(6.2) *To give an interior characterization of the movability.*

(6.3) *Does there exist a non-movable compactum X such that all its homology groups are isomorphic to the corresponding homology groups of a movable compactum Y?*

(6.4) *Let X and $A \subset X$ be movable compacta. Is the homology sequence of the pair (X, A) necessarily exact?*

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Incompleteness of Lp languages

by

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An **Lp** language is an extended first order language which is "rich enough" for the comparison of the cardinalities of two sets. In § 1 we develop a general theory about **Lp** languages, and show them to be incomplete (the set of valid formulae is not recursively enumerable) and incompact. Also, it will be shown that no obvious modification of the Löwenheim–Skolem–Tarski theorem holds in an **Lp** language.

In § 2, we consider four examples of **Lp** languages. The first one is an ad hoc invention from the definition. The remaining three were considered in the literature. We have repeated definitions of these languages so that this paper will be intelligible by itself. Some results are extended, some problems are solved, and some conjectures are refuted. These are mentioned in § 2.

§ 1. The general theory of Lp languages. We have in mind extensions of first order language made by the introduction of new quantifiers or quantifier variables. Semantical notions such as interpretation and satisfiability include those of possible new variables.

Definition of an **Lp** language. In an **Lp** language, there is a formula $[A, B]$, having no individual variables and no predicate symbols but A and B (these are monadic), and satisfying the following two conditions:

(**Lp** 1) *If $[A, B]$ is true in an interpretation, and if (the interpretation of) A is a subset of B while the complements of A and B are of the same power, then A is of strictly less power than B.*

(**Lp** 2) *For every strictly ascending sequence of cardinalities $\langle \varkappa_0, \varkappa_1, \ldots, \varkappa_\eta \rangle$ there is an interpretation of the variables in $[{}^*{}_1, {}^*{}_2]$ so that $[A, B]$ is true for all subsets A, B of a given domain D if the cardinalities of A, B, D are $\varkappa_\zeta, \varkappa_{\zeta+1}, \varkappa_\eta$, respectively, where $\zeta+1 < \eta$.*

We call this interpretation *right for* the given sequence of cardinalities.

We call a language *incomplete* if the set of valid formulae is not recursively enumerable; *incompact* if there is a set of formulae which is not simaltaneously satisfiable even though each finite subset is.

THEOREM 1. *Each* **Lp** *language is incomplete.*

Proof. We denote by $\mathbf{Od}(<, F)$ the conjunction of the following formulae:

(i) $$(x)\,\neg\, x < x\,,$$

(ii) $$(x)(y)(z)(x < y \mathbin{\&} y < z \supset x < z)\,,$$

(iii) $$(x)(y)(x < y \vee x = y \vee y < x)\,,$$

(iv) $$(x)\big[(y)(\mathbf{E}!z)\big(F(x, y, z) \mathbin{\&} (x < z \vee x = z)\big) \mathbin{\&} (z)\big(x < z \vee x = z \supset (\mathbf{E}!y) F(x, y, z)\big)\big],$$

(v) $$(x)(y)\big(x < y \supset \big[\hat{z}(z < y \mathbin{\&} z \neq x), \hat{z}(z < y)\big]\big)\,,$$

Here $[\hat{z}\varphi(z), \hat{z}\psi(z)]$ is the result of the substitution [1] of $\varphi(z)$ for $A(z)$ and $\psi(z)$ for $B(z)$ in the formula $[A, B]$.

We are to show that $\mathbf{Od}(<, F)$ characterises natural numbers with the less-than relation. That is, $\mathbf{Od}(<, F)$ is satisfiable and in every model the interpretation of $<$ is isomorphic to the less-than relation of natural numbers.

First, take the set of natural numbers as the domain, and let (the interpretation of) $<$ be the less-than relation, and for each number b let $F(b, *_1, *_2)$ be a one-one mapping of the terminal segment of b onto the natural numbers. Certainly (i)–(iv) are all true in this interpretation. Consider the sequence of cardinalities $\langle \varkappa_0, \varkappa_1, \ldots, \varkappa_\omega\rangle$ where $\varkappa_\zeta$ is the cardinality of ζ, and take the interpretation right for this sequence. For each number b, the cardinality of its initial segment is $\varkappa_b$ and the cardinality of this segment 'minus' an element is $\varkappa_b - 1$. Thus (v) is also true.

Conversely, in every model of $\mathbf{Od}(<, F)$, the interpretation of $<$ is a linear ordering and each proper terminal segment is of the same power as the domain. Hence the domain must be an infinite set. Now consider elements a, b such that $a < b$ and let $A = \{m;\ m \neq a, m < b\}$ and $B = \{m;\ m < b\}$. Certainly, A is a subset of B, and the complements of A and B are of the same power. Hence, by (v) and (**Lp** 1), A must be of smaller power than B. But A and B differ by only one element. So B must be a finite set. We have shown that $<$ is a linear ordering of an infinite set such that each of its initial segments is finite. Hence $<$ is isomorphic to the less-than relation on natural numbers.

A language which is capable of characterising the ordering of natural numbers is known to be incomplete [2].

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[1] in the sense of [1].
[2] Refer to, e.g., [4].

THEOREM 2. *Each* **Lp** *language is incompact.*

Proof. We use infinitely many individual constants $a_1, a_2, \ldots$ Consider the following set [**IC**] of formulae: (i)–(iv) in the proof of the previous theorem and

(i, j) $$a_j < a_i \mathbin{\&} [\hat{x}(x < a_j), \hat{x}(x < a_i)]$$
for all $i, j = 1, 2, \ldots$ and $i < j$.

Every finite subset of [**IC**] has a model. Indeed, a model can be the set of all ordinal numbers less than ω_ω, a finite number of designated cardinalities $\varkappa_1, \ldots, \varkappa_n$ in this domain, and an interpretation right for the sequence $\langle \varkappa_1, \ldots, \varkappa_n, \varkappa_{n+1}\rangle$ where $\varkappa_{n+1}$ is ω_ω.

On the other hand, the whole set [**IC**] has no model. For, otherwise, there would be a linear ordering such that the cardinalities of the initial segment determined by $a_1, a_2, \ldots$ would form an infinite descending chain: Contradiction, since the cardinalities are well-ordered.

Remark. The incompactness is an easy consequence of the characterizability of natural numbers with the usual ordering. We presented the above proof, since it is just as easy and does not depend on the other result. Thus a language which can distinguish infinite cardinalities but is "blind to" finite ones, is incompact, even though it may be incapable of characterising natural numbers.

THEOREM 3. *The Löwenheim–Skolem–Tarski theorem does not hold in any* **Lp** *language. More precisely, there is no infinite cardinality $\varkappa$ such that either a formula is satisfiable at* [3] *$\varkappa$ if it is satisfiable at some $\varkappa' \geqslant \varkappa$ or a formula is satisfiable at every $\varkappa' \geqslant \varkappa$ if it is satisfiable at $\varkappa$.*

Proof. We exhibit formulae [**Lt**] and [**Sc**] which are satisfiable exactly at limit and at successor cardinalities, respectively.

[**Lt**] is the conjunction of (i)–(iv) and

$$(x)(\mathbf{E}y)\big(x < y \mathbin{\&} [\hat{z}(z < x), \hat{z}(z < y)]\big)\,.$$

[**Lt**] is certainly satisfiable in a domain of limit cardinality $\varkappa_\lambda$, under the interpretation right for the sequence consisting of all cardinalities $\leqslant \varkappa_\lambda$.

Conversely, a model of [**Lt**] must have a linear ordering in which each initial segment must be included in another of greater cardinality. Thus the cardinality of the domain must be a limit cardinality.

[**Sc**] is the conjunction of (i)–(iv) and

$$(x)\big([\hat{z}(z < x), \hat{z}(z = z)]\big) \mathbin{\&} (\mathbf{E}x)(y)\Big[(u)\Big(u < y \supset (\mathbf{E}!\,z)\big(z < x \mathbin{\&} G(y, u, z)\big)\Big) \mathbin{\&} (z)\Big(z < x \supset (\mathbf{E}!\,u)\big(u < y \mathbin{\&} G(y, u, z)\big)\Big)\Big]\,.$$

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[3] 'Satisfiable at $\varkappa$' is short for 'satisfiable in some domain of cardinality $\varkappa$'.

By virtue of the last formula, each initial segment is of smaller power than the domain and there must be an initial segment which is at least of the same power as every initial segment. Hence a model of [**Sc**] must be of successor cardinality.

2. Examples of Lp languages.

2.1. Consider the language $\mathbf{L}_1$ which has a new binary quantifier $(\mathbf{L}x)[\varphi(x), \psi(x)]$, in addition to the usual features of a first order language with equality. The new quantifier has the following meaning: In every interpretation, $\lambda x \varphi(x)$ is of less power than $\lambda x \psi(x)$. Here $\lambda x \varphi(x)$ is the set of those elements which satisfy $\varphi(*)$ in the interpretation.

Evidently, if $(\mathbf{L}x)[A(x), B(x)]$ is true in an interpretation, then A is of less power than B. Thus ($\mathbf{Lp}$ 1) in § 1 is satisfied without further conditions on sets A, B and their complements. Also, the above interpretation of the new quantifier is right for every ascending sequence of cardinalities. Thus, $(\mathbf{L}x)[A(x), B(x)]$ serves as the formula $[A, B]$ in the definition of an **Lp** language.

2.2. $\mathbf{L}_2$ is the language considered in [2]. The new feature is a binary quantifier $(\mathbf{I}x)[\varphi(x), \psi(x)]$ with the meaning that $\lambda x \varphi(x)$ and $\lambda x \psi(x)$ are of the same power. We take $\neg\, (\mathbf{I}x)[A(x), B(x)]$ as $[A, B]$. If A is a subset of B and $\neg\, (\mathbf{I}x)[A(x), B(x)]$ is true, then A must be of less power than B. Also, the above interpretation is right for every ascending sequence of cardinalities. Hence, $\mathbf{L}_2$ is an **Lp** language.

The compactness (Endlichkeitsatz) and the Löwenheim–Skolem theorem were mentioned as interesting problems in [2]. Both are answered negatively by resultats in § 1.

2.3. $\mathbf{L}_3$ has the quantifier $\mathbf{Q}_{2,2,d_2}$ in [3]. The new quantifier is explained in [3] as follows: "A formula $(\mathbf{Q}_{2,2,d_2} x, y;\, v, w)\varphi(x, y;\, v, w)$ can be read as meaning that for every x there is a v, and for every y there is a w—depending only on y—such that $\varphi(x, y;\, v, w)$ holds; and this formula is equivalent to the second-order formula $(\mathbf{E}g)(\mathbf{E}h)(x)(y)\varphi\big(x, y;\, g(x), h(y)\big)$" (4). The formula

$$(\mathbf{Q}_{2,2,d_2} x, y;\, v, w)\big[\big((x = y) \equiv (v = w)\big) \,\&\, \varphi\big]$$

is equivalent to (5)

$$(\mathbf{E}g)\big[g \text{ is one-one} \,\&\, (x)(y)\varphi\big(x, y;\, g(x), g(y)\big)\big].$$

(4) We changed the notation slightly.

(5) For Ehrenfeucht's proof of this equivalence, refer to [3], p. 182.

For $[A, B]$, take

$$\neg\, (\mathbf{Q}_{2,2,d_2} x, y;\, v, w)\big[\big((x = y) \equiv (v = w)\big) \,\&\, \big(A(v) \equiv B(x)\big) \,\&\, y = y \,\&\, w = w\big].$$

This formula is equivalent to

$$\neg\, (\mathbf{E}g)\big[g \text{ is one-one} \,\&\, (x)\big(A\big(g(x)\big) \equiv B(x)\big)\big].$$

$[A, B]$ clearly satisfies the requirements for an **Lp** language.

$\mathbf{L}_3$ was shown by Ehrenfeucht to be incomplete. Actually, he showed that the notion of being a finite set is expressible in $\mathbf{L}_3$. Thus, he showed essentially the characterisability of the less-than relation on natural numbers and the incompactness (6) of this language. The result concerning the Löwenheim–Skolem–Tarski theorem seems to be a new contribution.

2.4. $\mathbf{L}_4$ has one unary quantifier *variable* ($\mathbf{Q}^*$). The meaning of $\mathbf{Q}$ was given in [4]. A recapitulation runs as follows: in a domain of cardinality $\varkappa$, an interpretation of $\mathbf{Q}$ is a set Q of ordered pairs of cardinalities whose sum is equal to $\varkappa$. $(\mathbf{Q}x)\varphi(x)$ is true in this interpretation if $\langle \mu, \nu \rangle \in Q$ where μ and ν are the cardinalities of the sets $\lambda x \varphi(x)$ and $\lambda x \neg\, \varphi(x)$, respectively.

$\mathbf{L}_4$ is an **Lp** language with $\neg\, \big((\mathbf{Q}x)A(x) \equiv (\mathbf{Q}x)B(x)\big)$ as $[A, B]$. First, we remark that if two sets A, B are of the same power, and if their complements are also of the same power, then $(\mathbf{Q}x)A(x) \equiv (\mathbf{Q}x)B(x)$ is true under every interpretation of $\mathbf{Q}$. So, if this formula is false, or $[A, B]$ is true, and the complements of A and B are of the same power, then A and B must be of different power, for every interpretation of $\mathbf{Q}$. If A is a subset of B in addition, then A must be of less power than B. Thus, the condition ($\mathbf{Lp}$ 1) is satisfied.

Given an ascending sequence of cardinalities $\langle \varkappa_0, \varkappa_1, \ldots, \varkappa_\eta \rangle$, consider the interpretation of $\mathbf{Q}$ consisting of ordered pairs $\langle \mu, \nu \rangle$ such that $\mu = \varkappa_\zeta$ for some *even* $\zeta < \eta$, and $\nu = \varkappa_\eta$. Then the truth values of $(\mathbf{Q}x)A(x)$ and $(\mathbf{Q}x)B(x)$ are different for all sets A and B if they are of power $\varkappa_\zeta$ and $\varkappa_{\zeta+1}$ for some $\zeta+1 < \eta$. So our $[A, B]$ satisfies ($\mathbf{Lp}$ 2) also.

Thomason proposed in [5] an axiomatic system for a language with *denumerably many* quantifier variables, and conjectured that every set of formulae has a model if it is consistent in his axiomatic system (*strong completeness*). Also, he conjectured a modified Löwenheim–Skolem theorem, stating that every set of formulae has a model of cardinality $\beth_\omega$ if it has an infinite model. Since strong completeness implies compactness as well as completeness, our results refute these conjecturs even for a language with only one quantifier variable.

(6) Refer to the Remark after Theorem 2.

Added in proof (July 1968): This paper was presented at the San Francisco meeting of the American Mathematical Society, January 1968.

In the meantime we noted that the following definition is much simpler and better suited for our work.

"In an **Lp** language, there is formula $[A, B]$ — here A and B are monadic predicate symbols — which satisfies the following two conditions:
(**Lp** 1)" — the same as in the text,

"(**Lp** 2) *In every infinite domain, there is an interpretation of the variables in* $[\ ,\ ]$ *so that* $[A, B]$ *is true if* B *is of the smallest cardinality after that of* A."

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