A note on local homogeneity and stability

by

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Definition 1. A point p of a topological space X is *unstable* (called labile in [5] and homotopocally labile in [1]) if for each neighborhood U of p there exists a continuous function $F\colon X\times I\to X$ ($I=[0,1]$ throughout this paper) satisfying

(1) $F(x,0)=x$ for each x in X;

(2) $F(x,t)=x$ for each x in $X\setminus U$ and t in I;

(3) $F(x,t)\in U$ for each x in U and t in I;

(4) $p\notin F(X\times\{1\})$.

A point p is *stable* if it is not unstable.

The purpose of this paper is to investigate the stability of points in locally homogeneous [8] and group-supporting spaces. The first section uses results of Borsuk and Jaworowski [1] and Montgomery [8] to show that each point in a finite dimensional locally homogeneous space is stable. The second section contains the result that each point of a locally compact connected group is stable. A corollary to these results is that a compact connected finite dimensional locally homogeneous semigroup S with an identity is a group.

If X is a topological space, G an abelian group, and n a non-negative integer, then $H^n(X;G)$ will denote the n-dimensional Alexander-Kolmogoroff cohomology group of X with coefficients in G. Dimension will mean cohomological dimension [2]. The additive group of integers will be denoted by Z, the integers modulo p by Zp, and the additive group of real numbers will be denoted by R. If $F\colon X\times I\to X$ is a function, then by $F_t\colon X\to X$ is meant the function defined by $F_t(x)=F(x,t)$ for each x in X.

§ 1. Stability in locally homogeneous spaces.

Definition 2. Let X be a topological space and $A\subset X$. If $h\in H^n(A;G)$ a *roof* for h is a closed set $R\subset X$ satisfying

(1) h is not extendable to $R\cup A$, i.e. h is not in the image of $H^n(R\cup A;G)$ under the homomorphism induced by the inclusion map from A into $R\cup A$;

(2) if S is a closed proper subset of R, then h is extendable to $S \cup A$.

It is known [2] that if X is a compact Hausdorff space, A a closed subset of X, and $h \in H^n(A; G)$ is not extendable to X, then h has a roof. The following theorem is a rephrasing of a result of Borsuk and Jaworowski ([1], p. 165) in the Alexander-Kolmogoroff cohomology theory.

THEOREM 1. *Let X be a compact space and A a closed subset of X. If R is a roof for $c \in H^n(A; G)$, then each point of $R \setminus A$ is stable in $R \cup A$.*

Proof. It is shown that $R \setminus A$ cannot be reduced. Suppose there is a function F: $R \cup A \times I \to R \cup A$ satisfying Definition 1. Let i: $A \to R \cup A$ and j: $A \to F_1(R \cup A)$ be the inclusion maps. The $F_1 i = j$ and hence $i^* F_1^* = j^*$: $H^n(F_1(R \cup A); G) \to H^n(A; G)$. Since $F_1(R \cup A) = F_1(R) \cup A$ and $F_1(R) \underset{\neq}{\subset} R$, there is a $c' \in H^n(F_1(R \cup A); G)$ such that $j^*(c') = c$; however, this implies that $i^*(F_1^*(c')) = c$ contradicting the assumption that R is a roof for c.

The next theorem is a rephrasing of Theorem 3, p. 263 of Montgomery [8]. First two definitions are needed. It is assumed in the following that (X, d) is a separable metric space.

DEFINITION 3. For $A \subset X$ let F: $A \times I \to X$ be a continuous function. Then F is called an *ε-family of homeomorphisms* on A if it satisfies

(1) $F_0(x) = x$ for each x in A;

(2) for fixed t in I, F_t is a homeomorphism;

(3) $d(x, F_t(x)) < \varepsilon$ for each x in A and t in I.

DEFINITION 4. The space X is called *locally homogeneous* if it is connected, locally compact and if each $p \in X$ is in an open set U satisfying: if $\varepsilon > 0$ there is $\delta > 0$ such that if a is in $\mathrm{Cl}(U)$ and b is in X and $d(a, b) < \delta$, then there is an ε-family of homeomorphisms F on $\mathrm{Cl}(U)$ such that $F_1(a) = b$.

THEOREM 2. *Let X be locally homogeneous and n-dimensional and let $p \in X$. There exists a neighborhood V of p such that if A is a compact set in V, $h \in H^{n-1}(A; Z_3)$, and R is a compact set in V which is a roof for h, then $R \setminus A$ is open in X.*

Remark on Proof. The proof of the above is essentially "dual" to the proof in Montgomery's paper [8]; however, liberal use of the homotopy axiom and the Mayer-Vietoris sequence seems necessary.

THEOREM 3. *Let X be locally homogeneous and n-dimensional. Then each point of X is stable.*

Proof. Since points of X have homeomorphic neighborhoods ([8], p. 264), it suffices ([1], p. 159) to show that some point of X is stable. Let $p \in X$ and choose a compact neighborhood V of p which is n-dimensional ([2], p. 214) and satisfies the conclusion of Theorem 2. There exists ([2], p. 211) in V compact sets R and A such that R is a roof for some $h \in H^{n-1}(A; Z_3)$. Since the space is locally homogeneous, it can be assumed that $p \in R \setminus A$. By Theorem 1, p is stable in $R \setminus A$ and hence by Theorem 2 is stable in X.

If X is an n-dimensional space which is a locally connected locally compact group or the coset space of such a group, then X is locally homogeneous [8]. Thus Theorem 3 applies to such spaces. In the next section we extend these results. First we give an application of Theorem 3 to topological semigroups.

THEOREM 4. *Let S be a compact connected metric semigroup with an identity* 1. *If S is not a group and is locally arcwise connected at* 1, *then* 1 *is unstable in S.*

Proof. If $\varepsilon > 0$, since multiplication in S is uniformly continuous, there exists $\delta > 0$ such that if $d(x, 1) < \delta$, then $d(xy, y) < \varepsilon$ and $d(yx, y) < \varepsilon$ for y in S. Let $U_\delta = \{x \in S \mid d(x, 1) < \delta\}$. Since S is locally arcwise connected at 1, there exists an arc A: $I \to U_\delta$ satisfying $A(0) = 1$ and $1 \notin A(1) \cdot S$ or $1 \notin S \cdot A(1)$. Assume that $1 \notin A(1) \cdot S$. Define F: $S \times I \to S$ by $F(s, t) = A(t) \cdot s$. It follows using ([1], p. 160) that 1 is unstable.

COROLLARY. *A locally homogeneous (necessarily connected) finite dimensional compact semigroup with an identity is a group.*

Remark. The above corollary is known [4] with locally homogeneous replaced by homogeneous. The author does not know of examples of locally homogeneous continua that are not homogeneous. The dimension restriction is necessary since the Hilbert cube is a locally homogeneous compact connected semigroup with identity that is not a group.

§ 2. Stability in locally compact groups. It was shown in [7] that the space of a compact connected group G is irreducible, i.e. there does not exist a continuous function F: $G \times I \to G$ such that F_0 is the identity function and $F_1(G) \underset{\neq}{\subset} G$. If compactness is weakened to local compactness, then the analogous result would be that a locally compact connected group is "locally irreducible", i.e. each point is stable. It is also true [7] that a connected coset space of a compact group is irreducible. Therefore one would suspect that each point of a connected coset space of a locally compact group is stable. Such results are helpful in considering the problem of what spaces are coset spaces of groups. In this section it is proved that each point of a locally compact connected group is stable, and the similar result for coset spaces is partially resolved.

DEFINITION 5. A *limit manifold* [4] is a compact space which is the inverse limit of an inverse system of manifolds $\{X_\alpha, \pi_{\alpha\beta}, D\}$ satisfying the following: the projections π_α: $X \to X_\alpha$ induces a non-zero homomorphism π_α^*: $H^n(X_\alpha; R) \to H^n(X; R)$ where n is the dimension of X_α.

THEOREM 5. *Let G be a locally compact connected group. Then each point of G is stable.*

Proof. The space G is homeomorphic to the product $K\times R^n$ of a compact connected group K and a euclidean space R^n [6]. Let $x=(h,0)$ with $h\in K$ and 0 the origin of R^n. Let B be the open unit ball in R^n. Assume that there exists a continuous function $F\colon K\times R^n\times I\to K\times R^n$ satisfying Definition 1 with $U=K\times B$. Define the relation $\mathcal{R}$ on $K\times R^n$ by $(k_1,x_1)\mathcal{R}(k_2,x_2)$ if and only if $k_1=k_2$ and $\|x_1\|\geqslant 1$ and $\|x_2\|\geqslant 1$ or $k_1=k_2$ and $x_1=x_2$. It is seen that $(K\times R^n)/\mathcal{R}$ is homeomorphic to $K\times S^n$ with $S^n=R^n/(R^n\setminus B)$, and hence F induces a continuous function $\hat{F}\colon K\times S^n\times I\to K\times S^n$ which has the property that $\hat{F}_0$ is the identity and $\hat{F}_1(K\times S^n)\subsetneq K\times S^n$. We show that this is impossible. To see this note that K is a limit manifold and is hence irreducible [7]. Let $K=\varprojlim\{K_a,\pi_{a\beta},D\}$ with $\pi_a\colon K\to K_a$ inducing a non-zero homomorphism $\pi_a^*\colon H^p(K_a)\to H^p(K)$ with $p=\dim K_a$. For $a\in D$ set $S^n=S^n_a$. Then $K\times S^n=\varprojlim\{K_a\times S^n_a,\pi_{a\beta}\times id,D\}$ and the projections can be seen to satisfy Definition 5. Thus $K\times S^n$ is irreducible. This is a contradiction and shows that $x=(k,0)$ is stable and hence each point of G is stable.

It is conjectured that each point of a coset space of a locally compact group is stable. This is true if the space is the product of a connected coset space of a compact group and a euclidean space. This follows exactly as the proof of Theorem 5 using [7]. Other cases follow from the following theorem:

THEOREM 6. *Let D be a totally disconnected space and M a locally euclidean space. Then each point of $D\times M$ is stable.*

Proof. It is easily seen that neighborhoods of the form $D\times V$ where V is a euclidean space cannot be reduced.

COROLLARY. *Let G be a separable metric locally compact group. If H s a subgroup of G such that G/H is finite dimensional, then each point of G/H is stable.*

Proof. This follows from the local structure of such spaces ([9], p. 239).

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