Isomorphism with a $C(Y)$ of the maximal ring of quotients of $C(X)$ *

by

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Using the R. Johnson–Y. Utumi definition of „ring of quotients", Fine, Gillman, and Lambek have studied the maximal ring of quotients of $C(X)$ (the ring of continuous real-valued functions on the completely regular Hausdorff space X) [2]. A principle result of theirs is that this ring, denoted $Q(X)$, is isomorphic to the ring of all continuous real-valued functions on dense open subsets of X, modulo identification of functions which agree on a dense open set. Using this realization, we shall prove: *$Q(X)$ is isomorphic to some $C(Y)$ iff the isolated points of X form a dense subset of X* (provided no measurable cardinals exist nearby). Some related problems will be dealt with, also.

In order to prove this theorem, and its relatives, it seems necessary to know at least a little about how $Q(X)$ can be represented on its space of maximal ideals. A convenient context is provided by the theory of φ-algebras of Henriksen and Johnson [5]. With this backdrop, the present proofs proceed quite naturally.

We shall assume a certain familiarity with $C(X)$ (as in [3]), and therefore with the Stone–Čech compactification βX. A sketch of the background on φ-algebras and on $Q(X)$ has been included.

φ-algebras. We indicate those features of φ-algebras which will be useful.

Let K be a compact space, and let $D(K)$ be the set of continuous functions f on K to $\bar{R}$, the two-point compactification of the reals R, for which $\mathfrak{R}(f) = f^{-1}(R)$ is dense. Let $f, g, h \in D(K)$. By definition, $f = g + h$ if $f(x) = g(x) + h(x)$ for $x \in \mathfrak{R}(g) \cap \mathfrak{R}(h)$. Sums of elements of $D(K)$ need not exist in $D(K)$. Similarly, $f = g \cdot h$ is defined, and similarly, products need not exist. But, with the obvious definitions, $g \vee h$, $g \wedge h$, and rg $(r \in R)$ always exist in $D(K)$.

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A φ-algebra is an archimedean lattice-ordered algebra over R, with an identity which is a weak order unit. A homomorphism of φ-algebras is an algebra homomorphism preserving the lattice operations. The kernels of (φ-algebra) homomorphisms are the absolutely convex ring ideals. The term "φ-subalgebra" of $D(K)$ is meant with respect to the operations in $D(K)$ discussed above.

1.1 ([5], 2.3). *A φ-algebra A is isomorphic to a φ-subalgebra of $D(\mathcal{M}(A))$, where $\mathcal{M}(A)$ is the set of maximal absolutely convex ring ideals of A carrying the Stone topology. Under this isomorphism, the identity of A becomes the constant function* 1; *the copy of A "0-1 separates" disjoint closed subsets of $\mathcal{M}(A)$.*

In the remainder of this section, and throughout most of the sequel, a φ-algebra A will be identified with its copy in $D(\mathcal{M}(A))$.

$\mathcal{R}(A) = \bigcap\{\mathcal{R}(f): f \in A\}$ is called the *real ideal space of A*; it consists of those $M \in \mathcal{M}(A)$ with $A/M = R$. There is a natural homomorphism μ (or μ_A) of A into $C(\mathcal{R}(A))$ defined by $\mu(f) = f|\,\mathcal{R}(A)$.

1.2. *μ is one-to-one iff $\mathcal{R}(A)$ is dense in $\mathcal{M}(A)$.*

This follows readily using the "separation condition" in 1.1.

When $\mathcal{R}(A)$ is dense, A is called a *φ-algebra of real-valued functions* ([5], § 4).

Consider a continuous map τ: $Y \to \mathcal{R}(A)$. Putting $\tau'(f) = f \circ \tau$, for $f \in C(\mathcal{R}(A))$ defines a homomorphism τ': $C(\mathcal{R}(A)) \to C(Y)$, and $\tau' \circ \mu$ is a homomorphism of A into $C(Y)$.

1.3. *Let a: $A \to B$ be a homomorphism of φ-algebras A and B. Then there is continuous τ: $\mathcal{R}(B) \to \mathcal{R}(A)$ for which $\tau' \circ \mu_A = \mu_B \circ a$.*

This is proved easily by mimicking the details in ([3], 10.6), which concerns the case when A and B are C's.

1.4. *If the φ-algebra A is isomorphic to some $C(Y)$, then A is isomorphic to $C(\mathcal{R}(A))$ by μ_A.*

This is noted in ([5], § 5); it is immediate from 1.3.

The phrases "isomorphic to some $C(Y)$" and "isomorphic to $C(\mathcal{R}(A))$" henceforth will be used interchangably.

A φ-algebra A is said to be *uniformly closed* if A is complete in the metric

1.5. $\varrho(f, g) = \sup\{|f(x)-g(x)| \wedge 1|: x \in \mathcal{R}(f) \cap \mathcal{R}(g)\}$.

(Completeness in ϱ is equivalent to an algebraic condition on A ([5], 3.1).)

If A is not uniformly closed, the completion in ϱ need not be a ring ([6], 1.8). This partly accounts for the form of the following:

1.6. *Let A and B be φ-algebras with $\mathcal{M}(A) = \mathcal{M}(B)$. Suppose B is the completion of A in ϱ. Then $\mathcal{R}(A) = \mathcal{R}(B)$.*

Since $A \subset B$, $\mathcal{R}(A) \supset \mathcal{R}(B)$. $\mathcal{R}(A) \subset \mathcal{R}(B)$ follows from the fact that $\varrho(f_n, f) \to 0$ iff f_n converges uniformly to f on $\bigcap \mathcal{R}(f_n)$.

$Q(X)$. We consider, for a moment, arbitrary commutative rings with identity, following ([2], Ch. 1). (The more general situation of modules is discussed nicely in [1].)

Let A be such a ring, and B an overring with the same identity as A. B is said to be a *ring of quotients of A* if for each $b \in B$, no element of B other than 0 annihilates $\{a \in A: ba \in A\}$ by multiplication. With this definition, A has a unique maximal ring of quotients which contains, and often properly, the "classical" ring of quotients, which is obtained, roughly speaking, by formal inversion of all non-zero-divisors of A [9].

Now let X be a completely regular Hausdorff space, and $Q(X)$ the maximal ring of quotients of $C(X)$. Consider the set $\bigcup\{C(V): V$ is dense and open in $X\}$ modulo the equivalence relation mentioned before. If $\hat{f}$ and $\hat{g}$ are the equivalence classes of $f \in C(V)$ and $g \in C(W)$, then $\hat{f}+\hat{g}$ is, by definition, the equivalence class of $h \in C(V \cap W)$ defined by $h(x) = f(x)+g(x)$. The product is defined similarly. It is shown in ([2], 2.6) that the ring so obtained is isomorphic to $Q(X)$. It is clear that $\hat{f} \wedge \hat{g}$ and $\hat{f} \vee \hat{g}$ can be defined in a similar way, and also $r\hat{f}$ for $r \in R$; and all operations are extensions of the corresponding operations in $C(X)$. Thus, $Q(X)$ becomes a φ-algebra, and a φ-algebraic extension of $C(X)$.

(Another extension of $C(X)$ is obtained by considering $\bigcup\{C(V): V$ is a dense cozero-set in $X\}$ and proceeding as above. [A cozero-set in X is a set of the form $\{x \in X: f(x) \neq 0\}$, for some $f \in C(X)$.] This is isomorphic to the "classical" ring of quotients of $C(X)$ ([2], 2.6); direct verification of this is easy. This ring is $Q(X)$ if each open set in X is a cozero-set, e.g., if X is metrizable. But frequently $Q(X)$ differs: let X be the one-point compactification of an uncountable discrete space D; D is the smallest dense open set in X, so that $Q(X) = C(D)$; X has no proper dense cozero-set, so the "classical" ring of quotients is $C(X)$.)

Ultimately, we shall examine homomorphisms of $Q(X)$ to a φ-algebra $C(Y)$, and, therefore, we shall want to know about $\mathcal{M}(Q(X))$. Much information can be obtained from [2]; there is considered the space of maximal *ring* ideals. There is no difference, because each maximal ring ideal is absolutely convex. A mimic of ([3], 5.5) establishes this.

(It is worth noting that each *ring* homomorphism of $Q(X)$ to a $C(Y)$ which carries 1 to 1 is a φ-algebra homomorphism. This is so because non-negative elements (in these rings) are squares, and therefore the order, and the lattice operations, are determined by the multiplication.)

In ([2], 11.15, etc.), it is shown that $\mathcal{M}(Q(X))$ is (homeomorphic to) the Stone representation space of the complete Boolean algebra of regular open subsets of βX (or X). This, and the proof of ([4], 3.2) suffice to conclude that

2.1. *$\mathcal{M}(Q(X))$ is the projective resolution of βX. That is, $\mathcal{M}(Q(X))$ is extremally disconnected, and there is a continuous map π of $\mathcal{M}(Q(X))$ onto βX which maps proper closed subsets of $\mathcal{M}(Q(X))$ onto proper subsets of βX (i.e., π is "irreducible").*

(Actually, the information 2.1 is derived more-or-less directly in ([2], 6.7 and 6.9).)

An extremally disconnected — henceforth, "e.d." — space has, by definition, the property that open sets have open closure; and, in an e.d. space, dense subsets, and open subsets, are C^*-embedded ([3], 1H and 6M). It follows that $D(K)$, for K e.d., is a uniformly closed φ-algebra ([5], 2.2, etc.). The completion, $\bar{Q}(X)$, of $Q(X)$ in the metric 1.5 is a subset of $D(\mathcal{M}(Q(X)))$. In fact,

2.2. $\bar{Q}(X) = D(\mathcal{M}(Q(X)))$.

(This is a disguised version of ([2], 5.5). We leave the translation to the reader.)

Now, with K compact e.d., $\mathcal{M}(D(K)) = K$ (remarked in ([5], 3.9), and so it follows from 1.6 that $\mathcal{R}(Q(X)) = \mathcal{R}(D(\mathcal{M}(Q(X))))$. We are led, therefore, to the following considerations.

The real ideals of $D(K)$. In this section, K will be a compact e.d. space, so that $D(K)$ is a uniformly closed φ-algebra. (M) denotes the assumption that the cardinal of $D(K)$, or equivalently, the cardinal of K, is non-measurable. (See [3], Ch. 12.)

3.1. (M) *$\mathcal{R}(D(K))$ is the set of isolated points of K.*

Proof. Each isolated point lies in $\mathcal{R}(D(K))$ because $\mathcal{R}(f)$'s are dense. For the converse, it suffices to show that (M) if p is not isolated in the e.d. space K, then there is $f \in C(K)$ with $f(p) = 0$ and f positive on a dense subset of K. Then, inverting f produces the desired function in $D(K)$. f is constructed by the (non-trivial) argument in ([3], 12H. 1-4). (The reference is to a proof of Isbell's theorem that (M) an e.d. P-space is discrete.)

3.2. (M) *the map μ: $D(K) \to C(\mathcal{R}(D(K)))$ is onto.*

Proof. Because of 3.1, $\mathcal{R}(D(K))$ is open in K and therefore C^*-embedded. So, if $f \in C(\mathcal{R}(D(K)))$, f extends over the closure of $\mathcal{R}(D(K))$ with values in $\bar{R}$. Assign the value 0 off the closure of $\mathcal{R}(D(K))$. Because this closure is open, the resulting function is continuous; its image under μ is f.

The following applies immediately to $\bar{Q}(X)$.

3.3. *$D(K)$ is a φ-algebra of real-valued functions iff $D(K)$ is isomorphic to some $C(Y)$.*

Proof. This follows from 3.2 and 1.3. But (M) isn't needed: if $\mathcal{R}(D(K))$ is dense, it is C^*-embedded, and this makes μ onto.

3.4. Remarks. (a) 3.1 is false without (M). Let K be the Stone–Čech compactification of a discrete space of measurable cardinal. Then $\mathcal{R}(D(K))$ is not discrete. See ([3], 12H. 7).

(b) Let Y be the one-point compactification of an uncountable discrete space. Y is not e.d., but $D(Y) = C(Y)$ is a φ-algebra. $\mathcal{R}(D(Y)) = Y$, and is not discrete.

(c) I don't know if (M) is needed in 3.2.

(d) Suppose Y is a compact space for which $D(Y)$ is a φ-algebra (i.e. each dense cozero-set is C^*-embedded ([5], 2.2)). It would be interesting to have a condition on Y equivalent to "$D(Y)$ is isomorphic to some $C(Z)$". This does not automatically follow from "$D(Y)$ is a φ-algebra of real-valued functions". The Baire functions on R is an example; see ([5], 5.1 and 3.5).

The main result. From 3.1, and previous results, we see that the condition that $Q(X)$ be isomorphic to a $C(Y)$ is concerned with the condition that the set of the isolated points of $\mathcal{M}(Q(X))$ be dense. The latter is translated into a property of X using the following:

4.1. *Let f be an irreducible closed continuous map of K onto Z (T_1-spaces). Then, the isolated points of K are in one-to-one correspondence with the isolated points of Z by f; and one set is dense iff the other is.*

Proof. If p is isolated in K, then $f(K-\{p\})$ is a proper closed subset of Z. Evidently, $f(K-\{p\})$ excludes only $f(p)$, so $\{f(p)\}$ is open. (This argument is ([8], 11.1).) Next, let x be isolated in Z. Then $f^{-1}(x)$ is open; we show it is a singleton. If not, there are $p, q \in f^{-1}(x)$ with $p \neq q$. Choose open U containing p but not q, and arrange it that $f(U) \subset \{x\}$ (by continuity). But $f(K-U) = Z$, and this contradicts irreducibility.

Finally, if the isolated points of K are dense, then so are the isolated points of Z, by continuity. The converse is immediate because f is closed and irreducible.

4.2. *The isolated points of $\mathcal{M}(Q(X))$ are in one-to-one correspondence with the isolated points of X. One set is dense iff the other is.*

Proof. The isolated points of βX are precisely those of X (denseness of X and ([3], 6.9 (d))). Now apply 4.1.

From 4.2, 3.1, 2.2, 1.6, and 1.3, it follows that each homomorphism of $Q(X)$ into $C(Y)$ is of the form $\tau' \circ \mu$, where μ can be regarded as restriction of the "functions" in the Fine–Gillman–Lambek realization of $Q(X)$ to the set of isolated points of X, and τ is a continuous map of Y into the set of isolated points of X.

4.3. (M) *μ: $Q(X) \to C(\mathcal{R}(Q(X)))$ is onto.*

Proof. It suffices that each function on the isolated points of X be extendible over some dense open subset of X: assign the value 0 off the closure of the set of isolated points.

Remark. The property of 4.3 is not shared by all „dense" sub-φ-algebras of $D(K)$, K e.d. Let K be the Stone–Čech compactification of an uncountable discrete space X. $\mu_{D(K)}$ is an isomorphism onto $C(X)$. Let A be the sub-φ-algebra of functions f with $f(X)$ countable.

4.4. THEOREM. *The following are equivalent (M).*

(1) *$Q(X)$ is a φ-algebra of real-valued functions.*

(2) *$\bar{Q}(X)$ is a φ-algebra of real-valued functions.*

(3) *$Q(X)$ is isomorphic to some $C(Y)$.*

(4) *$\bar{Q}(X)$ is isomorphic to some $C(Y)$.*

(5) *The isolated points of $\mathcal{M}(Q(X))$ are dense in $\mathcal{M}(Q(X))$.*

(6) *The isolated points of X are dense in X.*

Proof. $(1) \Longleftrightarrow (2)$ because $\mathcal{R}(Q(X)) = \mathcal{R}(\bar{Q}(X))$ (1.6, etc.). $(2) \Longleftrightarrow (5)$ by 3.1 (etc.). $(1) \Longleftrightarrow (3)$ by 4.3. $(2) \Longleftrightarrow (4)$ by 3.3 (and 2.2). $(5) \Longleftrightarrow (6)$ by 4.2.

We conclude with a related problem.

In ([2], 4.11) it is shown that the Dedekind completion of $C(X)$ is (isomorphic to) the subring of $\bar{Q}(X)$ of all C-bounded functions, i.e., those $f \in \bar{Q}(X)$ such that for some $g, h \in C(X)$, $g \leqslant f \leqslant h$.

4.5. THEOREM. (M) *$\bar{Q}(X)$ is the Dedekind completion of $C(X)$ iff X is discrete.*

Proof. If X is discrete, then $\bar{Q}(X) = C(X)$, and the result follows.

For the converse, let C' denote the Dedekind completion of $C(X)$. If $\bar{Q}(X) = C'$, then $\mathcal{R}(\bar{Q}(X)) = \mathcal{R}(C')$. Thus, using 3.1, $\mathcal{R}(C')$ is discrete. It is known that $\mathcal{R}(C')$ is the projective resolution of vX ([7], p. 236). (vX is the Hewitt realcompactification of X [3].) By 4.1, vX is discrete, and so X is also (from, say, ([3] 6.9 (d)) and the fact that $vX \subset \beta X$).

(Another proof utilizes specific knowledge of the embedding of $C(X)$ into $D(\mathcal{M}(Q(X)))$, namely: $C(X) \ni f \to f^\beta \circ \pi$, where π is the map of 2.1 and f^β denotes the Stone extension of f over βX into $\bar{R}$ ([3], 6.5). Then, if x is not isolated in X, any $p \in \pi^{-1}(x)$ is not isolated, and $p \notin \mathcal{R}(\bar{Q}(X))$. If $f \in \bar{Q}(X)$ has $f(p) = +\infty$, then f is not C-bounded.)

References

[1] G. D. Findlay and J. Lambek, *A generalized ring of quotients I, II*, Canad. Math. Bull. 1 (1958), pp. 77-85 and 155-167.

[2] N. J. Fine, L. Gillman, and J. Lambek, *Rings of quotients of rings of functions*, Montreal 1966.

[3] L. Gillman and M. Jerison, *Rings of continuous functions*, Princeton 1960.

[4] Andrew M. Gleason, *Projective topological spaces*, Ill. J. Math. 2 (1958), pp. 482-489.

[5] M. Henriksen and D. G. Johnson, *On the structure of a class of archimedean lattice-ordered algebra*, Fund. Math. 50 (1961), pp. 73-94.

[6] J. R. Isbell, *Algebras of uniformly continuous functions*, Ann. of Math. 68 (1958), pp. 96-125.

[7] J. E. Mack and D. G. Johnson, *The Dedekind completion of $C(X)$*, Pac. J. Math. 20 (1967), pp. 231-243.

[8] V. I. Ponomarev, *On spaces co-absolute with metric spaces*, Russian Math. Surveys (London Math. Soc. translations) 21 (1966), pp. 87-114.

[9] O. Zariski and P. Samuel, Commutative algebra, Vol. I, Princeton 1958.

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