On the differentiation of integrals in euclidean spaces

by

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1. Introduction. Let f be a continuous real-valued function of a real variable. Neugebauer [4] has shown that the two upper Dini derivates are equal, except, perhaps, on a set of the first category. Furthermore, according to a theorem of Young [7], an upper derivate on one side is at least as great as the lower derivate on the opposite side except for a set which is a most denumerable. Certain related results involving, for example, symmetric derivates are also valid [1].

The purpose of this article is to extend the theorems of Neugebauer and Young (and related theorems) in a natural way to derivates of integrals of summable functions in higher dimensional spaces. These analogues are established in section 3, below. Then, in section 4, we consider certain possible extensions of these results to other settings considered by Busemann and Feller [2], de Possel [5] and others.

2. Preliminaries. In this section we develop the definitions, notations and concepts which we use in section 3 below.

Let $\mathcal{A}$ be a family of bounded open sets in E_N, euclidean N-dimensional space, such that if $I \in \mathcal{A}$, then any open set homothetic to I is also in $\mathcal{A}$. Thus, the family $\mathcal{A}$ is closed under translations and (non-negative) dilations, but not necessarily under rotations. We write $I \sim J$ if J and I are homothetic. To each $I \in \mathcal{A}$, we associate a point x in the boundary of I such that if $I \sim J$, then the points associated with I and J are in corresponding positions. A sequence $\{I_n\}$ of sets in $\mathcal{A}$ is said to converge to a point x provided, for each n, x is the point associated with I_n and $\delta(I_n) \to 0$, where $\delta(I_n)$ denotes the diameter of I_n. Now let f be summable on compact subsets of euclidean N-space, μ denote Lebesgue N-dimensional measure and σ be defined by $\sigma(E) = \int_E f\,d\mu$ where E is any bounded measurable set. We shall write

$$\overline{D}\sigma(x) = \sup \limsup_{I_n \to x} \frac{\sigma(I_n)}{\mu(I_n)}$$

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and

$$\underline{D}\sigma(x) = \inf \liminf_{I_n \to x} \frac{\sigma(I_n)}{\mu(I_n)},$$

the supremum and infinum being taken over all sequences of sets $\{I_n\}$ in $\mathcal{A}$ converging to x. We shall call these two numbers the upper and lower derivates of σ at x. The family $\mathcal{A}$ is called a *differentiation basis*.

EXAMPLE. Let $\mathcal{A}$ denote the family of open intervals on the line. For each such interval, let the associated point be the left-hand end point. Then $I_n \to x$ means that for each n, I_n is an open interval having x as a left-hand end point, and $\delta(I_n) \to 0$. The upper and lower derivates of σ are then just the upper right and lower right Dini derivates of the function F given by $F(x) = \int_a^x f(t)\,dt$. Similar remarks would apply if we associated with each open interval its right-hand end point. The theorems of Neugebauer and Young have obvious interpretation in this setting.

To extend the theorems of Neugebauer and Young to higher dimensions, we consider a family $\mathcal{A}$ as decribed above. With each $I \in \mathcal{A}$ we associate two different points x and y and we establish a theorem relating the upper derivates obtained relative to the two notions of convergence obtained (Analogue of Neugebeuer's Theorem) and a theorem relating the upper derivate relative to one notion of convergence with the lower derivate relative to the other notion of convergence (analogue of Young's Theorem). To distinguish the two types of convergence, we shall use symbols such as $I_n \underset{1}{\to} x$ and $I_n \underset{2}{\to} y$. Furthermore, we shall use the notation $x \underset{1}{\in} I$ (or $y \underset{2}{\in} I$) to mean that x (or y resp.) is the point associated in the first sense (second sense resp.) with I. Finally, we shall use such symbols as $\overline{D}_1\sigma$ to indicate our differentiation is with respect to the first notion of convergence.

One more comment is in order. It was shown by de Possel that a necessary and sufficient condition that the Fundamental Theorem of Calculus holds for every bounded summable f (i.e., that $\mathcal{A}$ differentiates the integral of f back to f a.e.) is that the Weak Vitali Theorem holds for $\mathcal{A}$ (with the associated notion of convergence). Unless we explicitly state to the contrary, we shall assume that $\mathcal{A}$ is such a basis that the Weak Vitali Theorem holds with respect to each of the two notions of convergence. This theorem can be stated as follows:

Let $\mathcal{A}$ be a differentiation basis with $\to$ its notion of convergence. Let A be an arbitrary set and let a be an arbitrary number satisfying $0 < a < 1$. If $\mathcal{A}^*$ is a Vitali Cover for A, that is, a subset of $\mathcal{A}$ such that for every $x \in A$, there exists a sequence of sets in $\mathcal{A}^*$ which converges to x, then there exists a sequence $I_1, I_2, \ldots$ of sets in $\mathcal{A}^*$ such that (i) $\mu^*A = \mu^*(A \cap \bigcup I_n)$ and (ii) $\sum \mu I_n < \frac{1}{a}\mu^* A$. (Here μ^* denotes Lebesgue outer measure.)

3. Analogues to the Theorems of Neugebauer and Young. We turn now to our two main results. Throughout this section, the definitions and notations of section 2 apply.

THEOREM 1. *Let $\mathcal{A}$ be a differentiation basis consisting of bounded open sets in E_N, closed under homothetic transformation, and possessing the weak Vitali property with respect to the convergences $\underset{1}{\to}$ and $\underset{2}{\to}$. Let f be summable on compact sets and define σ by $\sigma(E) = \int_E f\,d\mu$. If $\mathcal{A}$ differentiates σ to f a.e. (with respect to both notions of convergence), then $\overline{D}_1\sigma = \overline{D}_2\sigma$ and $\underline{D}_1\sigma = \underline{D}_2\sigma$ except for a set of the first category.*

Proof. We assume first that f is non-negative; the case that f is non-positive is analogous and the general case then follows by decomposing the integral into its positive and negative parts. Let, then, $E = \{x\colon \overline{D}_1\sigma(x) < \overline{D}_2\sigma(x)\}$. We show that E is of the first category.

For each pair of rational numbers $r < s$ and each positive integer n, define a set E_{rsn} by

$$E_{rsn} = \{x\colon\ \sigma(I) \leqslant r\mu(I) \text{ whenever } x \underset{1}{\in} I \in \mathcal{A} \text{ and } \delta(I) < 1/n\} \cap \\ \cap \{x\colon \overline{D}_2\sigma(x) > s\}.$$

It is easy to verify that $E \subset \bigcup_{r,s,n} E_{rsn}$, so it suffices to show that E_{rsn} is nowhere dense for each triple (r, s, n). Suppose, then, that for some triple (r, s, n), the set E_{rsn} is dense in some non-empty open set G. Then there exists $J \in \mathcal{A}$, $J \subset G$ such that $\delta(J) < 1/n$ and $\sigma(J) > s\mu(J)$. Thus E_{rsn} is dense in J. Let $\mathcal{A}^*$ consist of those $I \in \mathcal{A}$ such that $I \subset J$ and $\sigma(I) \leqslant r\mu(I)$. We note first that if $x \in E_{rsn} \cap J$, $I \in \mathcal{A}$ and $x \underset{1}{\in} I \subset J$, then $I \in \mathcal{A}^*$. Thus, $\mathcal{A}^*$ forms a Vitali Cover for J. To see this, let $x \in J$. Let S be a closed sphere centered at x and contained in J. If $I \subset S$, and $x \underset{1}{\in} I$, then $\sigma(I) \leqslant r\mu(I)$. For, if instead $\sigma(I) > r\mu(I)$ and if I' is a translate of I of sufficiently small magnitude, then $\sigma(I') > r\mu(I')$. In particular, since E_{rsn} is dense in J, I' may be so chosen that $y \underset{1}{\in} I'$ for some $y \in E_{rsn} \cap J$ and $I' \subset J$. But this is impossible since such an I' must be in $\mathcal{A}^*$. Thus, the family $\mathcal{A}^*$ is a Vitali Cover of J. Since $\mathcal{A}$ possesses the weak Vitali property, there exists (taking $a = r/s$) a sequence $\{I_k\}$ of sets in $\mathcal{A}^*$ such that $\mu(J) = \mu(\bigcup I_k) \leqslant \sum \mu(I_k) < \frac{s}{r}\mu(J)$. Therefore $\sigma(J) = \sigma(\bigcup I_k) \leqslant \sum \sigma(I_k) \leqslant \sum r\mu(I_k) < s\mu(J)$, the equality following from the fact that σ is absolutely continuous with respect to μ. But this contradicts the fact that J was so chosen that $\sigma(J) > s\mu(J)$. It follows that E_{rsn} is nowhere dense and that E is of the first category. Analogously, we can show that the set $\{x\colon \bar{D}_1\sigma(x) > \bar{D}_2\sigma(x)\}$ is of the first category, and the theorem is established for upper derivates.

The part of the theorem dealing with lower derivates can be proved in a similar fashion, taking into account certain obvious changes in the definitions of the sets E, E_{rsn}, and J. There is one nontrivial difference, however. The analogous string of inequalities becomes $\sigma(J) = \sigma(\bigcup I_k) \geqslant \frac{r}{s}\sum\sigma(I_k) > \sum r\mu(I_k) \geqslant r\mu(J)$, contradicting the choice of J. Now, the fact that the sequence $\{I_k\}$ can be so chosen that the first inequality appearing in this string, $\sigma(\bigcup I_k) \geqslant \frac{r}{s}\sum\sigma(I_k)$ is valid depends on the fact that $\mathcal{A}$ possesses the weak Vitali property *with respect to* σ. That this is the case is a consequence of a result of Hayes and Pauc ([3], p. 245): if a differentiation basis differentiates an integral σ a.e. to its Radon-Nikodym derivative, and possesses the weak Vitali property with respect to μ, then it also possesses the weak Vitali property with respect to σ.

This completes the proof of Theorem 1.

Remark. We note that the weak Vitali property with respect to σ was used only in the part of the proof dealing with lower derivates. For positive functions, this property is not necessary for upper derivates. Thus, we can drop the requirement that $\mathcal{A}$ differentiates σ to f a.e. if we are interested only in upper derivates and we are dealing with positive f.

THEOREM 2 below is stated in a two-dimensional setting for simplicity. As stated, the theorem would be false in E_N for $N \geqslant 3$. Nonetheless, it is easy to see from the proof of this theorem, that there exist valid higher dimensional analogues which can be obtained by suitably restricting the types of allowable continua. The theorem would be valid in E_N if the term "non-degenerate continua" were suitably restricted. In this connection, see the comment immediately following the proof of Theorem 2.

THEOREM 2. *Under the same hypotheses as in Theorem* 1 (*in* E_2), *any collection of pairwise disjoint non-degenerate continua contained in the set* $A_1 = \{x\colon \bar{D}_1\sigma(x) < \underline{D}_2\sigma(x)\}$ *or the set* $A_2 = \{x\colon \underline{D}_1\sigma(x) > \bar{D}_2\sigma(x)\}$ *must be at most denumerable.*

Proof. It suffices to prove the Theorem for A_1. Fix $I_0 \in \mathcal{A}$ and let $\mathcal{A}_0$ be the family of sets homothetic to I_0. Thus $\mathcal{A}_0 \subset \mathcal{A}$, and $\mathcal{A}_0$ satisfies the several hypotheses put on $\mathcal{A}$. Throughout this proof, we shall deal with $\mathcal{A}_0$ rather than all of $\mathcal{A}$. For each pair of rational numbers $r < s$ and each positive integer n, let E_{rsn} be the set $\{x\colon \sigma(I) < r\mu(I)$ if $x \underset{1}{\in} I \in \mathcal{A}_0,\ \delta(I) < 1/n\} \cap \{x\colon \sigma(J) > s\mu(I)$ if $x \underset{2}{\in} I \in \mathcal{A}_0,\ \delta(I) < 1/n\}$. Then $A_1 \subset \bigcup_{r,s,n} E_{rsn}$.

Since any two sets in $\mathcal{A}_0$ are homothetic, there exists a direction θ such that if $I \in \mathcal{A}_0$, $x \underset{1}{\in} I$, $y \underset{2}{\in} I$, then the line segment determined by x and y points in the direction θ. Let L be a fixed line whose direction is perpendicular to the direction θ. Let K be any non-degenerate continuum contained in A_1. The set K cannot contain a line segment perpendicular to L (i.e., in the direction θ). To see this we need only observe that $K \subset \bigcup E_{rsn}$, so if K contained such a segment S, there would exist a triple (r, s, n) such that $S \cap E_{rsn}$ would be non-denumerable. There would then exist two points $x \in S \cap E_{rsn}$ and $y \in S \cap E_{rsn}$ and an $I \in \mathcal{A}_0$ such that $\delta(I) < 1/n$, $x \underset{1}{\in} I$ and $y \underset{2}{\in} I$. But the foregoing implies that $\sigma(I) < r\mu(I)$ and also that $\sigma(I) > s\mu(I)$, a contradiction since $r < s$. Thus K cannot contain a segment perpendicular to L. Therefore, the projection of K onto L is a non-degenerate interval. Now suppose A_1 contains non-denumerably many pairwise disjoint non-degenerate continua $K_\gamma\colon \gamma \in \Gamma$, where Γ is a non-denumerable index set. Each of these projects onto a non-degenerate interval of L. Thus, there is a point $z \in L$ which lies in non-denumerably many of these projections. Since these continua K_γ are pairwise disjoint, the points which project onto z are distinct. For each $\gamma \in \Gamma$, let x_γ be such a point. It follows that there exists a triple (r, s, n) such that the set $E_{rsn} \cap \{x_\gamma\colon \gamma \in \Gamma\}$ is non-denumerable. There exist two points x and y in this set and an $I \in \mathcal{A}_0$, such that $\delta(I) < 1/n$, $x \underset{1}{\in} I$ and $y \underset{2}{\in} I$. But, as before, this implies $\sigma(I) < r\mu(I)$ and also $\sigma(I) > s\mu(I)$, a contradiction.

The proof of Theorem 2 is complete.

A word about extending Theorem 2 to N-dimensional space. The proof of Theorem 2 shows that if by "degenerate continua" we mean any continua whose projection onto some $(N-1)$-dimensional hyperplane has empty interior, then Theorem 2 remains valid, the proof involving only minor modifications to the proof we gave. This comment applies in particular to the case $N = 1$, giving us Young's Theorem.

Remark. Theorem 2 shows that the set $A_1 = \{x\colon \bar{D}_1\sigma(x) < \underline{D}_2\sigma(x)\}$ cannot contain a non-denumerable collection of pairwise disjoint non-degenerate continua. This set can, of course, be non-denumerable. Let, for example, $\mathcal{A}$ consist of the family of all squares in E_2 having sides parallel to the coordinate axes. Let $I_n \underset{1}{\to} x$ mean that x is in the lower right-hand corner of I_n and $\delta(I_n) \to 0$. Let $I_n \underset{2}{\to} x$ mean that x is the lower left-hand corner of I_n and $\delta(I_n) \to 0$. Let f be the characteristic function of the right half plane (i.e., in rectangular coordinates, $\{(\xi, \eta)\colon \xi > 0\}$). Then $\bar{D}_1\sigma(x) = 0$ and $D_2\sigma(x) = 1$ for all $x = (\xi, \eta)$ such that $\xi = 0$. Thus $A_1 = \{(\xi, \eta)\colon \xi = 0\}$.

We note also that the set $E = \{x\colon \bar{D}_1\sigma(x) < \bar{D}_2\sigma(x)\}$ can contain non-denumerably many pairwise disjoint non-degenerate continua. Thus, let P be the usual Cantor set. Define a function g of a real variable as follows: $g(x) = 0$ if $x \in P$; g is continuous on $[0, 1]$; if (a, b) is an interval contiguous to P, and c is the midpoint of (a, b), then g is linear on $[a, c]$ and on $[c, b]$ and $2g(c) = b-a$. It is easy to verify that g is absolutely continuous, that $D^{-}g \leqslant 0$ on P and that $D^{+}g > 0$ on P. Furthermore, if D denotes any Dini derivative of g, then $|Dg(\xi)| \leqslant 1$ for all $\xi \in [0, 1]$. Let f be defined by $f(\xi) = g(\xi)+\xi$. Then f is an increasing absolutely continuous function. Finally, define F and σ by $F(\xi, \eta) = \eta f(\xi)$ and $\sigma(I) = F(\xi+h, \eta+h)-F(\xi, \eta+h)-F(\xi+h, \eta)+F(\xi, \eta)$, where I is the square with sides parallel to the coordinate axis and having principle vertices (ξ, η) and $(\xi+h, \eta+h)$. Then the extension of σ to the class of Lebesgue measurable sets is an absolutely continuous measure. Let $\mathcal{A}$ denote the family of squares with sides parallel to the coordinate axes and let $I_n \underset{1}{\to} x$ ($I_n \underset{2}{\to} x$) means $\delta(I_n) \to 0$ and x is in the lower right-hand (lower left-hand, resp.) corner of I_n. Then for every $x = (\xi, \eta)$, $0 \leqslant \xi \leqslant 1$, $\overline{D}_1\sigma(x) = D^{-}f(\xi)$ while $\overline{D}_2\sigma(x) = D^{+}f(x)$. Thus E consists of $\{(\xi, \eta)\colon \xi \in P\}$. This set is a non-denumerable family of lines.

4. On extensions to other settings. Thus far, our setting has been that of a certain type of differentiation basis in E_N with two different notions of convergence. One can also consider related questions on settings dealing with *two* differentiation bases. Busemann and Feller [2] developed a theory in E_N in which a differentiation bases is a family of bounded open sets which (for some of their theorems) is closed under homothetic transformations. Convergence of a sequence $\{I_n\}$ of these sets to a point means only that $x \in I_n$ for each n and that $\delta(I_n) \to 0$. The position of x in I_n is unimportant. Many authors have extended the theory to arbitrary measure spaces. The first of these was de Possel, who considered differentiation bases in abstract measure spaces with an abstract notion of convergence. Now, in such settings, one might ask questions of the following type: if $\mathcal{A}_1$ and $\mathcal{A}_2$ are different differentiation bases on the same space, what can be said about the sets where the derivates relative to $\mathcal{A}_1$ and $\mathcal{A}_2$ are different? The results one gets depend heavily on the particular notion one considers. We do not attempt here to investigate the question fully, but we do make two remarks.

Remark 1. Theorem 1 carries over to the Busemann-Feller setting with only minor modifications in the proof. Thus, if $\mathcal{A}_1$ and $\mathcal{A}_2$ are two different differentiation bases on E_N which possess the weak Vitali property, then the set $\{\overline{D}_1\sigma \neq \overline{D}_2\sigma\}$ is of the first category. The classical setting is the one in which $\mathcal{A}_1$ consists of the oriented squares and $\mathcal{A}_2$ consists of the two-dimensional intervals. There exist positive summable functions f whose integrals have upper derivates with respect to $\mathcal{A}_2$ everywhere to $+\infty$ ([6], p. 133). On the other hand, the derivatives relative to $\mathcal{A}_1$ must equal f a.e. Our Theorem 1 (in the Busemann-Feller setting), together with the remark immediately following the proof of that theorem, shows that for such an f, the upper derivate with respect to $\mathcal{A}_1$ is equal to $+\infty$ on a residual set. Thus, for such an f we have $\overline{D}_1\sigma(x) = \underline{D}_1\sigma(x) = f(x)$ a.e. and $\overline{D}_1\sigma(x) = +\infty$ on a residual set (of measure 0).

Remark 2. Theorem 1 is not valid in the general de Possel setting. For example, let $\mathcal{A}_1$ be the differentiation basis of open spheres in E_N with $\{I_n\} \to x$ meaning, as usual, that $x \in I_n$ and $\delta(I_n) \to 0$. Let S denote the closed unit sphere in E_N and let R be a zero measure residual subset of $E_N \sim S$. Let $\mathcal{A}_2$ consist of all sets which are either open spheres or the union of two open spheres one of which contains the origin and let $\{I_n\} \to x$ mean that $x \in I_n$ and $\delta(I_n) \to 0$ if $x \notin R$, $\mu(I_n) \to 0$ if $x \in R$. Let f be the characteristic function of S and σ be the integral of f. Then $\mathcal{A}_1$ and $\mathcal{A}_2$ both possess the weak Vitali property, and for $x \in R$, $\overline{D}_1\sigma(x) = 0$ while $\overline{D}_2\sigma(x) = 1$. Thus $\overline{D}_1\sigma < \overline{D}_2\sigma$ on a set of the second category.

References

[1] A. M. Bruckner and C. Goffmann, *The boundary behavior of real functions in the upper half plane*, Rev. Math. Pures Appl. 11 (1966), pp. 507-518.

[2] H. Busemann and W. Feller, *Zur Differentiation der Lebesgueschen Integrale*, Fund. Math. 22 (1934), pp. 226-256.

[3] C. Hayes and C. Pauc, *Full individual and class differentiation theorems in their relations to halo and Vitali properties*, Can. J. Math. 7 (1955), pp. 221-274.

[4] C. Neugebauer, *A theorem on derivates*, Acta. Sci. Math. (Szeged) 23 (1962), pp. 79-81.

[5] R. de Possel, *Dérivation abstraite des fonctions d'ensemble*, Journ. Math. Pures et Appl. 15 (1936), pp. 391-409.

[6] S. Saks, *Theory of the integral*, Warszawa 1937.

[7] G. Young, *On the derivates of a function*, Lond. Math. Soc. 2 (1916), pp. 360-384.

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