A new kind of compactness for topological spaces

by

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1. Introduction. A topological space X is said to be *compact* if it has the property that any open cover has a finite subcover; X is *countably compact* if every sequence of points has a limit point, or equivalently, if every countable open cover has a finite subcover. For many purposes the weaker notion of countable compactness suffices. However, for other purposes this is not the case. The most striking example of this is the Tychonoff theorem — the product of compact spaces is compact yet it is possible to have two countably compact spaces whose product is not countably compact [5]. It is our purpose here to introduce a new notion, D-compactness (see Definition 3.2), which is intermediate between countable compactness and compactness and yet which suffices to yield results which usually rely on the full strength of compactness. In particular we shall show that the product of any number of D-compact spaces is D-compact (hence countably compact). We shall show furthermore that D-compactness does not coincide with any of the familiar types of compactness, i.e. that it is in fact a new kind of compactness for topological spaces.

The framework in which we work is the theory of ultraproducts as originated by Łoś together with A. Robinson's Theory of Non-standard Analysis ([6] and [7]). The principal definitions and results are all stated in standard terms and the reader who is so inclined should be able to recast those proofs relying on Non-standard Analysis into a standard framework. However a certain amount of motivation may be lost in this process. In addition the results presented here have an interest from a purely model-theoretic point of view. Robinson has shown that a topological space is compact if in a moderately strong elementary extension (an *enlargement* in Robinson's terminology) every point is near-standard. We examine here the consequence of assuming that every point in a weaker kind of elementary extension is near-standard, which is in fact precisely the notion of D-compactness introduced here. It is interesting to contrast

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this situation with that of previous applications of Non-standard Analysis. In most cases the non-standard models used have been either arbitrary elementary extensions or else elementary extensions with the stronger property that they contain certain types of elements (e.g. enlargements or saturated models [4]) whereas here we explore the usefulness of considering only models which are relatively weak (namely countably indexed ultrapowers).

2. Ultraproducts and Non-standard Analysis. In this section is presented a brief introduction to ultraproducts and to that part of the theory of Non-standard Analysis needed later on. A full introduction to ultraproducts may be found in [1] or [3] while a more complete account of Non-standard Analysis is contained in [6] or [7]. Actually we need here only the simpler first order theory as discussed in [6] although some of the results used here are found in [7].

Let I denote the set of positive integers $\{1, 2, 3, \ldots\}$ and let D be a non-principal ultrafilter over I. If A is any non-empty set consider the set A^I of sequences $\langle a_i\rangle$ of elements of A. For $\langle a_i\rangle, \langle b_i\rangle \in A^I$ we define an equivalence relation as follows:

$$\langle a_i\rangle \sim \langle b_i\rangle \quad \text{if and only if} \quad \{i|\ a_i = b_i\} \in D .$$

Denoting the equivalence class to which $\langle a_i\rangle$ belongs by $\langle a_i\rangle/D$, the *D-power of A*, A_D is defined by:

$$A_D = \{\langle a_i\rangle/D|\ \langle a_i\rangle \in A^I\} . \tag{2.1}$$

For $a \in A$ we identify the equivalence class of the constant sequence $\{a, a, a, \ldots\}$ with a itself so we obtain $A \subseteq A_D$. Now let R be a relation over A so R is either a subset of A or a subset of the n-fold Cartesian product of A, $n \geqslant 2$. We define a corresponding relation R_D over A_D as follows:

$$(\langle a_i^{(1)}\rangle/D,\ \langle a_i^{(2)}\rangle/D, \ldots, \langle a_i^{(n)}\rangle/D) \in R_D \quad \text{if and only if} \quad \{i|\ (a_i^{(1)}, a_i^{(2)}, \ldots, a_i^{(n)}) \in R\} \in D . \tag{2.2}$$

To put this in the framework of first order model theory let $M = (A, R^{(\alpha)})$, where $R^{(\alpha)}$ is an enumeration of all the relations over A. M is called the *complete model* of A. Then the D-power of M, $M_D = (A_D, R_D^{(\alpha)})$, is an elementary extension of M. That is, if L is a first order language containing an individual constant for each element of A and a predicate symbol for each relation $R^{(\alpha)}$ over A, as well as the usual connectives, individual variables and quantifiers over these variables, then any sentence of L which is true in M is true also in M_D.

Now let (X, σ) be a topological space where X is the set of points and σ is the set of open sets. We adopt the usual convention of referring

to X itself as the topological space when no confusion is likely to arise as to the topology on X. The closure of a set P or sequence $\langle x_i\rangle$ of points of X is denoted by $\mathrm{cl}P$ or $\mathrm{cl}\langle x_i\rangle$. In the framework of model theory as described above we regard (X, σ) as a submodel of the complete model M of X since the open sets in σ are particular relations over X. Now let D be an ultrafilter over I. Letting $\sigma' = \{O_D|\ O \in \sigma\}$ we obtain the non-standard topological space (X_D, σ') as a submodel of the D-power of M, M_D.

Let $x \in X_D$. If $x \in X$ then x is called a *standard* point, otherwise it is called a *non-standard* point. For standard x the *monad* of x, $M(x)$ is defined by

$$M(x) = \bigcap_{x \in O \in \sigma} O_D .$$

(If X is a metric space then the monad of a point x in X is just the set of points of X_D which are infinitely close to x.) For $y \in X_D$, y is called *near-standard* if $y \in M(x)$ for some standard point x.

3. D-compactness. Before introducing the basic definitions of this paper we first state the following theorem of Robinson [7]:

THEOREM 3.1. *Suppose X is compact. Then every point of X_D is near-standard.*

Proof. Suppose there were a point $y \in X_D$ which was not near-standard. Then to every $x \in X$ we could find an open set O which contains x but such that $y \notin O_D$. Consider the collection of all such open sets as x ranges over X. This collection covers X so there must be a finite sub-collection, say $\{O^{(1)}, O^{(2)}, \ldots, O^{(n)}\}$ such that

$$\forall z \in X, \quad z \in O^{(1)} \text{ or } z \in O^{(2)} \text{ or } \ldots \text{ or } z \in O^{(n)} .$$

This statement can be formalized as a statement in L which must then be true in M_D, i.e.,

$$\forall z \in X_D, \quad z \in O_D^{(1)} \text{ or } z \in O_D^{(2)} \text{ or } \ldots \text{ or } z \in O_D^{(n)} .$$

But this contradicts the fact that y is a point in X_D which does not belong to any of the $O_D^{(i)}$, $1 \leqslant i \leqslant n$. Hence every point of X_D must be near-standard.

Now let $\langle x_i\rangle$ be a sequence of points of the topological space (X, σ) and let D be an ultrafilter over the positive integers I.

DEFINITION 3.1. An element x of X is a *D-limit point* of $\langle x_i\rangle$ if and only if given any neighborhood N of x, $\{i|\ x_i \in N\} \in D$.

This definition is of interest only in the case that D is non-principal since if D is principal, say $D = \{S \subseteq I|\ k \in S\}$ for some $k \in I$, then any sequence $\langle x_i\rangle$ has as D-limit point simply x_k. If D is non-principal then any set in D is infinite so any neighborhood about a D-limit point x of $\langle x_i\rangle$ must contain an infinite number of the x_i; thus x is in particular a limit point of $\langle x_i\rangle$.

THEOREM 3.2. *Let X be Hausdorff. If $\langle x_i\rangle$ is a sequence of points of X and D an ultrafilter over I, then $\langle x_i\rangle$ has at most one D-limit point.*

Proof. Suppose y and z are two distinct D-limit points of $\langle x_i\rangle$ and let N_1 and N_2 be disjoint neighborhoods about y and z respectively. Then since $N_1\cap N_2=\varphi$, $\{i|\ x_i\in N_1\}\cap\{i|\ x_i\in N_2\}=\varphi$. But this is impossible since the definition of D-limit point requires these sets both to be in D.

DEFINITION 3.2. X *is D-compact* if and only if every sequence of points of X has a D-limit point.

Clearly if X is D-compact for some non-principal ultrafilter D then X is in particular countably compact. (Note that for principal D, any space is D-compact.)

DEFINITION 3.3. X is *ultracompact* if and only if given any ultrafilter D over I, X is D-compact.

THEOREM 3.3. *X is D-compact if and only if every point of X_D is near-standard.*

Proof. (i) Suppose X is D-compact. Let $y\in X_D$ so $y=\langle x_i\rangle/D$ where $x_i\in X$. Let $x\in X$ be a D-limit point of $\langle x_i\rangle$ so given any open set O containing x, $\{i|\ x_i\in O\}\in D$. But then by the definition of O_D (2.2), $y=\langle x_i\rangle/D\in O_D$. Thus $y\in\bigcap_{x\in O}O_D$, i.e. $y\in M(x)$. Thus y is near-standard.

(ii) Now suppose every point of X_D is near-standard. Let $\langle x_i\rangle$ be a sequence of points of X. Consider the point $\langle x_i\rangle/D$ in X_D. $\langle x_i\rangle/D$ is near-standard so for some x, $\langle x_i\rangle/D\in M(x)$. This says that given any open set O containing x, $\langle x_i\rangle/D\in O_D$, i.e. $\{i|\ x_i\in O\}\in D$. Therefore x is a D-limit point of $\langle x_i\rangle$.

As a direct consequence of the above theorem together with Theorem 3.1 we obtain:

COROLLARY. *If X is compact then X is ultracompact.*

THEOREM 3.4. *Suppose X has the property that the closure of any countable set of points of X is compact. Then X is ultracompact.*

Proof. Suppose not. Let D be an ultrafilter such that X is not D-compact. Thus we have a sequence $\langle x_i\rangle$ of points of X with no D-limit point. Around each $p\in \mathrm{cl}\langle x_i\rangle$ let U_p be a neighborhood such that $\{i|\ x_i\in U_p\}\notin D$. Since D is an ultrafilter, $\{i|\ x_i\notin U_p\}\in D$. But the U_p's cover $\mathrm{cl}\langle x_i\rangle$ and $\mathrm{cl}\langle x_i\rangle$ is compact so

$$\{x_i\}\subseteq U_{p_1}\cup U_{p_2}\cup\ldots\cup U_{p_n}\,.$$

Restated,

$$\{i|\ x_i\notin U_{p_1}\cup U_{p_2}\cup\ldots\cup U_{p_n}\}=\varphi\,.$$

But $\{i|\ x_i\notin U_{p_1}\cup U_{p_2}\cup\ldots\cup U_{p_n}\}=\bigcap_{k=1}^{n}\{i|\ x_i\notin U_{p_k}\}\in D$ since D is closed under finite intersections. But then $\varphi\in D$ which is impossible, proving the theorem. The converse to Theorem 3.4 is true at least under the following circumstances.

THEOREM 3.5. *Let X be a completely regular Hausdorff space which is ultracompact. Then the closure of any sequence of points of X is compact.*

Proof. Embed X in a compact Hausdorff space Y (cf. [2], p. 145). Given a sequence $\langle x_i\rangle$ of points of X we may then consider its closure in Y, $\mathrm{cl}_Y\langle x_i\rangle$ as well as its closure in X, $\mathrm{cl}_X\langle x_i\rangle$. Since Y is compact and Hausdorff, $\mathrm{cl}_Y\langle x_i\rangle$ is compact so if it can be shown that $\mathrm{cl}_X\langle x_i\rangle=\mathrm{cl}_Y\langle x_i\rangle$ the theorem will have been proved.

Let y be a limit point (in Y) of $\langle x_i\rangle$. Let F be the set of all sets of integers of the form

$$\{i|\ x_i\in N\}\,,\qquad N \text{ a neighborhood of } y\,.$$

The intersection of any finite number of elements of F is clearly non-empty so we may obtain an ultrafilter D over I which contains F. Then y is a D-limit point of $\langle x_i\rangle$ since given any neighborhood N about y, $\{i|\ x_i\in N\}\in F\subseteq D$. But since X is ultracompact $\langle x_i\rangle$ already has a D-limit point in X. But then by Theorem 3.2 y must be in X. This shows $\mathrm{cl}_Y\langle x_i\rangle\subseteq X$ so $\mathrm{cl}_X\langle x_i\rangle=\mathrm{cl}_Y\langle x_i\rangle$ and hence $\mathrm{cl}_X\langle x_i\rangle$ is compact as required.

4. The Tychonoff Theorem. In this section the analog of the Tychonoff Theorem is proved for D-compactness, that is, that the product of D-compact spaces is D-compact. The proof itself is essentially the same as that given in [7] for the ordinary Tychonoff Theorem.

Suppose $\{(X^{(j)},\sigma_j)\}$ is a collection of topological spaces where j ranges over some index set J. Let (X,σ) be the product $\bigtimes_{j\in J}(X^{(j)},\sigma_j)$. Thus

$$X=\{f|\ f(j)\in X^{(j)}\} \tag{4.1}$$

and a base for the topology σ on X consists of all sets of the form

$$O=\{f|\ f(j_k)\in O^{(j_k)}\}\,,\qquad k=1,\ldots,m \tag{4.2}$$

where $O^{(j_k)}$ is an open set of $X^{(j_k)}$. Then for each j the continuous projection function $P^{(j)}$ from X onto $X^{(j)}$ is defined by

$$P^{(j)}(f)=f(j)\,.$$

Then we may rephrase (4.2) by saying

$$O=\{p\in X|\ P^{(j_k)}(p)\in O^{(j_k)}\}\qquad k=1,\ldots,m\,.$$

To put this in the framework of first order model theory, we let $A = \bigcup_{j \in J} X^{(j)} \cup X$ and let M be the complete model of A. Then the topological spaces $(X^{(j)}, \sigma_j)$ and the product space (X, σ) may be considered as submodels of M. Also the projection functions $P^{(j)}$ are relations of M. Now let D be an ultrafilter over I and let M_D be the D-power of M.

THEOREM 4.1. *Let $x \in X$, $y \in X_D$. Then*

$$y \in M(x) \text{ if and only if } P_D^{(j)}(y) \in M\big(P^{(j)}(x)\big) \text{ for all } j \in J .$$

Proof. (i) Suppose $y \in M(x)$. Given $j \in J$, let $O^{(j)}$ be an arbitrary open set of $X^{(j)}$ which contains $P^{(j)}(x)$. We need to show $P_D^{(j)}(y) \in O_D^{(j)}$. Since $P^{(j)}$ is continuous there is an open set O of X, $x \in O$, such that

$$\forall z \in X\big(z \in O \to P^{(j)}(z) \in O^{(j)}\big) .$$

This statement can be expressed as a sentence of L which interpreted in M_D says

$$\forall z \in X_D\big(z \in O_D \to P_D^{(j)}(z) \in O_D^{(j)}\big) .$$

Since $y \in M(x)$, in particular $y \in O_D$ so $P_D^{(j)}(y) \in O_D^{(j)}$ as required.

(ii) Suppose $P_D^{(j)}(y) \in M\big(P^{(j)}(x)\big)$ for all $j \in J$.

Let O be a basic open set of X which contains x, so there are open sets $O^{(j_k)} \in \sigma_{j_k}$, $k = 1, 2, \ldots, n$ such that

$$(4.3)\quad \forall z \in X\big(z \in O \leftrightarrow P^{(j_1)}(z) \in O^{(j_1)} \text{ or } P^{(j_2)}(z) \in O^{(j_2)} \text{ or} \ldots \text{or } P^{(j_n)}(z) \in O^{(j_n)}\big) .$$

The corresponding statement true about M_D is

$$(4.4)\quad \forall z \in X_D\big(z \in O_D \leftrightarrow P_D^{(j_1)}(z) \in O_D^{(j_1)} \text{ or } P_D^{(j_2)}(z) \in O_D^{(j_2)} \text{ or} \ldots \text{ or } P_D^{(j_n)}(z) \in O_D^{(j_n)}\big) .$$

Now $x \in O$ so $P^{(j_k)}(x) \in O^{(j_k)}$, $k = 1, 2, \ldots, n$. Thus since $P_D^{(j_k)}(y) \in M\big(P^{(j_k)}(x)\big)$, in particular $P_D^{(j_k)}(y) \in O_D^{(j_k)}$, $k = 1, 2, \ldots, n$. But then by 4.4, $y \in O_D$, showing $y \in M(x)$ as required.

THEOREM 4.2. *Let D be an ultrafilter over I. Suppose for each j that $X^{(j)}$ is D-compact. Then the product space X is D-compact.*

Proof. By Theorem 3.3 we need to show that every point of X_D is near-standard. Let $y \in X_D$. For any $j \in J$, $P_D^{(j)}(y) \in X_D^{(j)}$ and since $X^{(j)}$ is D-compact, $P_D^{(j)}(y)$ must be near-standard by Theorem 3.3. Thus there is a point $x_j \in X^{(j)}$ such that $P_D^{(j)}(y) \in M(x_j)$. Let x be the element of X whose jth coordinate is x_j for each $j \in J$, thus $P^{(j)}(x) = x_j$. Hence $P_D^{(j)}(y) \in M\big(P^{(j)}(x)\big)$ for all $j \in J$. Then by Theorem 4.1 $y \in M(x)$ so y is near-standard which shows X is D-compact.

5. Examples. In this section we show that for a given D, the notion of D-compactness does not coincide with the other usual notions of compactness. To this end we construct spaces (all of them Hausdorff and completely regular) which are

(i) ultracompact but not sequentially compact,

(ii) ultracompact but not compact,

(iii) countably compact but not D-compact for any non-principal ultrafilter D.

Finally in order to show that the characterization given by Theorems 3.4 and 3.5 does not work for D-compactness we construct for any D a separable completely regular Hausdorff space which is D-compact but not compact and hence has a countable set whose closure is not compact. In addition this provides by Theorem 3.5 a space which is D-compact but not ultracompact.

EXAMPLE 1. A space X which is ultracompact but not sequentially compact.

Let X be a space which is compact but not sequentially compact (e.g. the product of $2^{\aleph_0}$ two point discrete spaces). Then X is the required example since it is ultracompact by the corollary to Theorem 3.3.

EXAMPLE 2. A space X which is ultracompact but not compact.

Let X be the space of ordinals less than the first uncountable ordinal under the order topology. X is not compact but the closure of any countable set of points is compact so by Theorem 3.4 X is ultracompact.

The remaining two spaces are constructed as subspaces of the Stone–Čech compactification βI of the integers I (cf. [8]). We regard βI as the space of all ultrafilters D over I topologized by basic open sets of the form

$$\{D \mid S \in D\} , \qquad S \subseteq I .$$

If π is a permutation on I, then for $D \in \beta I$, we define $\pi(D)$ by $S \in \pi(D) \Longleftrightarrow$ for some $T \in D$, $S = \{\pi(i) \mid i \in T\}$. Clearly $\pi(D) \in \beta I$ and we say $\pi(D)$ is a permutation of D. For $i \in I$ we denote by $\hat{i}$ the principal ultrafilter $\{S \mid i \in S\}$. Then βI is a compact Hausdorff space in which $\{\hat{i} \mid i \in I\}$ is dense. If we denote by $\mathfrak{c}$ the cardinality of the continuum then $|\beta I| = 2^{\mathfrak{c}}$. A further fact we shall use about βI is that any infinite closed subset must contain $2^{\mathfrak{c}}$ points [5].

EXAMPLE 3. A space X which is countably compact but not D-compact for any non-principal ultrafilter D.

We shall construct X as a subspace of βI. First we construct a transfinite sequence $\langle X_\eta \rangle_{\eta < \aleph_1}$, of subsets of βI, with $|X_\eta| \leqslant \mathfrak{c}$ for all $\eta < \aleph_1$. The definition is by transfinite induction as follows:

(i) $X_0 = \{\hat{i}|\ i \in I\}$.

(ii) Suppose $\eta < \aleph_1$, and that X_α has been defined for all $\alpha < \eta$, $|X_\alpha| \leqslant \mathfrak{c}$.

We wish to define X_η. Note since $\eta < \aleph_1$ that $|\bigcup_{\alpha<\eta} X_\alpha| \leqslant \aleph_1 \cdot \mathfrak{c} = \mathfrak{c}$. Thus there are at most $\mathfrak{c}$ countable subsets of $\bigcup_{\alpha<\eta} X_\alpha$ and we index them by ordinals $< \mathfrak{c}$ obtaining a transfinite sequence $\langle S_\xi \rangle_{\xi<\mathfrak{c}}$. Now define $\langle x_\xi \rangle_{\xi<\mathfrak{c}}$ by transfinite induction.

Let $\zeta < \mathfrak{c}$ and suppose x_γ has been defined for $\gamma < \zeta$. Consider the set of permutations of the set $Z = (\bigcup_{\alpha<\eta} X_\alpha) \cup \{x_\xi\}_{\xi<\zeta}$.

$|Z| \leqslant \mathfrak{c}$ so the set of permutations of elements of Z, $P(Z)$, has cardinality $\mathfrak{c} \cdot \mathfrak{c} = \mathfrak{c}$. Now S_γ has $2^{\mathfrak{c}}$ limit points in βI so we define x_ζ to be a limit point of S_γ which is not in $P(Z)$. Then we define $X_\eta = \{x_\zeta\}_{\zeta<\mathfrak{c}}$. Finally let $X = \bigcup_{\eta<\aleph_1} X_\eta$ with the topology induced by βI. Note that we have constructed X in such a manner that if D is a non-principal ultrafilter in X, $D \in X - X_0$, then no non-trivial permutation of D is in X.

Let us first verify that X is countably compact. Let S be a countable subset of X. Then for some $\eta < \aleph_1$, $S \subseteq \bigcup_{\alpha<\eta} X_\alpha$. But in our construction of X_η we added a limit point of each S so S has a limit point lying in X_η and hence has a limit point lying in X. Thus X is countably compact. Next let D be a non-principal ultrafilter over I. We wish to show that X is not D-compact. By construction there is a non-trivial permutation π of I such that $\pi(D) \notin X$. We wish to show that the sequence

$$\widehat{\pi(1)}, \widehat{\pi(2)}, \widehat{\pi(3)}, \ldots \tag{5.1}$$

has no D-limit point in X for suppose it did have a D-limit point E in X. Let J be any set in E and let N be the neighborhood $\{F \in X|\ J \in F\}$. Then since E is a D-limit point of (5.1)

$$J' = \{i|\ \widehat{\pi(i)} \in N_J\} \in D\,. \tag{5.2}$$

But $\widehat{\pi(i)} \in N_J \Leftrightarrow J \in \widehat{\pi(i)} \Leftrightarrow \pi(i) \in J$. Thus $J' = \{i|\ \pi(i) \in J\}$. But this says that $\pi(J') = J$ where $J' \in D$ by (5.2). Thus we have obtained an arbitrary set $J \in E$ by permuting a set J' in D by π, so $E = \pi(D)$. But this is impossible since $\pi(D) \notin X$, hence the sequence (5.1) has no D-limit point so X is not D-compact.

Example 4. A separable completely regular Hausdorff space X which is D-compact for a given D but which is not compact.

We shall construct X as a subspace of βI. First we construct by induction a transfinite sequence $\langle X_\eta \rangle_{\eta<\aleph_1}$ of subsets of βI, with $|X_\eta| \leqslant \mathfrak{c}$ for all η as follows:

(i) $X_0 = \{\hat{i}|\ i \in I\}$.

(ii) Suppose $\eta < \aleph_1$ and that X has been defined for all $\alpha < \eta$, $|X_\alpha| \leqslant \mathfrak{c}$. Then since $\eta < \aleph_1$, $|\bigcup_{\alpha<\eta} X_\alpha| \leqslant \aleph_1 \cdot \mathfrak{c} = \mathfrak{c}$ so there are $\mathfrak{c}^{\aleph_0} = \mathfrak{c}$ sequences of elements of $\bigcup_{\alpha<\eta} X_\alpha$.

Let X_η be a set containing one D-limit point (in βI) of each such sequence so $|X_\eta| \leqslant \mathfrak{c}$. Finally let $X = \bigcup_{\eta<\aleph_1} X_\eta$ with the topology inherited from βI. We must establish three facts about X.

(i) X has a countable dense subset, namely X_0 since X_0 is dense in βI.

(ii) X is D-compact since given any sequence $\langle x_i \rangle$ of points of X, $\{x_i\} \subseteq X_\eta$ for some $\eta < \aleph_1$. But then by construction $X_{\eta+1}$ contains a D-limit point of $\langle x_i \rangle$ and since $X_{\eta+1} \subseteq X$ this shows X is D-compact.

(iii) X is not compact because if it were it would be a closed subset of βI and would have cardinality $2^{\mathfrak{c}}$. But $|X| = |\bigcup_{\eta<\aleph_1} X_\eta| \leqslant \aleph_1 \cdot \mathfrak{c} = \mathfrak{c}$. Hence X cannot be compact.

References

[1] H. J. Keisler, *Ultraproducts and elementary classes*, Proceedings of the Royal Academy of Sciences, Amsterdam, Ser. A, 64 (1962), pp. 477–495.

[2] J. L. Kelly, *General Topology*, Van Nostrand, Princeton, 1955.

[3] S. Kochen, *Ultraproducts in the theory of models*, Annals of Math., ser. 2, 79 (1961), pp. 221–261.

[4] W. A. J. Luxemburg, *A new Approach to the Theory of Monads*, Technical Report No. 1, Office of Naval Research, Contract No. 00014–66–C0009.A04, 1967.

[5] J. Novak, *On the cartesian product of two compact spaces*, Fund. Math. 40 (1953), pp. 106–112.

[6] A. Robinson, *Introduction to Model Theory and to the Metamathematics of Algebra*, North-Holland, Amsterdam, 1963.

[7] — *Non-standard Analysis*, North-Holland, Amsterdam, 1966.

[8] W. Rudin, *Homogeneity problems in the theory of Čech compactifications*, Duke Math. J. 23 (1955), pp. 409–420.

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