Embedding of the category of partially ordered sets into the category of topological spaces

by

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Let $U = (S(U), \leqslant_U)$ denote a partially ordered set ($S(U)$ the support of U — i.e. the underlying set, $\leqslant_U$ the corresponding order relation). Let U, V be two partially ordered sets, f a mapping of $S(U)$ in $S(V)$. f is called *isotone* when $x \leqslant_U y \Rightarrow f(x) \leqslant_V f(y)$, for all $x, y \in S(U)$. The category of all partially ordered sets with isotone mappings as morphisms will be denoted by $\mathfrak{U}$.

Under the topological space we mean the topological space in the sense of Bourbaki [1]. A topological space X will be denoted in details as $(S(X), \tau_X)$, where $S(X)$ denotes the support of X and τ_X the system of all open sets (τ_X is called the *topology of* X). The category of all topological spaces with continuous mappings as morphisms will be denoted by $\mathfrak{T}$.

As for notation concerning categories see [2]. Especially when a category $\mathfrak{K}$ is given and X, Y are two objects of $\mathfrak{K}$, $[X, Y]_{\mathfrak{K}}$ denotes the set of morphisms from X to Y in $\mathfrak{K}$. If X, Y are algebraic or topological structures, $S(X)$ and $S(Y)$ their underlying sets, then morphisms are to be taken as triples $\langle X, Y, f\rangle$ where $f \subset S(X) \times S(Y)$ is satisfying conditions for morphisms of a given category $\mathfrak{K}$. Such a "complication" is needed for getting disjoint sets of morphisms. Nevertheless, as usual, f will be often used instead of $\langle X, Y, f\rangle$.

Put

$$S([X, Y]_{\mathfrak{K}}) = \{f \mid \langle X, Y, f\rangle \in [X, Y]_{\mathfrak{K}}\}.$$

Recall here the notion of full embedding. Let two categories $\mathfrak{K}$ and $\mathfrak{L}$ be given, let F be a covariant functor from $\mathfrak{K}$ to $\mathfrak{L}$ with the following properties:

1. *X, Y being two objects in $\mathfrak{K}$, $\alpha, \beta \in [X, Y]_{\mathfrak{K}}$, $\alpha \neq \beta$, then $F(\alpha) \neq F(\beta)$, too.*

2. *If $X \neq Y$, then $F(X) \neq F(Y)$.*

3. *If $\varphi \in [F(X), F(Y)]_{\mathfrak{L}}$, there exists $\alpha \in [X, Y]_{\mathfrak{K}}$ such that $F(\alpha) = \varphi$.*

Then F is a full embedding of $\mathfrak{K}$ into $\mathfrak{L}$.

Let U be a partially ordered set and $x \in S(U)$. Define

$$A(x) = \{t |\ t \in S(U), t \leqslant_U x\},$$
$$E(x) = \{t |\ t \in S(U), x \leqslant_U t\}.$$

Let $\tau_l(U)$ be the topology on $S(U)$ with the subbase for open sets $\{A(x)|\ x \in S(U)\}$. $\tau_l(U)$ is called the *left topology for* U. A general left topology for U is a topology τ on $S(U)$ such that the intersection $\bigcap \{X| x \in X, X \in \tau\}$ of all neighbourhoods of x is $A(x)$. Similarly, dealing with $E(x)$ instead of $A(x)$, one gets the definitions of the right topology $\tau_r(U)$ and general right topology.

Lemma 1. *Let U be a partially ordered set. If $\tau_l(U)$ (or $\tau_r(U)$, respectively) is a general right topology (or general left topology, respectively), then U is an antichain, i.e. every two distinct elements are incomparable.*

Proof. For every $x \in S(U)$ it is $E(x) = A(x)$. So $E(x) = A(x) = \{x\}$. Theorems 2.6 and 2.7 in [3] give

Lemma 2. *Let U_1 and U_2 be two partially ordered sets, let $\tau(U_1)$ be a T_0-topology on $S(U_1)$, $\tau(U_2)$ a T_0-topology on $S(U_2)$.*

(i) *If none of U_1, U_2 is an antichain, then the following conditions are equivalent:*

(A) *$\tau(U_1)$ is the right topology for U_1 and $\tau(U_2)$ a general right topology for U_2 or $\tau(U_1)$ is the left topology for U_1 and $\tau(U_2)$ a general left topology for U_2.*

(B) *A map f: $S(U_1) \to S(U_2)$ is isotone if and only if it is continuous with respect to the topologies $\tau(U_1)$ and $\tau(U_2)$, i.e.*

$$S([U_1, U_2]_{\mathfrak{U}}) = S\big([(S(U_1), \tau(U_1)), (S(U_2), \tau(U_2))]_{\mathfrak{T}}\big).$$

(ii) *If U_1 is an antichain and* $\operatorname{card} S(U_2) \geqslant 2$ *then* (B) *is equivalent to*

(A′) *$\tau(U_1)$ is the discrete topology (therefore it is at the same time the right topology and the left topology for U_1).*

Let $F_l(U) = (S(U), \tau_l(U))$, $F_l(f) = f$ for every partially ordered set U and every isotone mapping f. Similarly F_r is defined. F_l and F_r are full embeddings of $\mathfrak{U}$ in $\mathfrak{T}$. This fact (following among others from lemma 2) is known in essential for a long time, see e.g. [1], chapter I, § 4, problem 3.

Now we prove

Theorem. *Every full embedding G of $\mathfrak{U}$ into $\mathfrak{T}$ is naturally equivalent to F_l or F_r.*

Proof. Let U be a partially ordered set, one-point set $A = \{a\}$ be also considered as an object of $\mathfrak{U}$. Let $x \in S(U)$, f_x: $A \to S(U)$ such that $f^x(a) = x$. As $\operatorname{card}[A, A]_{\mathfrak{U}} = 1$ and $\operatorname{card}[X, A]_{\mathfrak{U}} \geqslant 1$ for every partially ordered set X, so $\operatorname{card}[G(A), G(A)]_{\mathfrak{T}} = 1$ and $\operatorname{card}[G(X), G(A)]_{\mathfrak{T}} \geqslant 1$. Hence $\operatorname{card} S\big(G(A)\big) = 1$. Put $S\big(G(A)\big) = \{b\}$. Define $\varphi_{\mathfrak{U}}(x) = [G(f_x)](b)$. As G is a full embedding, φ_U is one-to-one mapping of $S(U)$ onto $S\big(G(U)\big)$. Let V be a partially ordered set too, f: $U \to V$ an isotone mapping.

Let $x \in S\big(G(U)\big)$, $y = \varphi_U^{-1}(x)$. Put $f(y) = y'$. Then $ff_y = f_{y'}$. So $G(f)G(f_y) = G(f_{y'})$, therefore $G(f)G(f_y)(b) = G(f_{y'})(b)$. Hence $G(f)(x) = f_V(y') = \varphi_V f \varphi_U^{-1}(x)$.

So

$$\text{(a)} \qquad [G(f)](x) = \varphi_V f \varphi_U^{-1}(x).$$

Define $\lesssim$ on $S\big(G(U)\big)$ so that

$$x \lesssim y \equiv \varphi_U^{-1}(x) \leqslant_U \varphi_U^{-1}(y).$$

Denote $\big(S\big(G(U)\big), \lesssim\big)$ as U'. Similarly V', for a partially ordered set V, is defined etc.

According to (a)

$$\text{(b)} \qquad S([U', V']_{\mathfrak{U}}) = S\big([G(U), G(V)]_{\mathfrak{T}}\big).$$

Now we shall prove that $\tau_{G(U)}$ is a T_0-topology for every partially ordered set U.

Admit that there exists a partially ordered set U_1 such that $\tau_{G(U_1)}$ is not a T_0-topology. There exist x and y in $S\big(G(U_1)\big)$, $x \neq y$, such that for all $O \in \tau_{G(U_1)}$ $x \in O$ is equivalent to $y \in O$. Define φ: $S\big(G(U_1)\big) \to S\big(G(U_1)\big)$ as $\varphi(x) = y$, $\varphi(y) = x$, $\varphi(z) = z$ otherwise. φ is clearly continuous. Then according to (b) φ is an isotone mapping of U_1' into itself. That means that x is incomparable with y. Let V_1 be a two point chain, $V_1 = \{u, v\}$, $u <_{V_1} v$. Put $u' = \varphi_{V_1}(u)$, $v' = \varphi_{V_1}(v)$. It is $u' \leqslant_{V_1'} v'$. Define ψ: $U_1' \to V_1'$ as follows: if $t \in A(x)$, then $\psi(t) = u'$, $\psi(t) = v'$, otherwise. ψ is clearly isotone. Nevertheless ψ cannot be continuous mapping of $G(U_1)$ in $G(V_1)$. Suppose on the contrary that ψ is continuous. As all open sets in $G(U_1)$ containing x (contain y) and similarly as all open sets in $G(U_1)$ containing y (contain x), all open sets in $G(V_1)$ containing $\psi(x)$ (or $\psi(y)$, respectively) contain $\psi(y)$ (or $\psi(x)$, respectively). According to the above consideration $\psi(x)$ is incomparable with $\psi(y)$, which is a contradiction to $u' \leqslant_{V_1'} v'$.

Hence $\tau_{G(U)}$ is a T_0-topology for every U.

Now we can make the use of lemma 2. Let none of U and V be an antichain. So none of U' and V' is an antichain. According to (i) and (b) $\tau_{G(U)} = \tau_l(U')$ or $\tau_{G(U)} = \tau_r(U')$.

Let the first case occur. Then $\tau_{G(V)}$ is a general left topology. Now interchange U and V. By lemma 1, $\tau_{G(V)}$ is the left topology for V'. Hence for every V being not an antichain $\tau_{G(V)} = \tau_l(V')$. By lemma 2 (ii) the same is valid for antichains.

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Accordingly in this case φ_U is a homeomorphism of $(S(U), \tau_l(U))$ onto $S(U'), \tau_l(U')) = G(U)$ and by (a) a natural equivalence between F and G has been established.

The second case is dual.

References

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Cup product, duality and periodicity for generalized group cohomology*

by

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Introduction. A finite permutation representation (G, X) of a group G consists of a finite non-empty set X, with G acting on the left, such that $(\varrho\sigma)x = \varrho(\sigma x)$ for all $\varrho, \sigma \in G$ and all $x \in X$ and such that $1x = x$ for all $x \in X$, where 1 denotes the identity element of G.

Out of a given arbitrary finite permutation representation (G, X), one can form the "standard complex" $C(X; G)$ (see [6], p. 135) which generalizes the standard complex for ordinary group cohomology. By means of this "standard complex", a „cohomology theory of finite permutation representations" was defined and investigated in [6], [7], [8] and [9].

Using recent developments in relative homological algebra, this "cohomology of permutation representations" was axiomatized and investigated in [4]. In this paper we continue this study.

In Chapter I we investigate the cup product in this relative homological algebra setting, thereby extending illuminating and giving new proofs for the results of [7].

In Chapter II we examine the results of [8] in this relative setting and generalize the results of [8] to arbitrary (i.e. not necessarily transitive) finite permutation representations.

In Chapter III we go on to investigate question of periodicity and to generalize the results on periodicity for ordinary group cohomology given in [2], Chapter XII, § 11.

If (G, X) is a finite permutation representation, then $f((G, X))$ will denote the finite collection $\mathfrak{H}$ of the subgroups of finite index in G which fix the points of X. Clearly $f((G, X)) = \mathfrak{H}$ is closed under conjugation by elements of G. Moreover, if we are given a finite collection $\mathfrak{H}$ of subgroups of G of finite index which is closed under conjugation, then there exists a finite permutation representation (G, X) such that $f((G, X)) = \mathfrak{H}$.

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