Continuation in metric spaces

by

M. Świerczewski, C. Sloyer (Delaware)
and S. Gulden (Lehigh)

The purpose of this paper is to formalize the topological aspects of “Riemann Surfaces” by constructing function elements on metric spaces and showing that the behavior is similar to the analogous surfaces over the complex plane.

We consider a metric space \((X, d)\) and a set \(Y\). For any \(x \in X\) and \(r \in E^*_\mathbb{R}\) (extended positive reals), if \(f: B(x, r) \to Y\) is a function, then \(f\) is called admissible. We call \(x\) the center of \(f\), denoted \(c(f)\), and \(r\) the radius of \(f\), denoted \(r(f)\). If \(F\) is a family of admissible functions from \(X\) to \(Y\), we call \(F\) a family of function elements iff it satisfies the following:

1. If \(f \in F\), then there does not exist a \(g \in F\) such that \(c(g) \in B(c(f), r(f))\), \(r(g) > r(f) - d(c(f), c(g))\), and \(f/\mathcal{S}(c(g), r(g)) = g\).

2. If \(f, g \in F\), \(W = \mathcal{S}(c(f), r(f)) \cap \mathcal{S}(c(g), r(g)) = \emptyset\), and there exists an \(a \in B^*_\mathbb{R}\), \(x \in X\) such that \(B(x, a) \subseteq W\) and \(f/\mathcal{S}(c(a), r(a)) = g/\mathcal{S}(c(x), r(x))\), then \(f/\mathcal{S}(c(f), r(f)) = g/\mathcal{S}(c(g), r(g))\).

A simple application of Zorn’s Lemma yields

**Theorem 1.** If \(F\) is a non-empty family of function elements, then there exists a maximal family containing \(F\).

Hereafter, such a maximal family of function elements will be called an \(A\)-family.

The maximality of an \(A\)-family immediately yields

**Theorem 2.** Let \(F\) be an \(A\)-family, \(f, g \in F\), \(y \in \mathcal{S}(c(f), r(f))\), and \(r' = r(f) - d(c(f), y)\). Then there exists a unique \(g \in F\) such that \(c(g) = y\) and \(g/\mathcal{S}(c(g), r(g)) = f/\mathcal{S}(c(f), r(f))\). Moreover, \(r(g) > r'\).

Suppose now that \(F\) is an \(A\)-family on \((X, d)\). Given an \(f \in F\) and \(a \in B^*_\mathbb{R}\) such that \(a < r(f)\), let \(N_a(f) = \{g \in F\mid d(c(g), c(f)) < a\} \) and \(f/\mathcal{S}(c(g), r(g)) = g/\mathcal{S}(c(x), r(x))\). We now define a topology \(\mathcal{T}\) for \(F\) as follows: \(U \in \mathcal{T}\) iff for each \(f \in U\), there exists an \(N_a(f) \subseteq U\).

It is easy to see that the topology obtained is Hausdorff. Indeed, if we
define $p: F \to X$ by $p(f) = c(f)$, then $p$ is a local homeomorphism so that $(F, p, X)$ becomes a topological sheaf. One can, of course, obtain the same sheaf by constructing the sheaf of germs of $F$-functions on $X$.

One can now define direct continuation, continuation, and continuation along a path in the obvious way (1). Standard methods then yield:

**Theorem 3.** If $f_1$ is a continuation of $f_2$ along a path $\alpha$, then $f_3$ is a continuation of $f_2$.

**Theorem 4.** If $f_1$ and $f_2$ are continuations of $f_3$ along a path $\alpha$, then $f_1 = f_2$.

**Theorem 5.** Let $f$ be an $A$-family over $(X, d)$, a path in $X$, and $a': I \to F$ a lift of $a$. Then there exists an $e$ in $E^*$ such that if $\beta$ is a path in $X$ with $d(\beta(0), a(t)) < e$ for all $t \in I$, then there exists $\beta': I \to F$ over $\beta$. Moreover, if $a(0) = \beta(0)$, and $a(1) = \beta(1)$, then $\beta'$ can be chosen such that $a(0) = \beta'(0)$ and $a(1) = \beta'(1)$.

We now obtain a generalization of the classic Poincaré–Volterra theorem.

**Theorem 6.** If $F$ is an $A$-family over a separable and locally convex metric space $X$, then for any path component $\gamma(f)$ of $F$ and $a \in X$, the set $F^{-1}(a) \cap \gamma(f)$ is at most countable.

Proof. Since $X$ is separable, there exists a countable set $A \subset X$ such that $A = X$. Let $\gamma(f)$ be any path component of $F$. Let $y_0 \in X$ be a point such that $p^{-1}(y_0) \cap \gamma(f) \neq \emptyset$. Let $y_0 = p(f)$ and $f' = p^{-1}(y_0) \cap \gamma(f)$. Since $f' \in \gamma(f)$, there exists a path $\tilde{f}$ in $F$ such that $\tilde{f}(0) = f$ and $\tilde{a}(1) = f'$. For each $t \in I$ with $t \neq 0$, there exists a $\delta_t \in E^*$ such that $f(t) = \gamma(f)$ and any two points of $B(\delta, \delta')$ can be connected by a unique segment which lies in $B(\delta, \delta')$ where $\delta(t) = p = \gamma(t) = \gamma(t)$. Since $\alpha$ is continuous, for each $t \in I$, there exists an $e_t \in E^*$ such that $\alpha(t) \in X$. The collection $\{S(t, e_t) | t \in I\}$ is an open cover of $F$. Since $I$ is compact, there exists a finite subcover $\{S(t_i, e_i) | i = 1, \ldots, m\}$. Let $B = \bigcup_{i=1}^{m} S(t_i, e_i)$, where $t_i \leq t_{i+1}$, $t_0 = 0$, and $t_m = 1$. It is easy to see that $B(t, e_i) \subset S(t, e_i)$ for $i = 0, \ldots, m$. Hence, $B(t, e_i)$ is an open neighborhood of some element of $X$. Therefore, $B(t, e_i) \cap F$ is a neighborhood of some element of $F$. For each $i$, $B_{i+1}(t, e_i) \cap F$.

$i = 0, \ldots, n-1$. Note that each consecutive pair $x_i$ and $x_{i+1}$ can be joined by a unique segment in $S(a(t_i), e_i)$. Let $a_i$ be the segment joining $x_i$ to $x_{i+1}$ for $i = 0, \ldots, n$. We construct a path $\beta$ in $X$ as follows:

Define $\beta(t) = a_i((t-t_i)/(t_{i+1}-t_i))$ for $t \in [t_i, t_{i+1}]$. For $t \in [t_i, t_{i+1}]$, we abbreviate $a_i((t-t_i)/(t_{i+1}-t_i))$ by $\beta_i(t)$. Since $\beta_i(t) = \beta_i((t_{i+1}-t)/t_{i+1}-t_i)$, $i = 0, \ldots, n-1$, $\beta$ is a map by the gluing lemma. Clearly, $d(\beta(t), a_i(t)) < \varepsilon$ for all $t \in I$.

By Theorem 5, $\beta$ can be lifted to a path $\tilde{\beta}: I \to F$ such that $\tilde{\beta}(0) = f$ and $\tilde{\beta}(1) = f'$.

Thus, to each element $f'$ of $p^{-1}(y_0) \cap \gamma(f)$ we have assigned a finite set $\{x_0, x_1, x_2, \ldots, x_{n-1}\} \subset A \cup \{y_0, y_1\}$ such that the segments joining $x_i$ to $x_{i+1}$, $i = 0, \ldots, n-1$, form a path which can be lifted to a path $\tilde{\beta}$ in $F$ with $\tilde{\beta}(0) = f$ and $\tilde{\beta}(1) = f'$. If the same finite set $\{x_0, x_1, x_2, \ldots, x_{n-1}\}$ is assigned to an $f'' \in p^{-1}(y_0) \cap \gamma(f)$ in the above manner, then $f'' = f'$ by Theorem 4. Thus, no two elements of $p^{-1}(y_0) \cap \gamma(f)$ can be assigned by our method the same finite subset of $A \cup \{y_0, y_1\}$. Since $\gamma(f)$ is countable and the number of finite subsets of a countable set is countable, $p^{-1}(y_0) \cap \gamma(f)$ is at most countable.

*Reçu par l’Édition le 6. 6. 1968.*

---