Continuation in metric spaces

by

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The purpose of this paper is to formalize the topological aspects of "Riemann Surfaces" by constructing function elements on metric spaces and showing that the behavior is similar to the analogous surfaces over the complex plane.

We consider a metric space (X, d) and a set Y. For any $x \in X$ and $r \in R^+_\infty$ (extended positive reals), if $f\colon S(x, r) \to Y$ is a function, then f is called *admissible*. We call x the *center* of f, denoted $c(f)$, and r the *radius* of f, denoted $r(f)$. If F is a family of admissible functions from X to Y, we call F a family of *function elements* iff it satisfies the following:

(1) If $f \in F$, then there does not exist a $g \in F$ such that $c(g) \in S\big(c(f), r(f)\big)$, $r(g) < r(f) - d\big(c(f), c(g)\big)$, and $f|S\big(c(g), r(g)\big) = g$.

(2) If $f, g \in F$, $W = S\big(c(f), r(f)\big) \cap S\big(c(g), r(g)\big) \neq \emptyset$, and there exists an $a \in R^+_\infty$, $x \in X$ such that $S(x, a) \subset W$ and $f|S(x, a) = g|S(x, a)$, then $f|W = g|W$.

An easy application of Zorn's Lemma yields

THEOREM 1. *If F is a non-empty of function elements, then there exists a maximal family containing F.*

Hereafter, such a maximal family of function elements will be called an A-family.

The maximality of an A-family immediately yields

THEOREM 2. *Let F be an A-family, $f \in F$, $y \in S\big(c(f), r(f)\big)$, and $r' = r(f) - d\big(c(f), y\big)$. Then there exists a unique $g \in F$ such that $c(g) = y$ and $g|W = f|W$ where $W = S\big(c(g), r(g)\big) \cap S\big(c(f), r(f)\big)$.* Moreover, $r(g) \geqslant r'$.

Suppose now that F is an A-family on (X, d). Given an $f \in F$ and $a \in R^+_\infty$ such that $a \leqslant r(f)$, let $N_a(f) = \{g \in F|\ d\big(c(g), c(f)\big) < a$ and $f|W = g|W$ where $W = S\big(c(g), r(g)\big) \cap S\big(c(f), r(f)\big)\}$. We now define a topology $\mathcal{T}$ for F as follows: $U \in \mathcal{T}$ iff for each $f \in U$, there exists an $N_a(f) \subset U$. It is easy to see that the topology obtained is Hausdorff. Indeed, if we define $p\colon F\to X$ by $p(f)=c(f)$, then p is a local homeomorphism so that (F, p, X) becomes a topological sheaf. One can, of course, obtain the same sheaf by constructing the sheaf of germs of F-functions on X.

One can now define direct continuation, continuation, and continuation along a path in the obvious way [1]. Standard methods then yield:

THEOREM 3. *If f_1 is a continuation of f_0 along a path a, then f_1 is a continuation of f_0.*

THEOREM 4. *If f_1 and f_2 are continuations of f_0 along a path a, then $f_1=f_2$.*

THEOREM 5. *Let f be an A-family over (X, d), a a path in X, and $a'\colon I\to F$ a lift of a. Then there exists an ε in R_∞^+ such that if β is a path in X with $d(\beta(t), a(t))<\varepsilon$ for all $t\in I$, then there exists $\beta'\colon I\to F$ over β. Moreover, if $a(0)=\beta(0)$, and $a(1)=\beta(1)$, then β' can be chosen such that $a'(0)=\beta'(0)$ and $a'(1)=\beta'(1)$.*

We now obtain a generalization of the classic Poincaré–Volterra theorem.

THEOREM 6. *If F is an A-family over a separable and locally convex metric space X, then for any path component $C(f)$ of F and $x\in X$, the set $p^{-1}(x)\cap C(f)$ is at most countable.*

Proof. Since X is separable, there exists a countable set $A\subset X$ such that $\bar{A}=X$. Let $C(f)$ be any path component in R. Let $y_1\in X$ be a point such that $p^{-1}(y_1)\cap C(f)\neq\emptyset$, $y_0=p(f)$ and $f'\in p^{-1}(y_1)\cap C(f)$. Since $f'\in C(f)$, there exists a path $\tilde{a}$ in F such that $\tilde{a}(0)=f$ and $\tilde{a}(1)=f'$. For $\varepsilon/4$ with the ε of Theorem 5, there exists a $\delta_t\in R_\infty^+$ for each $t\in I$ guaranteed by local convexity such that $\delta_t<\varepsilon/4$ and any two points of $S(a(t), \delta_t)$ can be connected by a unique segment which lies in $S(a(t), \varepsilon/4)$ where $a(t)=p\circ\tilde{a}(t)=c(\tilde{a}(t))$. Since a is continuous, for each $t\in I$ there exists an $\varepsilon_t\in R_\infty^+$ such that $a[S(t, \varepsilon_t)]\subset S(a(t), \delta_t)$. The collection $\{S(t, \varepsilon_t)|\ t\in I\}$ is an open cover of I. Since I is compact, there exists a finite subcover $\{S(t_1, \varepsilon_1)|\ i=1, \ldots, m\}$. Let $B=\{S(t_i, \varepsilon_i)|$ there does not exist an $S(t_j, \varepsilon_j)$ such that $S(t_i, \varepsilon_i)\subset S(t_j, \varepsilon_j)\}\cup\{S(0, \varepsilon_0), S(1, \varepsilon_1)\}$. Renumbering the t_i's, we have $B=\{S(t_i, \varepsilon_i)|\ i=0, \ldots, n\}$ where $t_i<t_{i+1}$, $i=0, \ldots, n-1$, $t_0=0$, and $t_n=1$. It is easy to see that for $S(t_i, \varepsilon_i)$, $S(t_{i+1}, \varepsilon_{i+1})\in B$, $S(t_i, \varepsilon_i)\cap S(t_{i+1}, \varepsilon_{i+1})\neq\emptyset$. Moreover, $a[S(t_i, \varepsilon_i)\cap S(t_{i+1}, \varepsilon_{i+1})]\subset a[S(t_i, \varepsilon_i)]\cap a[S(t_{i+1}, \varepsilon_{i+1})]\subset W_i\cap W_{i+1}$ where $W_i=S(a(t_i), \delta_{t_i})$. Thus, $W_i\cap W_{i+1}\neq\emptyset$ and hence $W_i\cap W_{i+1}$ is an open neighborhood of some element of $X=\bar{A}$. Therefore, $W_i\cap W_{i+1}\cap A\neq\emptyset$. Let $y_0=x_0, x_1, x_2, \ldots, x_{n+1}=y_1$ be such that $x_0\in W_0$, $x_{n+1}\in W_n$, and $x_{i+1}\in W_i\cap W_{i+1}\cap A$ for $i=0, \ldots, n-1$. Note that each consecutive pair x_i and x_{i+1} can be joined by a unique segment in $S(a(t_i), \varepsilon/4)$. Let a_i be the segment joining x_i to x_{i+1} for $i=0, \ldots, n$. We construct a path β in X as follows:

[1] See G. Springer, *Introduction to Riemann surfaces*, Massachusetts 1957.

Define $\beta(t)=a_1((t-t_i)/(t_{i+1}-t_i))$ for $t\in[t_i, t_{i+1}]$. For $t\in[t_i, t_{i+1}]$, we abbreviate $a_i((t-t_i)/(t_{i+1}-t_i))$ by $\beta_i(t)$. Since $\beta_i(t_{i+1})=\beta_{i+1}(t_{i+1})$, $i=0, \ldots, n-1$, β is a map by the glueing lemma. Clearly, $d(\beta(t), a(t))<\varepsilon$ for all $t\in I$.

By Theorem 5, β can be lifted to a path $\tilde{\beta}\colon I\to F$ such that $\tilde{\beta}(0)=f$ and $\tilde{\beta}(1)=f'$.

Thus, to each element f' of $p^{-1}(y_1)\cap C(f)$ we have assigned a finite set $\{x_0, x_1, x_2, \ldots, x_{n+1}\}\subset A\cup\{y_0, y_1\}$ such that the segments joining x_i to x_{i+1}, $i=0, \ldots, n$, form a path which can be lifted to a path $\tilde{\beta}$ in F with $\tilde{\beta}(0)=f$ and $\tilde{\beta}(1)=f'$. If the same finite set $\{x_0, x_1, x_2, \ldots, x_{n+1}\}$ is assigned to an $f''\in p^{-1}(y_1)\cap C(f)$ in the above manner, then $f'=f''$ by Theorem 4. Thus, no two elements of $p^{-1}(y_1)\cap C(f)$ can be assigned by our method the same finite subset of $A\cup\{y_0, y_1\}$. Since $A\cup\{y_0, y_1\}$ is countable and the number of finite subsets of a countable set is countable, $p^{-1}(y_1)\cap C(f)$ is at most countable.

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