The fixed point theorems of circle and toroid groups on lens spaces

by

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A transformation group of a space has been defined as a pair (G, Y) where G is a topological group, Y is a space and where further to each element $g \in G$ there is given a homeomorphism $g(y) = f(g; y)$ of Y onto itself satisfying

1) $f(g; y) = g(y)$ is simultaneously continuous in G and y;

2) $g_1(g_2(y)) = (g_1 g_2)(y)$.

G effectively acts on Y means that $F(g, Y) = Y$ implies $g = e$, the identity element in G, where $F(g, Y) = \{y \in Y|\ g(y) = y\}$, called a set of fixed points of $g \in G$. $F(G, Y) = \{y \in Y|\ g(y) = y, g \in G\}$ is a set of fixed points under G. $G_y = \{g \in G|\ g(y) = y \in Y\}$ is called an isotropy group at $y \in Y$. If H is a closed subgroup of G, we denote by (H) the set of conjugate subgroups $\{gHg^{-1}|\ g \in G\}$. We call the sets of the forms (H) G-orbit types. The subsets of Y which are unions of all orbits of a fixed type form a partitioning of Y into invariant subsets. We call this partitioning of Y, each subset labeled by the corresponding orbit type, the orbit structure of X.

If Y is a compact Hausdorff space whose cohomology ring is isomorphic to that of lens space and a group G acts effectively on Y, is it true that the fixed point set $F(G, Y)$ is a cohomology lens space if $F(G, Y) \neq \emptyset$? An affirmative answer is given if $G = Z_p$, where Z_p is a cyclic group of odd prime order [4]. That is $F(G, Y)$ is a set of cohomology lens spaces of lower dimensional. In general this type of questions are hard to answer, but putting additional conditions on the space Y and the group G, and the way the group G acts on Y we can get reasonable answer.

Here on, G will be either a circle groups or a toroid group T^n unless otherwise stated explicitly.

Let Y be a (2n+1)-dimensional compact, connected, locally pathwise connected, and semi-locally 1-connected space such that $\Pi_1(Y) = Z_p$, and

$$H^*(Y; Z_p) = H^*(L_{2n+1}(p); Z_p) = \Lambda[a] \otimes Z_p[x]/(x^{n+1}),$$

where $\Lambda[a]$ is an exterior algebra on one generator a of degree 1, $Z_p[x]/(x^{n+1})$ is a polynomial algebra on one generator of degree 2 and truncated in dimension $2n+2$, and $L_{2n+1}(p)$ is $(2n+1)$-dimensional lens space for odd prime p. Here cohomology group will mean Čech cohomology with compact supports.

LEMMA 1. *Let X be a compact Hausdorff space such that $H^*(X; Z_p) = H^*(S^{2n+1}; Z_p)$. If Z_p acts freely on X, then X/Z_p is a cohomology $(2n+1)$-lens space over Z_p.*

See Proposition 2.4 in [4].

LEMMA 2. *Let G act on Y so that $F(G, Y) \neq \emptyset$. Let X be the universal covering space of Y with respect to a base point $b \in F(G, Y)$. Then the action of G can be lifted onto X.*

Proof. Choose a path r from b ending at y. Let $x_0 \in \Pi^{-1}(b)$, where Π denotes the projection represent trivial loops. $g \in G$ acting on Y induces a map on the path space based at b onto itself, that is, $g(r)$ is also a path at b ending at $g(y)$. Take covering paths $\tilde{r}$ and $\widetilde{g}(r)$ over r and $g(r)$ from x_0 and ending at x and x', where $\Pi x = y$ and $\Pi x' = g(y)$. Let $t \in \Pi_1(Y)$ and a loop σ belong to the homotopy class t. Then there are uniquely determined paths $\tilde{\tilde{r}}$ and $\tilde{\tilde{g}}(r)$ at tx_0 covering r and $g(r)$, respectively, such that their end points are tx and tx', where $\Pi tx = y$ and $\Pi tx' = g(y)$.

Define $g(x) = x'$. This is a well-defined map. It follows that $g(t(x)) = t(g(x))$ as soon as we show that loop σ is homotopic to $g(\sigma)$. Let $W(s)$ be a path in G joining e to g^{-1}, where e is the identity element of G and $s \in [0,1]$. We defined a homotopy $g_s = W(s)g$. Then $g_0 = g$ and $g_1 = e$. During the homotopy, b is not moved. Hence the induced homomorphism $g_*: \Pi_1(Y) \to \Pi_1(Y)$ is trivial. That is, σ and $g(\sigma)$ are homotopic. Thus $g(tx) = t(gx)$. Now we would like to show that g is in fact a homoemorphism on X. The action of g is obviously one-to-one and onto. Let us take $g(x) \in U$, where $x \in X$ and U is open in X. Then since Π, the projection map of X onto Y, is an open map, we have that $\Pi(U)$ is open in Y and contains $\Pi(g(x))$. Since g is a homeomorphism on Y, there exists an open set V in Y such that $g(V) \subset \Pi(U)$. Since Π is continuous, there exists an open set W in X such that $\Pi(W) \subset V$. $g(W) \subset U$ since Π and g commute. This shows that g is continuous and the same argument shows that g^{-1} is also continuous on X. This method of proof is somewhat similar to that of [3].

THEOREM 1. *$H^*(F(S, Y)) = H^*(L_{2r+1}(p))$ over Z_p where $-1 \leqslant r \leqslant n$.*

Proof. If $F(S, Y)$ is empty, there is nothing to prove. Assume $F(S, Y) \neq \emptyset$. Now we construct the universal covering space X over Y with respect to b, where $b \in F(S, Y)$. We know that X is a cohomology sphere over Z_p such that $X/Z_p = Y$, where Z_p is the deck transformation group $\Pi_1(Y)$ (see Theorem 2.6 in [4] for this assertion).

By Lemma 2, there is a lifting of the action of S onto X such that it commutes with the deck transformations. Let $y \in F(S, Y)$ and $x \in \Pi^{-1}(y)$, where Π is the projection map from X onto Y. Then if $gx = x'$, $g \in S$, $\Pi(gx) = g(\Pi(x)) = g(y) = y$ since $y \in F(S, Y)$. Thus, $gx = x'$ is obtained from x by a deck transformation for each $g \in S$. Since S is connected and the deck transformations are discrete, $gx = x = x'$. Thus $\Pi^{-1}(F(S, Y)) \subset F(S, X)$. On the other hand, if $gx = x$ in X, then $\Pi gx = \Pi x = g\Pi(x)$. Therefore, $\Pi(F(S, X)) \subset F(S, Y)$. Thus, $F(S, Y) = \Pi(F(S, X))$ and $\Pi^{-1}(F(S, Y)) = F(S, X)$. That is, in order to find $F(S, Y)$, we need only to find $F(S, X)$ and project it down on Y.

Let $x \in \Pi^{-1}(y)$. Then $S \supset G_y \supset G_x$ since if $gx = x$, then $\Pi(gx) = \Pi(x) = y = g(\Pi(x)) = g(y)$. Thus if G_y is a finite group, then G_x is a finite group for each $x \in \Pi^{-1}(y)$. If $G_y = S$, then $y \in F(S, Y)$, and $G_x = S$ for $x \in \Pi^{-1}(y)$. A finite group has a finite number of finite subgroups, we have a finite orbit structure on X for S. By well-known theorem in [1] and [2], we have $F(S, X)$ is a $(2r+1)$-dimensional cohomology sphere over Z_p for some r, where $2r+1 \leqslant 2n+1$. Projecting $F(S, X)$ back on Y, we have the desired result, that is, $\Pi(F(S, X)) = F(S, Y)$, which is a $(2r+1)$-dimensional cohomology lens space over Z_p by Lemma 1.

THEOREM 2. *Let T^n be a toroid group operation on Y such that $F(T^n, Y) \neq \emptyset$. Then*

$$H^*(F(T^n, Y); Z_p) = H^*(L_{2r+1}(p); Z_p), \quad \textit{where} \quad r \leqslant n.$$

Proof. Proof is very similar to that of Theorem 1 and we omit here.

Note. It will be interesting to try to eliminate some undesirable conditions on Y.

References

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