf}\varkappa < a \leqslant 2^{\varkappa}$, are similar to cf $\varkappa$ in the extension $M[G]$; all cardinals less than or equal to cf $\varkappa$ are preserved.* (4)

(3) We use $\mu\beta\ldots$ for: the least ordinal β such that ...

(4) This result, that $2^{\varkappa}$ collapses to cf $\varkappa$ when $\varkappa$ is singular, was obtained independently by Yates and Paris.

Proof. Exactly as above, $2^{\varkappa}$ collapses to $\operatorname{cf}\varkappa$ in the extension. In this case we can only say that the union of a set of M of compatible conditions of cardinality less than $\operatorname{cf}\varkappa$ is again a condition; so cardinals less than or equal to $\operatorname{cf}\varkappa$ will be preserved.

Note. In this case, the set of conditions P will have cardinality in M, $\varkappa^{\varkappa}$; this may be greater than $2^{\varkappa}$, and we do not know whether this also collapses to $\operatorname{cf}\varkappa$.

COROLLARY 4. *If the real cardinal of $2^{\varkappa}$ of M is greater than the real cardinal of $\operatorname{cf}\varkappa$ of M, then there is no set generic over M for this notion of forcing.*

Proof. By Corollaries 2 and 3.

In the case of $\varkappa$ singular, another notion of forcing is immediately suggested, which turns out to be simpler to deal with than the notion above: namely, to take as conditions, those partial functions in M from $\varkappa$ into $\{0,1\}$, whose domain is bounded by an ordinal less than $\varkappa$. (Clearly this coincides with the previous notion for regular $\varkappa$.) Assuming G' is generic over M for this second notion, we can prove:

THEOREM 5. *If a is an ordinal with $\operatorname{cf}\varkappa < a \leqslant 2^{\varkappa}$, then in the extension $M[G']$, a is similar to $\operatorname{cf}\varkappa$; all cardinals outside this range are preserved.*

Proof. The proof that for a cardinal $a < \varkappa$ of M, 2^{a} collapses to $\operatorname{cf}\varkappa$ in $M[G']$, can be taken over from Theorem 1 without change (though it is essentially simpler in this case); and the proof that cardinals less than or equal to $\operatorname{cf}\varkappa$ are preserved is as in Corollary 3.

To see that cardinals greater than $2^{\varkappa}$ are now preserved we simply note that the set P' of conditions in the new sense has cardinality $2^{\varkappa}$ in M.

Added in proof: Since presenting this paper, the author has been informed, that some of these results were known previously: in particular the case $\varkappa = \aleph_1$ was known to Vopěnka. Also Jech has pointed out that the question noted after Corollary 3 can be answered negatively using results of Engelking and Karlowicz.

References

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[2] P. J. Cohen, *Set Theory and the Continuum Hypothesis*, Benjamin, N.Y., 1966.

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Modified Vietoris theorems for homotopy

by

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1. Introduction. Smale's Vietoris theorem for homotopy [9] and its various generalizations ([5], [8]) impose local connectivity conditions on the fibres of the given map $p\colon X \to Y$; in this paper we obtain versions that depend on the manner that the fibers of p are embedded in X rather than on their actual structure.

In the first part (§ 2) we study a condition, called PC_X^n, on the embedding of a set A in a space X; in particular (2.4) suitable conditions on A itself are sufficient (but not necessary) for A to be PC_X^n. In § 3, 4, upper semi-continuous decompositions of a space X into PC_X^n subsets having a paracompact decomposition space Y are characterized; under certain assumptions (4.4–4.7), for example, when Y is metrizable, then Y must have strong local properties. The Vietoris-type theorems for $p\colon X \to Y$ are given in § 5; the general result (5.1) can be improved considerably if either Y is dominated by a polytope (5.2) or if Y has suitable local properties. Some applications are given in § 6.

2. Proximally n-connected sets. In writing homotopy groups, the base point will be omitted unless explicitly needed. Let $A \subset B$; for $n \geqslant 1$ we denote by $\pi_n(A|B)$ the image of $\pi_n(A)$ in $\pi_n(B)$ under the homomorphism induced by the inclusion map; $\pi_0(A|B) = 0$ will denote that any two points of A can be joined by a path in B.

2.1. DEFINITION. Let X be a Hausdorff space. The set $A \subset X$ is called *proximally n-connected in X* (written: n-PC_X) if for each neighborhood $U(A)$ of A in X there is a neighborhood $V(A) \subset U$ of A in X such that $\pi_n(V|U) = 0$. The set A is PC_X^n if it is k-PC_X for all $0 \leqslant k \leqslant n$; and A is PC_X^{∞} if it is PC_X^n for every $n \geqslant 0$.

This notion reduces to that of LC^n ([1], [2], [6]) whenever A is a single point, in that a_0 is PC_X^n if and only if X is LC^n at a_0. No 0-PC_X set can be embedded into two disjoint open subsets so, in particular, a closed 0-PC_X subset of a normal X is necessarily connected. Other than this, even the

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