On movable compacta

by

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The purpose of this note is to distinguish among all compacta a class of spaces, called *movable compacta*, such that if a movable compactum X fundamentaly dominates another compactum Y (in the sense defined in [1], p. 233 then Y is also movable. It follows, in particular, that if two compacta X, Y lying in the Hilbert cube Q, are fundamentally equivalent (in the sense of [1], p. 233), that is if they have the same fundamental shape, then either both are movable, or both non-movable. Hence the movability is a topological property depending only on the shape of the compactum (similarly as several other global topological invariants, as the homology groups in the sense of Čech or Vietoris, or the fundamental groups, as defined in [1], p. 251). The class of all movable compacta is rather large; it contains in particular all compact ANR-sets, and also all plane compacta. However, in the 3-dimensional Euclidean space E^3 there exist compacta which are not movable (for instance the solenoids of van Dantzig).

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1. Definition and examples. Let X be a compactum lying in the Hilbert cube Q (that is in the subset of the Hilbert space consisting of all points $(x_1, x_2, \ldots)$ with $0 \leqslant x_n \leqslant 1/n$ for $n = 1, 2, \ldots$). It is said to be *movable*, if for every neighborhood U of X (in Q) there exists a neighborhood $U_0 \subset U$ of X (in Q) such that for every neighborhood W of X (in Q) there is a map

$$\varphi\colon\ U_0 \times \langle 0, 1\rangle \to U$$

satisfying the condition

(1.1) $\quad \varphi(x, 0) = x$ *and* $\varphi(x, 1) \in W$ *for every point* $x \in U_0$.

The following examples illustrate the sense of this notion:

(1.2) *Every compact ANR-set* $X \subset Q$ *is movable.*

In fact, if $X \in \mathrm{ANR}$, there exists a neighborhood G of X (in Q) and a retraction $r\colon\ G \to X$. Consider now a neighborhood U of X. Since $r(x) = x$

for every point $x \in X$, there is a neighborhood $U_0 \subset G$ of X such that $x \in U_0$ implies that the segment $|xr(x)|$ lies in U. Setting

$$\varphi(x, t) = (1-t)x + tr(x) \quad \text{for every } (x, t) \in U_0 \times \langle 0, 1\rangle,$$

we get a homotopy $\varphi\colon U_0 \times \langle 0, 1\rangle \to U$ such that

$$\varphi(x, 0) = x \text{ and } \varphi(x, 1) = r(x) \text{ for every point } x \in U_0.$$

It follows that for every neighborhood W of X the condition (1.1) is satisfied. Hence X is movable.

(1.3) *Every solenoid of van Dantzig is not movable.*

First let us give a purely geometric definition of a solenoid. Let A denote the anchor ring which one obtains in the Euclidean 3-space E^3 by rotating around the axis consisting of points of the form $(\frac{1}{2}, \frac{1}{4}, x_3)$ the circular disk lying in the plane $x_1 = \frac{1}{2}$ and having $(\frac{1}{2}, \frac{1}{3}, \frac{1}{6})$ as its center and $\frac{1}{13}$ as its radius. It is clear that if $(x_1, x_2, x_3) \in A$, then

$$0 < x_1 < 1, \quad 0 < x_2 < \tfrac{1}{2}, \quad 0 < x_3 < \tfrac{1}{3}.$$

Observe that the first Betti group $H_1(A)$ of A is cyclic infinite and there exists in the circle S with center $(\frac{1}{2}, \frac{1}{4}, \frac{1}{6})$ and radius $\frac{1}{12}$, lying in the plane $x_3 = \frac{1}{6}$, a true 1-dimensional cycle $\underline{\gamma}$ being a representative of a generator of the group $H_1(A)$.

Consider now a natural number $k > 1$ and let C_k denote the curve defined as the subset of E^3 consisting of all points of the form

$$\left(\tfrac{1}{2} + (\tfrac{1}{12} + \tfrac{1}{14}\sin u)\cos ku,\ \tfrac{1}{4} + (\tfrac{1}{12} + \tfrac{1}{14}\sin u)\sin ku,\ \tfrac{1}{6} + \tfrac{1}{14}\cos u\right)$$

where $0 \leqslant u \leqslant 2\pi$. One sees easily that there exists a positive number ε_k such that the set A_k consisting of all points $x \in E^3$ with $\varrho(x, C_k) \leqslant \varepsilon_k$ lies in the interior of A and it is homeomorphic to A. Consequently, there exists a homeomorphism

(1.4) $\quad h_k\colon A \to A_k$.

Observe that h_k maps the true 1-dimensional cycle $\underline{\gamma}$ onto a representative of a generator of the group $H_1(A_k)$ and that

(1.5) $\quad h_k(\underline{\gamma}) \sim k \cdot \underline{\gamma}$ in A.

Now let $k_1, k_2, \ldots$ be a sequence of integers $\geqslant 2$ and let $k_0 = 1$. Denote by g_0 the identity map of A onto itself and set

(1.6) $\quad g_m = h_{k_1} h_{k_2} \ldots h_{k_m}$ for every $m = 1, 2, \ldots$

Then g_m is a homeomorphism mapping the anchor ring A onto a set B_m, and since $h_{k_{m+1}}(A)$ lies in the interior $\mathring{A}$ of A, we infer that

$$B_{m+1} = g_{m+1}(A) = g_m[h_{k_{m+1}}(A)] \subset g_m(\mathring{A}) \subset \mathring{B}_m.$$

Let us show that

(1.7) $\quad g_m(\underline{\gamma}) \sim \left(\prod_{\mu=0}^{m} k_\mu\right) \cdot \underline{\gamma}$ in A,

for $m = 0, 1, \ldots$ If $m = 0$, then g_0 is the identity map of A onto itself and $\prod_{\mu=0}^{m} k_\mu = k_0 = 1$. Hence (1.7) is for $m = 0$ obvious. Assume that (1.7) holds for an index m. Then

$$g_{m+1}(\underline{\gamma}) = g_m[h_{k_{m+1}}(\underline{\gamma})] \sim g_m(k_{m+1} \cdot \underline{\gamma}) = k_{m+1} g_m(\underline{\gamma}) \sim \left(\prod_{\mu=0}^{m+1} k_\mu\right) \cdot \underline{\gamma} \text{ in } A,$$

and we infer by induction that (1.7) holds for every $m = 0, 1, \ldots$

It follows by the inclusion $B_{m+1} \subset \mathring{B}_m$ that the set $S(k_1, k_2, \ldots) = B_1 \cap B_2 \cap \ldots$, called the *solenoid corresponding to the sequence* $k_1, k_2, \ldots$, is a not empty continuum.

Suppose, contrary to (1.3), that $X = S(k_1, k_2, \ldots)$ is movable. Consider E^3 as a subset of the Hilbert space E^ω, by identifying every point $(x_1, x_2, x_3) \in E^3$ with the point $(x_1, x_2, x_3, 0, 0, \ldots) \in E^\omega$. It is clear that the anchor ring A lies in the set $Q^3 = E^3 \cap Q$. Setting

$$p(x_1, x_2, x_3, x_4, \ldots) = (x_1, x_2, x_3, 0, \ldots) \quad \text{for every point } (x_1, x_2, x_3, \ldots) \in Q,$$

observe that the set $U = p^{-1}(A)$ is a neighborhood of X in Q. It follows that there exists a neighborhood $U_0 \subset U$ of X such that for every neighborhood W of X there is a homotopy $\varphi\colon U_0 \langle 0, 1\rangle \to U$ satisfying the condition (1.1). Consider an index m such that $B_m \subset U_0$. Setting $W = p^{-1}(B_{m+1})$, we infer that φ is a deformation of the set B_m in the set U into a subset of W. It is clear that setting

$$\psi(x, t) = p[\varphi(x, t)] \quad \text{for every } (x, t) \in B_m \times \langle 0, 1\rangle,$$

we get a homotopy which contracts the set B_m in the set A to a subset of the set $p(W) = B_{m+1}$. Hence the true cycle $g_m(\underline{\gamma})$, being a representative of a generator of the group $H_1(B_m)$ is homologous in A to a true cycle lying in B_{m+1}, consequently to a true cycle of the form $q \cdot g_{m+1}(\underline{\gamma})$, where q is an integer. We infer by (1.7) that

$$k_1 \cdot k_2 \ldots k_m \cdot \underline{\gamma} \sim q \cdot k_1 \ldots k_m \cdot k_{m+1} \cdot \underline{\gamma} \text{ in } A,$$

which is impossible, because $\underline{\gamma}$ is a representative of a generator of the group $H_1(A)$ and $q \cdot k_{m+1} \neq 1$, because $k_{m+1} \geqslant 2$. Thus the supposition that X is a movable compactum leads to a contradiction.

2. Components of movable compacta. Let us prove the following

(2.1) THEOREM. *A compactum $X \subset Q$ is movable if every component of it is movable.*

Proof. Assume that every component of a compactum $X \subset Q$ is movable. Consider a neighborhood U of X. Then for every component X_μ of X there is an open neighborhood $\hat{V}_\mu$ of X_μ such that its boundary is disjoint to X and that for every neighborhood W_μ of X_μ there exists a homotopy $\varphi_{\mu,W_\mu}\colon \hat{V}_\mu \times \langle 0,1\rangle \to U$ such that

$$\varphi_{\mu,W_\mu}(x,0) = x \quad \text{and} \quad \varphi_{\mu,W_\mu}(x,1) \in W_\mu \text{ for every point } x \in \hat{V}_\mu.$$

Since X is compact, there is a finite system of indices $\mu_1, \mu_2, \ldots, \mu_k$ such that $\hat{V} = \hat{V}_{\mu_1} \cup \hat{V}_{\mu_2} \cup \ldots \cup \hat{V}_{\mu_k}$ is a neighborhood of X. Setting

$$\hat{V}_i = \hat{V}_{\mu_i} - \overline{\bigcup_{j<i} \hat{V}_{\mu_j}} \quad \text{for } i = 1, 2, \ldots, k,$$

we get a system of open and disjoint sets $V_1, V_2, \ldots, V_k$ such that the set $V = \bigcup_{i=1}^{k} V_i$ is a neighborhood of the set X.

Now let W be a neighborhood of X. Setting $W_\mu = W$ and

$$\varphi_W(x,t) = \varphi_{\mu_i,W}(x,t) \quad \text{for every } (x,t) \in V_i \text{ and } i = 1, 2, \ldots, k,$$

we get a homotopy $\varphi_W\colon V \times \langle 0,1\rangle \to U$ such that

(2.2) $\varphi_W(x,0) = x$ and $\varphi_W(x,1) \in W$ for every point $x \in V$.

Thus we have shown that for every neighborhood U of X there is a neighborhood V of X such that for every neighborhood W of X there exists a homotopy $\varphi_W\colon V \times \langle 0,1\rangle \to U$ satisfying the condition (2.2). Hence X is movable and this completes the proof of Theorem (2.1).

Remark. Observe that the theorem converse to (2.1) is not true. In fact, keeping the notations of § 1, one shows easily that the set

$$S(k_1, k_2, \ldots) \cup \bigcup_{m=1}^{\infty} (B_m \setminus \mathring{B}_m)$$

is a movable compactum, but its component $S(k_1, k_2, \ldots)$ is not movable.

3. Fundamental domination and the movability. One says ([1], p. 233) that a compactum $X \subset Q$ *fundamentaly dominates* another compactum $Y \subset Q$ (notation: $X \underset{F}{\geqslant} Y$) if there exists a fundamental sequence ([1], p. 225) $\underline{f} = \{f_k, X, Y\}$ and a fundamental sequence $\underline{g} = \{g_k, Y, X\}$ such that the composition $\underline{fg} = \{f_k g_k, Y, Y\}$ is a fundamental sequence homotopic to the fundamental identity sequence $\underline{i}_Y = \{i, Y, Y\}$. Let us prove the following

(3.1) Theorem. *If X, Y are two compacta lying in the Hilbert cube Q and if X is movable, then the relation $X \underset{F}{\geqslant} Y$ implies that Y is also movable.*

Proof. Since $X \underset{F}{\geqslant} Y$, there exist two fundamental sequences

$$\underline{f} = \{f_k, X, Y\} \quad \text{and} \quad \underline{g} = \{g_k, Y, X\} \text{ with } \underline{fg} = \{f_k g_k, Y, Y\} \simeq \underline{i}_Y.$$

Let V be a neighborhood (in Q) of the set Y. The homotopy $\underline{fg} \simeq \underline{i}_Y$ implies that there is a neighborhood $V_1 \subset V$ of Y and an index k_1 such that

(3.2) $f_k g_k / V_1 \simeq i/V_1$ in V for every $k \geqslant k_1$.

Moreover, there exists a neighborhood U of X (in Q) and an index k_2 such that

(3.3) $f_k/U \simeq f_m/U$ in V_1 for every $k, m \geqslant k_2$.

Consider now an arbitrary neighborhood G of Y (in Q). Then there exists a neighborhood W of X and an index k_3 such that

(3.4) $f_k/W \simeq f_m/W$ in G for every $k, m \geqslant k_3$.

Since the compactum X is movable, there is a neighborhood U_0 of X contained in U and such that the inclusion map of U_0 into U is homotopic to a map $a\colon U_0 \to U$ satisfying the condition

(3.5) $a(U_0) \subset W$.

This means that there exists a homotopy

(3.6) $\varphi\colon U_0 \times \langle 0,1\rangle \to U$

such that

(3.7) $\varphi(x,0) = x$ and $\varphi(x,1) = a(x)$ for every point $x \in U_0$.

If we recall that $\underline{g}$ is a fundamental sequence, we infer that there exists a neighborhood $V_0 \subset V_1$ of the set Y (in Q) and an index $k_0 \geqslant k_1, k_2, k_3$ such that

(3.8) $g_{k_0}(V_0) \subset U_0$.

Let us set

(3.9) $\psi(y,t) = f_{k_0}\varphi(g_{k_0}(y), t)$ for every $(y,t) \in V_0 \times \langle 0,1\rangle$.

It follows by (3.8) and (3.6) that $\varphi(g_{k_0}(y), t) \in U$, and by (3.9) and (3.3) that $\psi(y,t) \in V_1 \subset V$ for every $(y,t) \in V_0 \times \langle 0,1\rangle$. Hence

$$\psi\colon V_0 \times \langle 0,1\rangle \to V$$

is a homotopy joining the map $\beta_0\colon V_0 \to V$, given by the formula $\beta_0(y) = \psi(y,0)$ for every point $y \in V_0$, with the map $\beta\colon V_0 \to V$ given by the formula $\beta(y) = \psi(y,1)$ for every point $y \in V_0$.

Now let us observe that $\beta_0(y) = f_{k_0}\varphi(g_{k_0}(y), 0) = f_{k_0} g_{k_0}(y)$ for every point $y \in V_0 \subset V_1$. Since $k_0 \geqslant k_1$, we infer by (3.2) that

(3.10) The map $\beta_0\colon V_0 \to V$ is homotopic to the inclusion map $j\colon V_0 \to V$.

Moreover, (3.8), (3.7), (3.5) and (3.4) imply that

$$\beta(y) = f_{k_0}\varphi(g_{k_0}(y), 1) \in f_{k_0}\varphi(U_0, 1) = f_{k_0} a(U_0) \subset f_{k_0}(W) \subset G \quad \text{for } y \in V_0.$$

Since the homotopy ψ joins in V the maps β_0 and β, we infer by (3.10) that the inclusion map $j: V_0 \to V$ is homotopic to the map $\beta: V_0 \to V$ satisfying the condition $\beta(V_0) \subset G$.

Thus we have shown that for every neighborhood V of the set Y (in Q) there is a neighborhood V_0 of Y such that for every neighborhood G of the set Y (in Q) the inclusion map $j: V_0 \to V$ is homotopic to a map $\beta: V_0 \to V$ satisfying the condition $\beta(V_0) \subset G$. Hence the compactum Y is movable and the proof of Theorem (3.1) is finished.

(3.11) Corollary. *The movability of a compactum $X \subset Q$ depends only on the fundamental shape of X.*

It follows, in particular, that the movability is a topological property. Thus we can omit in the definition of the movability the hypothesis that X is a subset of the Hilbert cube and say that an arbitrary compactum is movable if it is homeomorphic to a compactum $X \subset Q$ movable in the previous sense.

Let us recall, that the fundamental absolute neighborhood retracts (called also FANR-sets) may be defined ([2], p. 67) as fundamental retracts of ANR-sets lying in the Hilbert cube Q. That is, for every set $Y \in$ FANR lying in Q there exists in Q and ANR-set $X \supset Y$ and a fundamental retraction $\underline{r}: X \to Y$, that is a fundamental sequence $\underline{r} = \{r_k, X, Y\}$ satisfying the condition $r_k(y) = y$ for every point $y \in Y$ and for $k = 1, 2, \ldots$ If we denote, for every $k = 1, 2, \ldots$, by g_k the identity map $i: Q \to Q$, then we see at once that the fundamental sequence $\underline{g} = \{g_k, Y, X\}$ satisfies the condition $\underline{r}\underline{g} \simeq \underline{i}_Y$. Consequently $X \underset{F}{\geqslant} Y$ and we obtain, by Theorem (3.1) the following

(3.12) Corollary. *Every FANR-set is movable.*

(3.13) Corollary. *Every factor of a movable compactum is movable.*

(3.14) Corollary. *The solenoids of van Dantzig are not FANR-sets.*

4. Cartesian product of movable compacta. Let us prove the following

(4.1) Theorem. *The Cartesian product of a finite or countable number of compacta X_i is movable if and only if all compacta X_i are movable.*

Proof. Let us represent the Hilbert cube Q as the Cartesian product $Q_1 \times Q_2 \times \ldots$, where Q_i is homeomorphic to Q for $i = 1, 2, \ldots$ Then every point $x \in Q$ may be written in the form $[x_1, x_2, \ldots]$, where $x_i \in Q_i$ for $i = 1, 2, \ldots$

Now consider a compactum X being the Cartesian product of the movable compacta $X_1, X_2, \ldots$ Since the movability is a topological property, we may assume that $X_i \subset Q_i$ for $i = 1, 2, \ldots$ Hence $X = X_1 \times X_2 \times \ldots \subset Q$.

Let U be a neighborhood of X in Q. Then for every $i = 1, 2, \ldots$ there exists a neighborhood U_i of X_i in Q_i and an index n_0 such that

$$U_1 \times U_2 \times \ldots \subset U, \qquad U_i = Q_i \quad \text{for every } i > n_0 .$$

Since X_i is movable, there exists a neighborhood $V_i \subset U_i$ of the set X_i in Q_i such that for every neighborhood W_i of X_i (in Q_i) there is a homotopy $\varphi_i: V_i \times \langle 0, 1\rangle \to U_i$ such that

$$\varphi_i(x_i, 0) = x_i \text{ and } \varphi_i(x_i, 1) \in W_i \text{ for every point } x_i \in V_i .$$

Let us observe that we can assume that $V_i = Q_i$ for $i > n_0$. In fact, it suffices to select a point $a_i \in X_i$ and to set for every $i > n_0$:

$$\varphi_i(x_i, t) = t \cdot a_i + (1-t) \cdot x_i \quad \text{for } (x_i, t) \in Q_i \times \langle 0, 1\rangle ,$$

in order to get a required homotopy φ_i.

Then the set $V = V_1 \times V_2 \times \ldots$ is a neighborhood of X in the space Q. Consider now a neighborhood W of the set X in Q. We can select a neighborhood W_i of X_i (in Q_i) so that $W_1 \times W_2 \times \ldots \subset W$ and that there is an index $n_1 \geqslant n_0$ such that $W_i = Q_i$ for every $i > n_1$. Setting

$$\varphi(x, t) = [\varphi_1(x_1, t), \varphi_2(x_2, t), \ldots] \quad \text{for every point } x = [x_1, x_2, \ldots] \in V$$

and for $0 \leqslant t \leqslant 1$, we get a homotopy

$$\varphi: V \times \langle 0, 1\rangle \to U$$

such that $\varphi(x, 0) = x$ and $\varphi(x, 1) \in W$ for every point $x \in V$.

Thus we have shown that X is movable. It remains to recall Corollary (3.12) in order to finish the proof.

5. Weakly contractible compacta. A compactum $X \subset Q$ is said to be *weakly contractible to a compactum* $Y \subset X$, if for every neighborhood U of X and for every neighborhood V of Y there is a homotopy $\varphi: X \times \langle 0, 1\rangle \to U$ such that $\varphi(x, 0) = x$ and $\varphi(x, 1) \in V$ for every point $x \in X$.

Let us prove two following lemmas:

(5.1) Lemma. *If a compactum $X \subset Q$ is weakly contractible to a compactum $Y \subset X$, then for every neighborhood U of X and for every neighborhood W of Y there is a neighborhood U_0 of X and a homotopy $\varphi: U_0 \times \langle 0, 1\rangle \to U$ such that $\varphi(x, 0) = x$ and $\varphi(x, 1) \in W$ for every point $x \in U_0$.*

Proof. It is clear that one can replace in the proof of Lemma (5.1) the given neighborhood U of X by any neighborhood of X contained in U, and also to replace the given neighborhood W of Y by any neighborhood of Y contained in W. Thus we may assume that U is an ANR-set and W is open.

Since X is weakly contractible to Y, there exists a homotopy

$$\hat{\varphi}: X\times\langle 0,1\rangle\to U$$

such that

$$\hat{\varphi}(x,0)=x \quad \text{and} \quad \hat{\varphi}(x,1)\in W \quad \text{for every point } x\in X.$$

The set U being an ANR-set and the identity map $i\colon U\to U$ being an extension of the restriction $\hat{\varphi}/X\times(0)$, we infer by the homotopy extension theorem (see, for instance [3], p. 94) that there is a homotopy $\psi\colon U\times\langle 0,1\rangle\to U$ being an extension of the homotopy $\hat{\varphi}$. Since the values of $\varphi/[X\times(1)]$ belong to the open set W, we infer that there exists a neighborhood $U_0\subset U$ of X (in Q) such that the restriction $\varphi=\psi|U_0\times\langle 0,1\rangle$ is a homotopy satisfying the required conditions.

(5.2) Lemma. *If $X\supset Y\supset Z$, where X is a compactum weakly contractible to Y and Y is a compactum weakly contractible to the compactum Z, then X is weakly contractible to Z.*

Proof. Let U be a neighborhood of X. Then U is also a neighborhood of Y and we infer by Lemma (5.1) that for every neighborhood W of Z there is a neighborhood V_0 of Y and a homotopy $\psi\colon V_0\times\langle 0,1\rangle\to U$ such that

$$\psi(x,0)=x \text{ and } \psi(x,1)\in W \text{ for every point } x\in V_0.$$

Moreover, there is a homotopy $\varphi\colon X\times\langle 0,1\rangle\to U$ such that $\varphi(x,0)=x$ and $\varphi(x,1)\in V_0$ for every point $x\in X$. Setting

$$\chi(x,t)=\varphi(x,2t) \quad \text{for every point } x\in X \text{ and for } 0\leqslant t\leqslant \tfrac{1}{2},$$

$$\chi(x,t)=\psi[\varphi(x,1),2t-1] \quad \text{for every point } x\in X \text{ and for } \tfrac{1}{2}\leqslant t\leqslant 1,$$

we get a homotopy $\chi\colon X\times\langle 0,1\rangle\to U$ satisfying the condition $\chi(x,0)=x$ and $\chi(x,1)\in W$ for every point $x\in X$. Thus X is weakly contractible to Z.

(5.3) Theorem. *If $X=\bigcap_{n=1}^{\infty}X_n$, where X_n are movable compacta and X_n is weakly contractible to X_{n+1} for every $n=1,2,\ldots$, then X is movable.*

Proof. Let U be a neighborhood of X. Then there is an index n such that U is a neighborhood of X_n. Since X_n is movable, there exists a neighborhood U_0 of X_n such that for every neighborhood V of X_n there is a homotopy

$$\varphi_V\colon U_0\times\langle 0,1\rangle\to U$$

such that

$$\varphi_V(x,0)=x \text{ and } \varphi_V(x,1)\in V \text{ for every point } x\in U_0.$$

Consider now a neighborhood W of X. Then there is an index $m\geqslant n$ such that W is a neighborhood of X_m. It follows by Lemma (5.2) that X_n is weakly contractible to X_m, and we infer by Lemma (5.1) that one can select the neighborhood V of the set X_n so, that there is a homotopy

$$\hat{\varphi}\colon V\times\langle 0,1\rangle\to U$$

such that

$$\hat{\varphi}(x,0)=x \text{ and } \hat{\varphi}(x,1)\in W \text{ for every point } x\in V.$$

Now let us set:

$$\varphi(x,t)=\varphi_V(x,2t) \quad \text{for every point } x\in U_0 \text{ and for } 0\leqslant t\leqslant\tfrac{1}{2},$$

$$\varphi(x,t)=\hat{\varphi}[\varphi_V(x,1),2t-1] \quad \text{for every point } x\in U_0 \text{ and for } \tfrac{1}{2}\leqslant t\leqslant 1.$$

One sees easily that $\varphi\colon U_0\times\langle 0,1\rangle\to U$ is a homotopy such that $\varphi(x,0)=x$ and $\varphi(x,1)=\hat{\varphi}[\varphi_V(x,1),1]\in W$ for every point $x\in U_n$. Hence X is movable.

The following example illustrates the last theorem:

(5.4) Example. Let $a=(0,\frac{1}{2},0,\ldots)$, $b_0=(0,0,\ldots)$, $b_n=(1/n,0,0,\ldots)$ for $n=1,2,\ldots$ and let X denote the union of the segments $|b_0b_1|$ and $|ab_n|$ with $n=0,1,\ldots$ Setting $X_n=X\cup|ab_0b_n|$, where $|ab_0b_n|$ denotes the triangle with vertices a,b_0 and b_n, one sees easily that $X_n\in\mathrm{ANR}$ and X_n is contractible (hence also weakly contractible) to X_{n+1} for every $n=1,2,\ldots$ Moreover $X=\bigcap_{n=1}^{\infty}X_n$. It follows by Example (1.3) and by Theorem (5.3) that X is movable.

It is known ([1], p. 235) that every plane continuum decomposing the plane E^3 into $\aleph_0$ regions fundamentally dominates every plane continuum. Thus, combining Example (5.4) with Theorem (3.1) and Theorem (2.1), we get the following

(5.5) Corollary. *Every plane compactum is movable.*

6. Problems. Let us formulate some open problems concerning movable compacta;

(6.1) *Is it true that every movable component of a compactum X is necessarily a fundamental retract of X?*

Let us remark, that a not movable component of a compactum X can be not a fundamental retract of X. For instance, if we keep the notations used in Sec. 1 and if we set

$$X=S(k_1,k_2,\ldots)\cup\bigcup_{k=1}^{\infty}(B_k-\dot{B}_k),$$

then $S(k_1,k_2,\ldots)$ is a component of X which is not a fundamental retract of X.

(6.2) *To give an interior characterization of the movability.*

(6.3) *Does there exist a non-movable compactum X such that all its homology groups are isomorphic to the corresponding homology groups of a movable compactum Y?*

(6.4) *Let X and $A \subset X$ be movable compacta. Is the homology sequence of the pair (X, A) necessarily exact?*

References

[1] K. Borsuk, *Concerning homotopy properties of compacta*, Fund. Math. 62 (1968), pp. 223-254.

[2] — *Fundamental retracts and extensions of fundamental sequences*, ibid. 64 (1968), pp. 55-85.

[3] — *Theory of Retracts*, Warszawa 1967.

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Incompleteness of Lp languages

by

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An **Lp** language is an extended first order language which is "rich enough" for the comparison of the cardinalities of two sets. In § 1 we develop a general theory about **Lp** languages, and show them to be incomplete (the set of valid formulae is not recursively enumerable) and incompact. Also, it will be shown that no obvious modification of the Löwenheim–Skolem–Tarski theorem holds in an **Lp** language.

In § 2, we consider four examples of **Lp** languages. The first one is an ad hoc invention from the definition. The remaining three were considered in the literature. We have repeated definitions of these languages so that this paper will be intelligible by itself. Some results are extended, some problems are solved, and some conjectures are refuted. These are mentioned in § 2.

§ 1. The general theory of Lp languages. We have in mind extensions of first order language made by the introduction of new quantifiers or quantifier variables. Semantical notions such as interpretation and satisfiability include those of possible new variables.

DEFINITION OF AN **Lp** LANGUAGE. In an **Lp** language, there is a formula $[A, B]$, having no individual variables and no predicate symbols but A and B (these are monadic), and satisfying the following two conditions:

(**Lp** 1) *If $[A, B]$ is true in an interpretation, and if (the interpretation of) A is a subset of B while the complements of A and B are of the same power, then A is of strictly less power than B.*

(**Lp** 2) *For every strictly ascending sequence of cardinalities $\langle \varkappa_0, \varkappa_1, \ldots, \varkappa_\eta \rangle$ there is an interpretation of the variables in $[{*}_1, {*}_2]$ so that $[A, B]$ is true for all subsets A, B of a given domain D if the cardinalities of A, B, D are $\varkappa_\zeta, \varkappa_{\zeta+1}, \varkappa_\eta$, respectively, where $\zeta+1 < \eta$.*

We call this interpretation *right for* the given sequence of cardinalities.

We call a language *incomplete* if the set of valid formulae is not recursively enumerable; *incompact* if there is a set of formulae which is not simaltaneously satisfiable even though each finite subset is.