ANR-spaces which are deformation retracts of some polyhedra

by

M. Moszyńska (Warszawa)

Contents

1. Introduction. This paper concerns compact ANRs (see [2], p. 100). The question is what assumptions are to be made in order that a given 2-dimensional ANR be a deformation retract of some 2-dimensional polyhedron (see [2], p. 11).

A homogeneously 2-dimensional polyhedron P is said to be a 2-dimensional Σ-pseudomanifold provided every 1-simplex of P is a face of exactly two 2-simplexes of P. For example, every 2-dimensional pseudomanifold ([4], p. 252) (thus any 2-dimensional manifold as well) is a Σ-pseudomanifold.

Given an arbitrary polyhedron P, a subset P' of P is said to be a subpolyhedron of P whenever there is a triangulation $T = \{\sigma_i\}_{i=1,\dots,k}$ of P such that $P' = \bigcup_{\nu} |\sigma_{i_\nu}|$. Obviously, we can find such two polyhedra P and P' that $P' \subset P$ but P' is not a subpolyhedron of P. However, if P is assumed to be a 2-dimensional Σ-pseudomanifold, then every polyhedron P' which is a subset of P is a subpolyhedron of P.

We are going to show that any ANR-set X lying on 2-dimensional Σ-pseudomanifold P is a strong deformation retract ([1]) of some subpolyhedron of P (Theorem (8.1)).

We begin by proving the particular case of this theorem, for P which is a manifold (Theorem (6.1)). Afterwards, by means of splitting operation (see [6]), we generalize this result to an arbitrary Σ-pseudomanifold.

2. Polyhedral approximation of the continuum lying on a 2-dimensional manifold. Let Y be a homogeneously n-dimensional polyhedron with a given triangulation T. A *boundary* $\dot{Y}$ of Y is understood as the union of $n-1$ simplexes of T with the following property:

every $n-1$ simplex in $\dot{Y}$ is a face of exactly one n-simplex of T.

The set $\mathring{Y} = Y - \dot{Y}$ is said to be an *interior* of the polyhedron Y.

By an arc we mean the image L of the closed interval $\langle 0, 1\rangle$ by an arbitrary homeomorphism $h\colon \langle 0,1\rangle \to L$. The set of end-points $\{h(0), h(1)\}$ of L is denoted by $\dot{L}$, the interior of L (i.e. $h\big((0,1)\big)$), by $\mathring{L}$. A 1-dimensional polyhedron Γ is said to be a *closed curve*, whenever its boundary $\dot{\Gamma}$ is empty.

Now, let us consider a 2-dimensional manifold M with a given metric ϱ. For an arbitrary subset A of M and for $\xi > 0$ let $K(A, \xi)$ denote the set $\{x \in M\colon \varrho(x, A) < \xi\}$. Let X be a non-degenerate continuum in M, Z^i — the components of the set $M - X$ and let $\varepsilon > 0$. A homogeneously 2-dimensional subpolyhedron Y of M is said to be a *polyhedral ε-approximation of X* provided that

1° $X \subset Y$;

2° $Y \subset K(X, \varepsilon)$;

3° $\dot{Y}$ is a union of $n = n(\varepsilon)$ closed curves $\Gamma^1, \ldots, \Gamma^m$ such that

a) $\Gamma^i \subset Z^i$ for $i = 1, \ldots, n$; b) $\Gamma^i = \Gamma^i_1 \cup \ldots \cup \Gamma^i_{k_i}$, all the sets Γ^i_j being arcs with end-points in X, with interiors in $M - X$ and with diameters less than ε;

4° the sets $Z^i \cap (M - Y)$ are connected for $i = 1, \ldots, n$;

5° if $\overline{\overline{\{Z^i\}}} = m$, then $n(\varepsilon) = m$.

We have the following

(2.1) LEMMA. *Let M be a 2-dimensional manifold. For any non-degenerate continuum $X \subset M$ and for every $\varepsilon > 0$ there is a polyhedral ε-approximation Y of X.*

([1]) As regards ANR-spaces, the following two conditions are equivalent: 1° Y is a strong deformation retract of X, 2° Y is a deformation retract of X (see [3]).

3. Proof of Lemma (2.1) in the case of X non-decomposing the manifold M. In this case, Lemma (2.1) is a modification of Borsuk's Lemma (13.3) of [2] p. 132, which concerns plane continua. The proof of this particular case of our Lemma (2.1) is similar to the proof of Borsuk's lemma and therefore we do not put it here in detail. Let us only remark that the proof of Borsuk's Lemma (13.3) was based on Lemma (13.2) ([2] p. 132), which concerns the k-dimensional Euclidean space. In our proof, this Lemma (13.2) should be replaced by a corresponding statement which deals with manifolds (see (3.3) below). To formulate this statement, we start with defining a special property of metric space $\langle M, \varrho\rangle$:

The space M has the *Λ-property* provided given a continuum X non-decomposing M, a point $b \in M - X$ and $\varepsilon > 0$, there is $\eta = \eta(\varepsilon) > 0$ which satisfies the following condition:

(3.1) *for any compact $Q \subset M - \{b\}$*

$$\mathrm{Fr}\, Q \subset K(X, \eta) \Rightarrow Q \subset K(X, \varepsilon)\,.$$

One can easily verify that

(3.2) *For compact metric spaces, the Λ-property is a topological invariant.*

Now, the statement mentioned above is the following:

(3.3) *Every locally connected continuum has the Λ-property.*

Let us observe that in the case of the k-dimensional sphere, proposition (3.3) is simply another form of Borsuk's Lemma (13.2). In general, for an arbitrary locally connected continuum, by Bing's theorem [1] and by our proposition (3.2), we can assume ϱ to be a convex metric. Then we can easily prove our statement (3.3) applying the idea used by K. Borsuk in the proof of (13.2). Obviously, (3.3) holds in particular for a manifold. Thus, for X non-decomposing the manifold M, Lemma (2.1) is taken as proved. Moreover, as in (13.3) of [2], Y is a subpolyhedron of M in a triangulation the simplexes of which have diameters less than $\eta(\varepsilon)$ ($\eta(\varepsilon)$ satisfying (3.1)).

4. Proof of Lemma (2.1) in the general case. Let M be a 2-dimensional manifold with a convex metric ϱ (by Bing's theorem [1] such a metric does exist). Consider an arbitrary non-degenerate continuum X in M and take $\varepsilon > 0$. Assume the set $M - X$ to be non-empty and denote its components by Z^i (the collection $\{Z^i\}$ is finite or countable). Let $b_i \in Z^i$ for each i. Observe that

(4.1) *There is an index $n = n(\varepsilon)$ such that $Z^k \subset K(X, \varepsilon)$ for $k > n$.*

In fact, if $\overline{\overline{\{Z^i\}}} = m$, then (4.1) is satisfied for $n(\varepsilon) = m$. Let the family $\{Z^i\}$ be infinite and let us suppose that

$$\bigwedge_n \bigvee_{k_n > n} \bigvee_{z_n \in Z^{k_n}} \varrho(z_n, X) \geqslant \varepsilon .$$

Since ϱ is a convex metric, for every $z_n, z_{n'} \in M$ there is a segment $|z_n z_{n'}|$ in M. Let $z_j \in Z^{k_j}$ for $j = n, n'$. Since $Z^{k_n} \cap Z^{k_{n'}} = 0$, the two sets $\mathrm{Fr}\, Z^{k_n}$ and $|z_n z_{n'}|$ have a point x in common. Then $x \in X \cap |z_n z_{n'}|$, and therefore

$$\varrho(z_n, z_{n'}) \geqslant \varrho(z_n, x) \geqslant \inf_{x' \in X} \varrho(z_n, x') = \varrho(z_n, X) \geqslant \varepsilon .$$

But the last inequality contradicts the compactness of M and thus proves (4.1).

Now, let us define

$$X^i = X \cup \bigcup_{j \neq i} Z^j , \qquad i = 1, 2, \ldots$$

Each of the sets X^i is a non-degenerate continuum which does not decompose the manifold M, i.e. $M - X^i$ is connected for every i; moreover, $b_i \in M - X^i$. Thus, Lemma (2.1) holds for X^i (see Section 3). According to (3.3) M has the Λ-property and thus for every i there is $\eta_i = \eta_i(\varepsilon)$ satisfying condition (3.1).

Let Y^i be a polyhedral ε-approximation of X^i $(i = 1, 2, \ldots)$. Y^i is a subpolyhedron of M for a triangulation T_i with simplexes of diameters less than η_i.

Let us set $\eta_0 = \min[\eta_1, \ldots, \eta_n]$, $n = n(\varepsilon)$ satisfying (4.1). We can find a triangulation T_0 with simplexes of diameters less than η_0 and such that each Y^i $(i = 1, \ldots, n)$ is a union of some simplexes of T_0.

Setting

$$Y \underset{\mathrm{Df}}{=} \bigcap_{i=1}^{n} Y^i$$

we obtain a subpolyhedron Y of the manifold M. As we have already remarked, condition (4.1) holds in particular for $n(\varepsilon) = m = \overline{\overline{\{Z^i\}}}$. Hence condition 5° is satisfied, and thus it remains to verify conditions 1°–4°. To this effect, let us notice that

$$X = \bigcap_{i=1}^{\infty} X^i . \tag{4.2}$$

Therefore $X \subset \bigcap_{i=1}^{n} X^i$. Since $X^i \subset Y^i$ for $i = 1, 2, \ldots$, we have $X \subset Y$, i.e. Y satisfies condition 1°.

Besides, we have

$$\bigcap_{i=1}^{n} X^i = X \cup \bigcup_{j>n} Z^j . \tag{4.3}$$

Now, take an arbitrary point $y \in Y$ and show that $\varrho(y, X) < \varepsilon$. In fact,

$$Y = \bigcap_{i=1}^{n} Y^i = \bigcap_{i=1}^{n} X^i \cup [\bigcap_{i=1}^{n} Y^i - \bigcap_{j=1}^{n} X^j]$$
$$= \bigcap_{i=1}^{n} X^i \cup [\bigcap_{i=1}^{n} Y^i \cap (M - \bigcap_{j=1}^{n} X^j)] = \bigcap_{i=1}^{n} X^i \cup [\bigcap_{i=1}^{n} Y^i \cap \bigcap_{j=1}^{n} Z^j] ,$$

whence, by (4.3), we obtain

$$Y = X \cup \bigcup_{j>n} Z^j \cup [\bigcap_{i=1}^{n} Y^i \cap \bigcup_{j=1}^{n} Z^j] .$$

If $y \in X$, then $\varrho(y, X) = 0$;

if $y \in \bigcup_{j>n} Z^j$, then $\varrho(y, X) < \varepsilon$ by (4.1);

if $y \in \bigcap_{i=1}^{n} Y^i$ and simultaneously $y \in \bigcup_{j=1}^{n} Z^j$, then there is a $j \leqslant n$ such that $y \in Z^j \cap Y^j$; therefore $\varrho(y, X^j) < \varepsilon$ (since Y^j is an ε-approximation of X^j), and $\varrho(y, X^j) = \varrho(y, X)$ (since $y \in Z^j$), and so $\varrho(y, X) < \varepsilon$ as well. Thus condition 2° is verified.

In order to verify condition 3° it suffices to show

$$\dot{Y} = \bigcup_{i=1}^{n} \dot{Y}^i . \tag{4.4}$$

Let us prove (4.4).

$$\dot{Y} = Y \cap \overline{M - Y} = \bigcap_{i=1}^{n} Y^i \cap \overline{M - \bigcap_{j=1}^{n} Y^j}$$
$$= \bigcap_{i=1}^{n} Y^i \cap \bigcup_{j=1}^{n} \overline{M - Y^j} = \bigcup_{j=1}^{n} \bigcap_{i=1}^{n} (Y^i \cap \overline{M - Y^j}) .$$

If $i \neq j$ then $\overline{M - Y^j} \subset \overline{Z}^j \subset X^i \subset Y^i$, and so $Y^i \cap \overline{M - Y^j} = \overline{M - Y^j}$; if $i = j$ then $Y^i \cap \overline{M - Y^j} = \dot{Y}^j$. Hence $\dot{Y} = \bigcup_{j=1}^{n} (\dot{Y}^j \cap \overline{M - Y^j}) = \bigcup_{j=1}^{n} \dot{Y}^j$, which proves (4.4).

Now, let us observe that

$$Z^i \cap (M - Y) = Z^i \cap (M - \bigcap_{j=1}^{n} Y^j) = Z^i \cap \bigcup_{j=1}^{n} (M - Y^j) = M - Y^i ;$$

since Y^i is an ε-approximation of X^i, the set $M - Y^i$ is connected. Then condition 4° is satisfied. This completes the proof of Lemma (2.1).

5. Two lemmas on strong deformation retracts.

(5.1) Lemma. *If A, B are two absolute retracts and $A \subset B$, then A is a strong deformation retract of B. Moreover, every retraction $r\colon B \to A$ is a strong deformational one.*

Proof. Since $A \in \mathrm{AR}$ and $A \subset B$, there is a retraction $r\colon B \to A$. Let $i\colon A \to B$ be the inclusion. Define a map $\varphi_0\colon B \times \{0,1\} \cup A \times I \to B$, where $I = \langle 0, 1\rangle$:

$$\varphi_0(x,t) \underset{\mathrm{Df}}{=} \begin{cases} \mathrm{ir}(x) & \text{for} \quad (x,t) \in B \times \{0\}, \\ x & \text{for} \quad (x,t) \in B \times \{1\} \cup A \times I. \end{cases}$$

Since $B \times \{0,1\} \cup A \times I \subset B \times I$ and $B \in \mathrm{AR}$, there is a map $\varphi\colon B \times I \to B$ such that $\varphi_0 \subset \varphi$ (i.e. $\varphi(x) = \varphi_0(x)$ for $x \in B \times \{0,1\} \cup A \times I$). We have

$$\varphi(x,0) = \varphi_0(x,0) = \mathrm{ir}(x), \quad \varphi(x,1) = \varphi_0(x,1) = x \quad \text{for } x \in B,$$

and

$$\varphi(x,t) = \varphi_0(x,t) = x \quad \text{for} \quad (x,t) \in A \times I.$$

Hence r is a strong deformational retraction. This completes the proof.

(5.2) Lemma. *If $Y = X \cup \bigcup_{\nu=1}^{n} C_\nu$, all the sets C_ν and $X \cap C_\nu$ being absolute retracts and $C_\nu \cap C_{\nu'} \subset X$ for $\nu \neq \nu'$, then X is a strong deformation retract of Y. Moreover, if $r_\nu\colon C_\nu \to X \cap C_\nu$ is a retraction for $\nu = 1, \ldots, n$, then the map $r\colon Y \to X$ defined by the formula*

$$r(x) \underset{\mathrm{Df}}{=} \begin{cases} r_\nu(x) & \text{for } x \in C_\nu, \\ x & \text{for } x \in X \end{cases}$$

is a strong deformational retraction.

Proof. According to Lemma (5.1) the set $X \cap C_\nu$ is a retract of C_ν for $\nu = 1, \ldots, n$. Let us take retractions $r_\nu\colon C_\nu \to X \cap C_\nu$. By (5.1) r_ν are strong deformational retractions, i.e. there exist maps $\varphi_\nu\colon C_\nu \times I \to C_\nu$ such that $\varphi_\nu(x,0) = r_\nu(x)$, $\varphi_\nu(x,1) = x$ for $x \in C_\nu$ and $\varphi_\nu(x,t) = x$ for $(x,t) \in (X \cap C_\nu) \times I$. Setting

$$\varphi(x,t) \underset{\mathrm{Df}}{=} \begin{cases} \varphi_\nu(x,t) & \text{for } (x,t) \in C_\nu \times I, \\ x & \text{for } (x,t) \in X \times I, \end{cases}$$

we obtain the map $\varphi\colon Y \times I \to Y$ such that $\varphi(x,0) = r(x)$, $\varphi(x,1) = x$ for $x \in Y$ and $\varphi(x,t) = x$ for $(x,t) \in X \times I$.

Hence r is a strong deformational retraction.

6. Deformational properties of ANRs lying on 2-dimensional manifolds.

The theorem which we shall prove now is a particular case of Theorem (8.1).

(6.1) Theorem. *Let M be a 2-dimensional manifold. If $X \in \mathrm{ANR}$ and X is a subset of M, then there exists a subpolyhedron Y of M such that X is a strong deformation retract of Y.*

Proof. Let us assume X to be connected.

If the manifold M is not homeomorphic to S^2, then there exists a simple closed curve which is not contractible in M. Moreover, since M is locally 1-connected, the inferior of diameters of all such curves is positive. Hence, for any 2-dimensional manifold M, the following positive number $\zeta_0 = \zeta_0(M)$ can be defined:

$$\text{(6.2)} \quad \zeta_0 \underset{\mathrm{Df}}{=} \begin{cases} \inf \delta(\Omega) & (\Omega \text{ being a simple closed curve non-contractible in } M), \text{ if } M \underset{\mathrm{top}}{\neq} S^2, \\ 1 & \text{if } M \underset{\mathrm{top}}{=} S^2. \end{cases}$$

If a simple closed curve Ω is contractible in M, then there is a disk $A = A(\Omega)$ in M, with the boundary $\mathrm{Fr} A = \Omega$. Moreover,

$$\text{(6.3)} \quad \bigwedge_{a>0} \bigvee_{\eta(a)} \delta(\Omega) < \eta \Rightarrow \delta\big(A(\Omega)\big) < a.$$

Besides, since $X \in \mathrm{ANR}$, the set $M - X$ has a finite number of components $Z^1, \ldots, Z^m$.

Let us define $\zeta_i = \zeta_i(M, X)$ for $i = 1, 2$:

$$\text{(6.4)} \quad \zeta_1 \underset{\mathrm{Df}}{=} \eta\big(\min_{1 \leqslant k \leqslant m} \delta(Z^k)\big),$$

$$\text{(6.5)} \quad \zeta_2 \underset{\mathrm{Df}}{=} \tfrac{1}{2}\min(\zeta_0, \zeta_1).$$

Since X is a locally connected continuum, the following condition is satisfied (see [4], p. 129):

$$\text{(6.6)} \quad \bigwedge_{\zeta>0} \bigvee_{\varepsilon'=\varepsilon'(\zeta)} \bigwedge_{y_1, y_2 \in X} [\varrho(y_1, y_2) < \varepsilon' \Rightarrow \bigvee_{L=L(y_1,y_2)} \delta(L) < \zeta],$$

L being an arc in X, with the end-points y_1, y_2.

Let us set

$$\text{(6.7)} \quad \varepsilon \underset{\mathrm{Df}}{=} \min\big(\zeta_2, \varepsilon'(\zeta_2)\big).$$

According to Lemma (2.1) there is a polyhedral ε-approximation Y of X. We are going to show that the set Y is the desired one, i.e. X is a deformation retract of Y. To this effect, let us take the arc Γ_j^i (determined by condition 3°, § 2, for $j = 1, \ldots, k_i$, $i = 1, \ldots, m$) and let x_j^i, x_{j+1}^i be the end-points of Γ_j^i ($x_{k_i+1}^i = x_1^i$). By condition 3° mentioned above, $\delta(\Gamma_j^i) < \varepsilon$, and then, by (6.7), $\varrho(x_j^i, x_{j+1}^i) < \varepsilon \leqslant \varepsilon'(\zeta_2)$. So, by (6.6), there is an arc $L_j^i = L_j^i(x_j^i, x_{j+1}^i)$ in X, such that $\delta(L_j^i) < \zeta_2$.

Since the two arcs Γ_j^i, L_j^i have common end-points and disjoint interiors, the set

$$\Omega_j^i \underset{\mathrm{Df}}{=} \Gamma_j^i \cup L_j^i$$

is a simple closed curve. Moreover, $\delta(\Omega_j^i) \leqslant \delta(L_j^i)+\delta(\Gamma_j^i) < \zeta_2+\varepsilon$ and thus, by (6.7), we obtain

(6.8) $$\delta(\Omega_j^i) < 2\zeta_2 .$$

Condition (6.8) together with (6.5) and (6.2) implies the contractibility of Ω_j^i in M. Let A_j^i be a disk in M which is bounded by Ω_j^i. Observe that, for any i, j $(j = 1, \ldots, k_i,\ i = 1, \ldots, m)$ the two sets A_j^i and $\bigcup_{k\neq i} Z^k$ are disjoint. In fact, $\Omega_j^i \cap Z^k \neq 0$ if and only if $k = i$; therefore, if there is a $k \neq i$ such that $A_j^i \cap Z^k \neq 0$, then $Z^k \subset A_j^i$, but this is impossible, because conditions (6.8), (6.5), (6.4) and (6.3) imply $\delta(A_j^i) < \min_{1\leqslant k\leqslant m} \delta(Z^k)$.

Hence

(6.9) $$A_j^i \cap A_{j'}^{i'} \subset X \quad \textit{for} \quad (i,j) \neq (i',j') .$$

By our construction, the polyhedron Y is of the form

(6.10) $$Y = X \cup \bigcup_{j,i} A_j^i .$$

Let us set

$$X_j^i \underset{\mathrm{Df}}{=} X \cap A_j^i$$

and show that

(6.11) $$X_j^i \in \mathrm{AR} \quad \textit{for every} \quad j, i .$$

To this effect, it suffices to prove that $X_j^i \in \mathrm{ANR}$, X_j^i is connected and it does not decompose the disk A_j^i. Since $X_j^i \cap \overline{X - X_j^i} = L_j^i \in \mathrm{AR}$ and $X_j^i \cap \overline{X - X_j^i} \subset X$, there is a retraction $f_j^i\colon X \to X_j^i \cap \overline{X - X_j^i}$. A map $g_j^i\colon X \to X_j^i$ defined by the formula

$$g_j^i(x) = \begin{cases} f_j^i(x) & \text{for } x \in \overline{X - X_j^i}, \\ x & \text{for } x \in X_j^i, \end{cases}$$

is a retraction as well. Then, $X \in \mathrm{ANR}$ implies $X_j^i \in \mathrm{ANR}$. Moreover, X_j^i is connected, since any two points of X_j^i can be joined in X by an arc having no points in common with $\mathring{\Gamma}_j^i$. At last X_j^i does not decompose A_j^i, since A_j^i does not contain Z^k $(k = 1, \ldots, m)$. Thus (6.11) is verified.

According to Lemma (5.2), it follows by (6.9)–(6.11) that X is a strong deformation retract of Y. This completes the proof of Theorem (6.1) for connected X.

If X is not connected, then a number of its components is finite. In this case we construct a polyhedral ε-approximation for each component of X. Assuming the triangulation of M to be sufficiently fine, we can make any two of these polyhedrons disjoint. Then the proof of Theorem (6.1) is reduced to the case of a connected X which has already been discussed.

7. 2-dimensional Σ-pseudomanifold as a continuous image of a union of 2-dimensional manifolds. Given a triangulation T of a polyhedron P and a vertex p of T, let us define the *star* $S(p)$ *of the point* p *relative to* T:

$$S(p) \underset{\mathrm{Df}}{=} \bigcup_i |\sigma_i| ,$$

where $\{\sigma_i\}$ is the totality of simplexes with a common vertex p. Let P be a 2-dimensional Σ-pseudomanifold. A point $x \in P$ is said to be *regular* whenever there is a neighbourhood $D(x)$ of x in P, $D(x)$ being a disk. Otherwise, x is said to be *singular*. The set of all regular points of P will be denoted by $\hat{P}$; the set of all singular points — by P^*.

By the definition of Σ-pseudomanifold it follows that

(7.1) *The set P^* is finite*

and

(7.2) *If the triangulation T of P is assumed to be sufficiently fine, then the star $S(p)$ is a union of disks $D_j(p)$ $(j = 1, \ldots, m)$ for each $p \in P^*$. Moreover, $D_j(p) \cap D_{j'}(p) = \{p\}$ for $j \neq j'$ and p is an inner point of $D_j(p)$ for $j = 1, \ldots, m$.*

In order to prove (7.2) it suffices to observe that

1° the boundary $\dot{S}(p)$ of the star $S(p)$ is a union of a finite number of disjoint closed simple curves, and

2° $S(p)$ is a cone over $\dot{S}(p)$ with the vertex p.

Let us establish the following

(7.3) Lemma. *Given a 2-dimensional Σ-pseudomanifold P, there exist 2-dimensional manifolds $M_1, \ldots, M_n$ and a map of their disjoint union $f\colon \bigcup_{i=1}^{n} M_i \xrightarrow{\text{onto}} P$ such that*

(1) $\overline{\overline{f^{-1}(x)}} > 1$ *for each* $x \in P^*$,

(2) $\overline{\overline{f^{-1}(P^*)}} < \aleph_0$,

(3) $f|f^{-1}(\hat{P})$ *is a topological imbedding.*

Proof. By (7.1) the set P^* is finite; let $P^* = \{p_1, \ldots, p_k\}$. By (7.2) we can assume each two of the stars $S(p_i)$ to be disjoint and

$$S(p_i) = \bigcup_{j=1}^{m_i} D_j(p_i) ,$$

the sets $D_j(p_i)$ being disks, $D_j(p_i) \cap D_{j'}(p_i) = \{p_i\}$ for $j \neq j'$, and $p_i \in \mathring{D}_j(p_i)$ for $j = 1, \ldots, m_i$ $(i = 1, \ldots, k)$. Consider the set

$$C \underset{\mathrm{Df}}{=} \overline{P - \bigcup_{i=1}^{k} S(p_i)} .$$

Let D_j^i ($j = 1, \ldots, m_i$; $i = 1, \ldots, k$) be disks disjoint one with another, not intersecting the set C. Take the homeomorphisms

$$h_j^i\colon D_j(p_i) \to D_j^i$$

and define the space N and the map $h\colon N \xrightarrow{\text{onto}} P$ as follows:

$$N \underset{\text{Df}}{=} C \cup \bigcup_{j,i} D_j^i, \qquad \text{(Fig. 1)} \tag{7.4}$$

$$h(x) \underset{\text{Df}}{=} \begin{cases} x & \text{for } x \in C, \\ (h_j^i)^{-1}(x) & \text{for } x \in D_j^i,\ j = 1, \ldots, m_i,\ i = 1, \ldots, k. \end{cases} \tag{7.5}$$

Denote $q_j^i = h_j^i(p_i)$.

Now, let M be a quotient space obtained from the space N by the following identification:

$$x \sim y \Leftrightarrow (x = y) \vee [x \in \dot{C} \wedge y \in \dot{D}_j^i \wedge h(y) = x], \qquad \text{(see [5], p. 9)}. \tag{7.6}$$

One can easily see that the space M is a compactum locally homeomorphic to the Euclidean plane E^2. On the other hand, the set M is of a finite

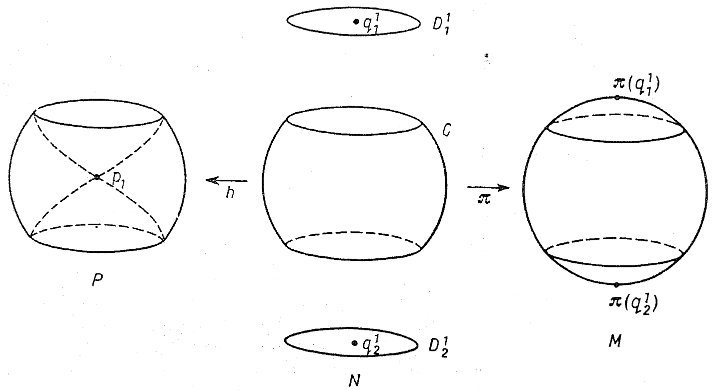

Fig. 1

number of components as well as the set $P - P^*$. Hence M is a union of a finite number of 2-dimensional manifolds $M_1, \ldots, M_n$.

Let $\pi\colon N \to M$ be the natural projection (see [5], p. 9). Setting

$$f(z) \underset{\text{Df}}{=} h\pi^{-1}(z) \quad \text{for every} \quad z \in M \tag{7.7}$$

we obtain a map $f\colon M \xrightarrow{\text{onto}} P$ satisfying the desired three conditions ((1)–(3)). In fact, by (7.5) and (7.6) f is single-valued. To verify the continuity

of f, take an arbitrary open subset U of P and $V = f^{-1}(U)$. By (7.7) we have $\pi^{-1}(V) = h^{-1}(U)$. Since h is continuous, $\pi^{-1}(V)$ is an open subset of N. Therefore, by the definition of a quotient topology (see [5], p. 9), V is open in M. Moreover, $f|f^{-1}(\hat{P})$ is one-to-one, and $f^{-1}(p_i) = \bigcup_{j=1}^{m_i} \pi(q_j^i)$ for each $p_i \in P^*$. Then $\overline{\overline{f^{-1}(p_i)}} = m_i$ and $\overline{\overline{f^{-1}(P^*)}} = \sum_{i=1}^{k} m_i < \aleph_0$ and hence conditions (1)–(3) are satisfied.

8. Deformational properties of ANRs lying on 2-dimensional Σ-pseudomanifolds. We are ready now to prove

(8.1) THEOREM. *Let P be a 2-dimensional Σ-pseudomanifold. If $X \in$ ANR and X is a subset of P, then there is a subpolyhedron Y of P such that X is a strong deformation retract of Y.*

Proof. Let $\hat{P}$ be the set of regular points and P^* — the set of singular points of P.

According to Lemma (7.3), there is a space M which is a union of 2-dimensional manifolds $M_1, \ldots, M_n$, and a map $f\colon M \xrightarrow{\text{onto}} P$ satisfying conditions (1)–(3).

Let $\omega_f\colon P \leftarrow M$ be a splitting of the space P, i.e. $\omega_f(x) = f^{-1}(x)$ (see [6]). Consider the set

$$X' \underset{\text{Df}}{=} \omega_f(X).$$

By Theorem (4.1) of [6], the set X' is again an absolute neighbourhood retract. Moreover, X' is a proper subset of M.

Take $X'_\nu \underset{\text{Df}}{=} X' \cap M_\nu$ for $\nu = 1, \ldots, n$. If X'_ν is a proper subset of M_ν, then by Theorem (6.1) there exists a subpolyhedron Y'_ν of M_ν such that X'_ν is a strong deformation retract of Y'_ν. Otherwise, if $X'_\nu = M_\nu$, we put $Y'_\nu \underset{\text{Df}}{=} M_\nu$. The set $Y' \underset{\text{Df}}{=} \bigcup_{\nu=1}^{n} Y'_\nu$ is a subpolyhedron of M and X' is a strong deformation retract of Y'.

Obviously, assuming the triangulation of P to be sufficiently fine, we can make the following equality satisfied:

$$X' \cap \omega_f(P^*) = Y' \cap \omega_f(P^*). \tag{8.2}$$

Now, let us put

$$Y \underset{\text{Df}}{=} f(Y').$$

It follows from the properties of the map f, that Y is a subpolyhedron of the Σ-pseudomanifold P. To finish the proof, we must show that X is a strong deformation retract of Y.

By the definition of the splitting operation, (8.2) implies $\omega_f(X \cap P^*) = X' \cap \omega_f(P^*) = Y' \cap \omega_f(P^*) = \omega_f(Y \cap P^*)$.

Hence

$$X \cap P^* = f\omega_f(X \cap P^*) = f\omega_f(Y \cap P^*) = Y \cap P^*,$$

and thus applying Theorem (3.1) of [6], we infer that X is a strong deformation retract of Y. This completes the proof of Theorem (8.1).

Remark 1. The polyhedron Y is a homogeneously 2-dimensional one.

Remark 2. Let us assume P to be a manifold. Observe that $A_j^i \subset K(X_j^i, \varepsilon)$ for every i, j (see § 2 and (6.10)). For sufficiently small ε there are retractions r_j^i: $A_j^i \to X_j^i$ satisfying the condition $r_j^i(A_j^i - X_j^i) \subset \mathrm{Fr}\, X_j^i$ ([2] p. 139). By Lemma (5.1) all r_j^i are strong deformational retractions and therefore by Lemma (5.2) we get a strong deformational retraction r: $Y \to X$ such that $r(Y - X) \subset \mathrm{Fr}\, X$. Obviously, mapping a union of manifolds onto a Σ-pseudomanifold (see § 7) we obtain the same result for an arbitrary Σ-pseudomanifold.

Using these two remarks we get the following

(8.3) COROLLARY. *Let P be a 2-dimensional Σ-pseudomanifold. If $X \in \mathrm{ANR}$, $X \subset P$, then there exist a homogeneously 2-dimensional subpolyhedron Y of P and a strong deformational retraction r: $Y \to X$ such that $r(Y - X) \subset \mathrm{Fr}\, X$.*

PROBLEM. Can the assumption on P in Theorem (8.1) be replaced by the following weaker one: P is a homogeneously 2-dimensional polyhedron, the set $\dot{P} \cap X$ being of a finite number of components?

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Lifting trees under light open maps *

by

J. H. Carruth (Knoxville, Tenn.)

The purpose of this paper is to prove the nonmetric analog of a theorem due to G. T. Whyburn concerning the liftability of dendrites under light open maps. The proof uses the nonmetric analog of an arc lifting theorem of Whyburn which is due to R. J. Koch.

A continuum is *hereditarily unicoherent* provided the intersection of any two of its subcontinua is connected. A *tree* is a locally connected hereditarily unicoherent continuum. An *arc* is a continuum with exactly two noncutpoints. The closure of a set A will be denoted by A^* and the void set by $\square$.

THEOREM. *Suppose f is a light open map from a compact Hausdorff space X onto a topological space Y. If T is a tree in Y and $a \in f^{-1}(T)$, then there exists a continuum K in X such that $a \in K$ and f maps K topologically onto T.*

Proof. Clearly, it can be assumed that $Y = T$ and $X = f^{-1}(T)$. Let $\mathcal{C}$ be the collection of all continua M in X such that $a \in M$ and f restricted to M is a homeomorphism into T. Then $\{a\} \in \mathcal{C}$ so that $\mathcal{C} \neq \square$. Let $\mathcal{M}$ be a maximal tower in $\mathcal{C}$, let $A = \bigcup \mathcal{M}$, and let $K = A^*$. We show that K is the desired continuum in two parts. First it is shown that f is one-to-one on K and second it is shown that f maps K onto T.

For the first part fix $p \in f(K)$. It suffices to show that $f^{-1}(p) \cap K$ is a single point. Let $\mathcal{U}$ be a basis for the topology of T at p consisting of open connected sets. The proof that $f^{-1}(p) \cap K$ is a single point depends on the following four facts:

(i) $f^{-1}(U) \cap A$ is connected for each $U \in \mathcal{U}$.

(ii) $f^{-1}(p) \cap K \neq \square$.

(iii) $f^{-1}(p) \cap K \subset \liminf\{f^{-1}(U) \cap A\colon U \in \mathcal{U}\}$.

(iv) $\limsup\{f^{-1}(U)\colon U \in \mathcal{U}\} \subset f^{-1}(p)$.

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