Decomposable inverse limits with a single bonding map on $[0,1]$ below the identity

by

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1. Introduction. It has been known for some time that the collection of all limits of inverse sequences of mappings from the interval $[0,1]$ onto $[0,1]$ is the collection $C$ of all non-degenerate chainable continua (compact, connected, metric spaces) [4]. The study of the collection $S$ of all limits of inverse sequences with a single bonding map on $[0,1]$ is more recent. Henderson [5] showed that the pseudo-arc is such a limit, while Mahavier [6] showed that not every chainable continuum is.

We let $B$ denote the collection of all limits of inverse sequences with a single bonding map $f$ on $[0,1]$ such that if $0 < x < 1$, $f(x) < x$, and note that Henderson's paper also shows that the pseudo-arc is an element of $B$.

In this paper, we characterize the decomposable elements of $B$ (Theorem 1), and show that $B$ is a proper subcollection of $S$, since the $\sin \frac{1}{x}$ continuum is not an element of $B$ (by Theorem 3), but is the inverse limit with single bonding map $f$, where $f(0) = 0, f(\frac{1}{4}) = 1, f(1) = \frac{1}{2}$, and $f$ is linear on $[0, \frac{1}{4}]$ and on $[\frac{1}{2}, 1]$.

2. Preliminaries and main theorem. For a discussion of inverse limits, the reader is referred to [2], and for chainable continua, to [1]. A $\delta$-regular $\epsilon$-chain is a chain such that each link of it is of diameter less than $\epsilon$, and the distance between any two non-intersecting links of it is greater than $\delta$. A regular chain is a chain which is, for some $\delta > 0$ and some $\epsilon > 0$, a $\delta$-regular $\epsilon$-chain. For more on this, see [3].

If $f$ is a continuous function from $[0,1]$ onto $[0,1]$, then $\lim f$ denotes the limit of the inverse sequence with $f$ as the only bonding map. The distance between two points $(x_1, x_2, \ldots)$ and $(y_1, y_2, \ldots)$ of $\lim f$ is $\sum_{n=1}^{\infty} |x_n - y_n| \cdot 2^{-n}$.

Definition. The continuum $M$ is said to have property $S$ with respect to the points $A$ and $B$ of $M$ if and only if there exists a reversibly continuous transformation $\theta$ from $M$ onto $M$ such that $\theta(A) = A$, $\theta(B) = B$. 
and if $\varepsilon > 0$ and $\delta > 0$ then there exists a positive integer $m$ such that if the distance from $B$ to the point $P$ of $M$ is greater than $\delta$, then the distance from $A$ to $\theta^m(P)$ is less than $\varepsilon$.

**Theorem 1.** If $M$ is a decomposable continuum then in order that there exist a continuous function $f$ from $[0, 1]$ onto $[0, 1]$ such that if $0 < x < 1$ then $f(x) < x$, and such that $M$ is topologically equivalent to $\lim f$, it is both necessary and sufficient that (1) $M$ be chainable, (2) $M$ be irreducible between two of its points, $A$ and $B$, and (3) $M$ have property $S$ with respect to $A$ and $B$.

3. The conditions are necessary. Suppose $f$ is a continuous function from $[0, 1]$ onto $[0, 1]$ such that if $0 < x < 1$, then $f(x) < x$. Then $\lim f$ is chainable and irreducible from the point $A(0, 0, ...)$ to the point $B(1, 1, ...)$. Let $\theta$ denote the reversely continuous transformation from $\lim f$ onto $\lim f$ such that if $P(p_1, p_3, ...) = \lim f$ is a point of $\lim f$, then

$$\theta(P) = (f(p_1), f(p_2), ...) = (f(p_1), p_1, p_3, ...).$$

**Lemma 1.** If $\varepsilon > 0$, and $n$ is a positive integer, there is a positive number $\delta$ such that if $P(p_1, p_3, ...) = \lim f$ is a point of $\lim f$ at a distance from $B$ greater than $\delta$, then $1 - p_k > \delta'$.\]

**Theorem 2.** The continuum $\lim f$ has property $S$ with respect to the points $A(0, 0, ...)$ and $B(1, 1, ...)$.\]

**Proof.** Suppose $\varepsilon > 0$ and $\delta > 0$. There exist (1) a positive integer $n$ such that $(\delta) < \varepsilon$, (2) a number $\delta' > 0$ such that if $P(p_1, p_3, ...) = \lim f$ is a point of $\lim f$ at a distance from $B$ greater than $\delta$, then $1 - p_k > \delta'$ (by Lemma 1), (3) a number $x$ such that $0 < x < 1$ and if $x < x < 1$, then $f(x) > 1 - \delta'$, (4) a number $k$ such that $0 < k < 1$ and if $x < x < x < k$, then $f(x) > k$, and (5) a positive integer $m$ such that $k^n < \varepsilon^2/2$.

Now, if $P(p_1, p_3, ...) = \lim f$ is a point of $\lim f$ at a distance from $B$ greater than $\delta$, then $1 - p_k > \delta'$, and $p_k < \varepsilon$. Thus

$$\frac{x'}{x} > f''(p_1) > f''(p_3) > ... > f''(p_{k-1}).$$

The distance from $A$ to $\theta^m(P) = (f''(p_1), f''(p_3), ...) = \lim f$ is easily shown to be less than $f''(p_{k-1}) + (\delta) < \varepsilon$.

**Theorem 3.** If $\lim f$ is decomposable, and irreducible from the point $P$ to the point $Q$, then $P$ is one of the points $A$ and $B$, and $Q$ is the other.

**Proof.** The continuum $\lim f$ is irreducible either from $A$ to $P$, or from $B$ to $P$. If $\lim f$ is irreducible from $A$ to $P$ then, since $\lim f$ is decomposable, there is an open set $R$ that contains $A$ such that $\lim f$ is not irreducible from $A$ to any point of $R$. If $P$ is distinct from $B$ then by
is linear on \([a_j, a_{j+1}]\), the interval \(f^n([a_j, a_{j+1}])\) is a subset of the interval \([a_1, a_{j+1}]\), and
\[f^n(a_{j+1}) - f^n(a_j) \leq (g^n(a_{j+1} - a_j))\]

**Lemma 4.** If \(i, j, \) and \(n\) are positive integers \((n_0 < i < j)\), and \(o^n(l_i)\) is a subset of \(l_i\), then \(f^n(l_i)\) is a subset of \(l_i\) (denotes the closure of \(l_i\)).

**Definition.** Let \(H\) denote the set \(B \cup B_2 \cup B_3 \cup \ldots\) and \(C\) denote the collection \(C_2 \cup C_3 \cup \ldots\).

**Lemma 5.** If \(e > 0\), \(n > 0\), and \(i > 0\), and \(P\) is a point of \(H\) that lies in \(o^n(l_i)\), then there is a positive integer \(j\) such that if \(e > E_j\), and \(j\) is a positive integer such that \(o^{n+j}(l_i)\) contains \(P\), then \(f^n(l_i)\) is a subset of \(s_i\) and the length of the interval \(f^n(l_i)\) is less than \(e\).

**Proof.** It is easy to show that if \(e > n_0\) and that there is a positive integer \(E\) such that if \(e > E\), then \(f^n(l_i)\) lies in \(l_i\) from which \(o^n(l_i)\) lies in \(l_i\) and by lemma 4, \(f^n(l_i)\) lies in \(s_i\), and \(2\) the number \(e_j\) is greater than each of the numbers \((l_j(a_j - a_{j+1}))\) and \((l_j(a_j - a_{j+1}))\). Hence, with the aid of lemma 3, each of the numbers \(f^n(a_{j+1}) - f^n(a_j)\) and \(f^n(a_{j+1}) - f^n(a_j)\) is less than \(e/\epsilon\). Since \(f^n\) is linear on both \([a_{j+1}, a_j]\) and \([a_j, a_{j+1}]\), the length of the interval \(f^n(l_i)\) is less than \(e\).

**Definitions.** Let \(V\) denote the collection of all sequences \(v = v_1, v_2, \ldots\) such that for each positive integer \(n_0\) \((n_0 \geq 1)\) there is a link \(l_i\) (to be denoted by \(L_0(v)\)) of \(C\) such that \(v_0 = o^n(l_i)\) \((0 < n < 1)\), \(2\) \(v_0\) lies in \(v_0\). If \(v\) is a sequence of \(V\), let \(L_0(v)\) denote the point of \(H\) common to all the elements of \(v\). The sequence \(v\) will be said to determine \(L_0(v)\).

**Theorem 5.** Suppose \(v\) is a sequence in \(V\) and for each \(n\) \(L_0(v)\) denotes \(L_0(v)\). Then if \(v\) is a positive integer, each term of the sequence \(s[L_0(v)], f[s[L_0(v)]], f^2[s[L_0(v)]]\), \(\ldots\) contains the closure of the next, and there is only one common number to all the elements of this sequence.

This theorem follows easily from lemmas 4 and 5.

**Definition.** If \(n\) is a positive integer and \(v\) is a sequence in \(V\) and for each \(i\) \(L_i(v)\) denotes \(L_0(v)\), then let \(x_0(v)\) denote the number common to all the elements of the sequence \(L_0(v), f(L_0(v)], f^2(L_0(v)], \ldots\).

**Theorem 6.** If \(v\) is a sequence in \(V\) and \(v\) is a positive integer, then \(x_0(v) = f^n(x_0(v))\).

**Theorem 7.** If the sequences \(v\) and \(v'\) of \(V\) both determine the point \(P\) of \(H\), then \(x_0(v') = x_0(v')\) for each \(n\).

**Theorems 5, 6, and 7** justify the following:

**Definition.** Let \(T_1\) denote the transformation from \(H\) into \(limf\) such that if \(P\) is a point of \(H\), then \(a(n)\) if \(P = E_1\), \(T_1(P) = (1, 1, \ldots)\).
(b) if \( P \neq B \), then \( T_k(P) = (x_1, x_2, \ldots) \), where for each \( n \), \( x_n = x_0(v) \), for any sequence \( v \) of \( V \) that determines \( P \).

**Lemma 6.** If \( l \) is an element of \( C \) and \( P \) is a point of \( H \) in \( \omega^\omega \), for some positive integer \( n \), and \( T_k(P) = (x_1, x_2, \ldots) \), then \( x_n \) belongs to \( s(l) \).

**Theorem 8.** \( T_k \) is reversibly continuous.

**Proof.** With the aid of lemma 6, it is easy to show that \( T_k \) is reversible. Since \( H \) is compact, we need show only that \( T_k \) is continuous.

Suppose \( P \) is a point of \( H \), \( T_k(P) = (x_1, x_2, \ldots) \), and \( R \) is an open set in \( \text{limf} \) that contains \( T_k(P) \).

Suppose \( P \neq B \), \( v \) is a sequence of \( V \) that determines \( P \), \( L_i \) denotes \( L_i(v) \) for each \( i \), \( \epsilon > 0 \) and \( n \) is a positive integer such that if \( Q \) is a point of \( \text{limf} \) and \( |x_n - x_{n+1}| < \epsilon \), then \( Q \) is in \( R \). By lemma 5, \( Q \) is a point of \( \text{limf} \) and \( |x_q - x_{q+1}| < \epsilon \), then \( Q \) is in \( R \).

By lemma 4, \( Q \) is a point of \( \text{limf} \) and \( |x_q - x_{q+1}| < \epsilon \), then \( Q \) is in \( R \), and \( n \) is a positive integer such that \( 1 - x_q < \epsilon \), \( L_i \) denotes the set \( H \cap (B \cup L_{i+1} \cup L_{i+2} \cup \cdots) \)

which is open with respect to \( H \). If \( Q \) is a point of \( D_i \), then either \( Q = B \) or \( Q \) is in \( L_i \) for some \( i \geq n \), so that by lemma 6, \( Q \) is in \( s(l_i) \). In any case, \( 1 - x_q < \epsilon \), \( Q \) lies in \( R \).

**Definition.** For each \( n \), let \( T_{n+1} \) denote the transformation from \( \omega^\omega \) into \( \text{limf} \) such that if \( P \) belongs to \( \omega^\omega \), then \( T_{n+1}(P) = (f(x_1), f(x_2), \ldots) \).

**Theorem 9.** If the point \( P \) belongs to \( \omega^\omega \), then \( T_n(P) = T_{n+1}(P) \).

**Definition.** Let \( T \) denote the transformation from \( M \) into \( \text{limf} \) such that if \( P \) belongs to \( M \), then (1) if \( P = A \), \( T(P) = (0, 0, \ldots) \), and (2) if \( P \neq A \), then \( T(P) = T_{n+1}(P) \), for every positive integer \( n \) such that \( \omega^\omega \) contains \( P \).

**Lemma 7.** There is a positive integer \( n \) such that if \( P \) is a point of \( M \), then \( T(P) = (x_1, x_2, \ldots) \).

**Theorem 10.** \( T \) is reversibly continuous.

**Proof.** Clearly \( T \) is reversible, and since \( M \) is compact, it suffices to show that \( T \) is continuous. So suppose \( P \) is a point of \( M \) and \( R \) is a region in \( \text{limf} \) that contains \( T(P) \).

If \( P = A \), then with the aid of lemma 7 it is not difficult to show that there exists a positive integer \( n \) such that if \( D \) denotes the open set \( M \), then \( T(D) \) lies in \( R \).

If \( P \neq A \), there is a positive integer \( n \) such that \( P \) lies in an open subset \( Q \) of \( \omega^\omega \). Then \( T(P) = T_{n+1}(P) \), and there is an open subset \( D \) of \( Q \) containing \( P \) such that \( T_{n+1}(D) \) lies in \( R \). But \( T_{n+1}(D) = T(D) \).

**Theorem 11.** \( T(M) = \text{limf} \).

**Proof.** Since \( f(0) = 0 \) and \( f(1) = 1 \), \( \text{limf} \) is irreducible from the point \( (0, 0, \ldots) \) to the point \( (1, 1, \ldots) \). Hence \( T(M) = \text{limf} \).

Theorems 10 and 11 show that \( M \) and \( \text{limf} \) are topologically equivalent.

**References**


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Reçu par la Redaction le 23. 1. 1968