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On ω-models which are not β-models

by

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In this paper we shall prove a theorem which, roughly speaking, says that β-models for the second-order arithmetic (see [1]) cannot be distinguished from ω-models by elementary sentences. Although this result is by no means surprising, the proof of it is not immediately obvious. In section 6 we state a similar result for models of the Zermelo–Fraenkel set theory and give a solution of a problem concerning the existence of models which are $\aleph_\tau$-standard but are not $\aleph_{\tau+1}$-standard. This problem was formulated in [3].

1. Syntax. In our formal language we shall use $\vee$, &, $\rightarrow$, $\neg$, $\equiv$ as propositional connectives, (E), ( ) as quantifiers. Variables will be denoted by Roman letters and the predicate of identity by "$\approx$". We shall use the abbreviation $(\mathrm{E}!\,\mathrm{x})F$ for $(\mathrm{Ez})(\mathrm{x})[(\mathrm{x}\approx\mathrm{z})\equiv F]$. (1)

We shall consider a first order theory T which has the primitive predicates N, S, E, A, P and possibly still other predicates. N, S will have one argument, E two and A, P three. We read N(x) as "x is an integer", S(x) as "x is a set of integers", E(x, y) as "x is an element of y", A(x, y, z) as "x is the sum of y and z" and P(x, y, z) as "x is the product of y and z".

In order to make our formulae more readable we introduce a number of simplifications.

We shall abbreviate $(\mathrm{x})[\mathrm{N}(\mathrm{x})\rightarrow\ldots]$ as $(\mathrm{x})_{\mathrm{N}}\ldots$ and $(\mathrm{Ex})[\mathrm{N}(\mathrm{x})\&\ldots]$ as $(\mathrm{Ex})_{\mathrm{N}}\ldots$; we also use similar symbols for quantifiers limited to S. Sometimes even the index N or S can be omitted, because we shall use lower case Roman letters a, b, ..., n as variables "ranging over elements of N" and upper case Roman letters X, Y, ..., F, ... as "variables ranging over elements of S". (Letters x, y, ... will be used whenever the domain

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(1) We use in the meta-language the abbreviations $(\exists x)$, $(\forall x)$, and $\Rightarrow$ for "there is an x", "for every x", and "if .., then...". The symbol "&" will also be used as an abbreviation of "and" and the symbol "ϵ" as an abbreviation of "is an element of".

of variability is unrestricted). Also a formula F in which the variable a occurs will be thought of as an abbreviation of $N(a)\to F$ and similarly for formulae with other variables b, c, ... Similar remarks apply to formulae with the free variables X, Y, ... Finally we write "$x \in y$" for $E(x, y)$.

The axioms will be interspersed with definitions (numbered D1, D2, ...). At each point when axioms formulated up to this place allow one to derive a theorem of the form $(E!x)\,F(x, \ldots)$ we shall allow a definition of the form $f(\ldots) = (\iota x)\,F(x, \ldots)$; the symbol f will be allowed to occur in subsequent axioms.

Of course, all these abbreviations and simplifications are really not necessary: with some patience it would be possible to write all axioms in the "official" language of the first order logic.

I. Arithmetical axioms.

1. $[A(x, y, z) \vee P(x, y, z)] \to N(x) \,\&\, N(y) \,\&\, N(z)$.
2. $(E!a)A(a, b, c) \,\&\, (E!a)P(a, b, c)$.

D1. $b+c = (\iota a)A(a, b, c)$, $b\cdot c = (\iota a)P(a, b, c)$.

3. $(E!a)A(a, a, a)$.

D2. $0 = (\iota a)A(a, a, a)$.

4. $(E!a)[\neg(a \approx 0) \,\&\, P(a, a, a)]$.

D3. $1 = (\iota a)(\neg(a \approx 0) \,\&\, P(a, a, a))$.

5. $\neg(a+1 \approx 0)$.
6. $(a+1 \approx b+1) \to (a \approx b)$.
7. $a+0 \approx a$.
8. $a+(b+1) \approx (a+b)+1$.
9. $a\cdot 0 \approx 0$.
10. $a\cdot(b+1) \approx (a\cdot b)+a$.

II. Set-theoretic axioms.

1. $\neg S(a)$.
2. $(x \in y) \to N(x) \,\&\, S(y)$.
3. $(a)[(a \in X) \equiv (a \in Y)] \to (X \approx Y)$.

III. Axiom of induction.

$$(0 \in X) \,\&\, (a)[(a \in X) \to (a+1 \in X)] \to (a \in X)\,.$$

IV. Axiom scheme of comprehension.

$$(EX)(a)[(a \in X) \equiv \Phi];$$

in this axiom Φ may be any formula in which the variable X does not occur freely.

D4. $\{a\colon \Phi\} = (\iota X)(a)[(a \in X) \equiv \Phi]$.

In D4 we assume that Φ does not contain X as a free variable; of course, D4 is not a single definition but a scheme.

From the above axioms one can deduce the theorem

$$(E!c)\big((c+c) \approx (a+b)\cdot[(a+b)+1]\big)\,,$$

and hence we can formulate the definitions

D5. $(a, b) = (\iota c)\{(c+c) \approx (a+b)\cdot[(a+b)+1]\}$.

D6. $X^{(a)} = \{b\colon (a, b) \in X\}$.

V. Axiom scheme of choice.

$$(a)(EX)\Phi \to (EY)(a)(EX)[(X \approx Y^{(a)}) \,\&\, \Phi].$$

In this scheme Φ is any formula in which the variable Y is not free.

It is known that axiom scheme V implies IV but we shall not use this fact in our considerations.

2. Auxiliary formal theorems and definitions. In this section we collect some further abbreviations and definitions and formulate a few theorems which can be proved in the basis of axioms I-V.

D7. $aXb \equiv (a, b) \in X$.

D8. $\mathrm{Ord}(X) \equiv (a)(aXa) \,\&\, (a)(b)(c)[(aXb) \,\&\, (bXc) \to (aXc)] \,\&\, (a)(b)[(aXb) \vee (a \approx b) \vee (bXa)] \,\&\, (a)(b)[(aXb) \,\&\, (bXa) \to (a \approx b)]$.

D9. $\mathrm{Bord}(X) \equiv \mathrm{Ord}(X) \,\&\, (Y)(a)\big((a \in Y) \to (Eb)\{(b \in Y) \,\&\, (c)[(c \in Y) \to (bXc)]\}\big)$.

Obviously Ord defines "orderings of N" and Bord "well-orderings of N".

D10. $\mathrm{Fn}(X) \equiv (a)(E!b)(aXb) \,\&\, (a)(a')(b)[(aXb) \,\&\, (a'Xb) \to (a \approx a')]$.

This formula defines "one-one mappings of N into N".

D11. $\mathrm{Imb}(F, X, Y) \equiv \mathrm{Fn}(F) \,\&\, (a)(a')(b)(b')\{aFb \,\&\, a'Fb' \to [(aXa') \equiv (bYb')]\}$.

This formula defines the notion: F is an isomorphic imbedding of the relation aXa' in the relation bYb'.

D12. $X \precsim Y \equiv (EF)\mathrm{Imb}(F, X, Y)$.

It is very easy to show that the transitivity of $\precsim$ is provable in T:

$$(X \precsim Y) \,\&\, (Y \precsim Z) \to (X \precsim Z)\,.$$

We mention still that for each integer $n \geqslant 1$ it is possible to define a formula Q_n with $n+1$ free variables $a, a_1, \ldots, a_n$ such that the following theorems are provable:

(*) $(E!a)Q_n(a, a_1, \ldots, a_n)$;

(**) $(E!a_1, \ldots, a_n)Q_n(a, a_1, \ldots, a_n)$.

Thus Q_n allows us to define a "one-one mapping of N^n onto N". The definition of Q_n proceeds by induction:

$$Q_1(a, a_1) \equiv (a \approx a_1);$$

$$Q_{n+1}(a, a_1, \ldots, a_n, a_{n+1}) \equiv (Eb)\big[Q_n(b, a_1, \ldots, a_n) \,\&\, \big(a \approx (b, a_{n+1})\big)\big] .$$

In view of (*) and (**) we can admit for each n and each $i \leqslant n$ the definition

D13. $\mathrm{pr}_i^n(a) = (\iota a_i)(Ea_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n)Q_n(a, a_1, \ldots, a_n)$.

3. Relational systems. We shall denote by L the first order language in which formulae of T are written. Since we shall also deal with various extensions of L, we shall recall here some definitions from model theory in case of an arbitrary first order language L^* whose expressions contain not only predicates but individual constants as well.

A relational system $\mathfrak{M}$ of type L^* is an ordered pair $\langle A, \mu\rangle$ where A is a set and μ a function; the domain of μ is the set of all primitive predicates and of individual constants of L^* and $\mu(c) \in A$ if c is an individual constant, $\mu(\varrho) \subseteq A^n$ if ϱ is an n-ary predicate other than $\approx$ and $\mu(\approx) = \{\langle x, x\rangle : x \in A\}$. We use capital German letters to denote relational systems. Instead of $\mu(N)$ we shall write $N_{\mathfrak{M}}$ and similarly for other (primitive or defined) predicates other than $\approx$. The values of various terms in $\mathfrak{M}$ will be denoted by a suffix $\mathfrak{M}$ added to the term; e.g. $(a, b)_{\mathfrak{M}}$ denotes the value of the term (a, b) for the assignment of a to the variable a and of b to the variable b.

The semantical notions of satisfaction, model, elementary extension, reduct, diagram, etc. are defined as usual. The notion of definability will be used in the following sense. A relation $R \subseteq A^n$ is definable in $\mathfrak{M}$ if there are an integer k, a sequence $b_1, \ldots, b_k$ of elements of A and a formula F of L^* with $n+k$ free variables such that $\langle a_1, \ldots, a_n\rangle \in R$ if and only if $\models_{\mathfrak{M}} F[a_1, \ldots, a_n, b_1, \ldots, b_k]$ for arbitrary $a_1, \ldots, a_n$ in A.

If L^* contains the predicates N, A, P of T, then the relational system $\langle N_{\mathfrak{M}}, \mu'\rangle$ where $\mu'(A) = A_{\mathfrak{M}}$ and $\mu'(P) = P_{\mathfrak{M}}$ is called the *arithmetical part* of $\mathfrak{M}$.

A model $\mathfrak{M}$ of T is called an *ω-model* if its arithmetical part is isomorphic to the standard model $\mathfrak{A}_0$ of arithmetic. In this case we shall usually identify the arithmetical part of $\mathfrak{M}$ with $\mathfrak{A}_0$ and each X in $S_{\mathfrak{M}}$ with the set of integers n which together with X satisfy the formula $n \in X$ in $\mathfrak{M}$.

A model $\mathfrak{M}$ of T is called a *β-model* if for each X in $S_{\mathfrak{M}}$ the condition $\models_{\mathfrak{M}} \mathrm{Bord}[X]$ implies that the relation $\{\langle m, n\rangle \in N_{\mathfrak{M}}^2 : \models_{\mathfrak{M}} mXn\}$ well orders the set $N_{\mathfrak{M}}$. (Strictly speaking, we should have written $\models_{\mathfrak{M}} \mathrm{Bord}(X)[X]$ and $\models_{\mathfrak{M}} (mXn)[m, X, n]$ instead of $\models_{\mathfrak{M}} \mathrm{Bord}[X]$ and $\models_{\mathfrak{M}} mXn$ but we shall use the simplified way of writing whenever possible.) It is known (and easy to prove) that β-models are ω-models but not conversely.

4. The pigeon-hole principle. As is well known this principle says that if many object are put into a small number of drawers, then at least one drawer contains many objects. In our case the objects will be well orderings of integers and the number of drawers will be denumerable.

Let Φ be a formula of L in which U is a free variable and Ψ a formula of L in which U and a are free variables. We shall write these formulae as $\Phi(U)$ and $\Psi(U, a)$ although we do not exclude the possibility that one or both of these formulae contain free variables other that U and a.

Let A be the conjunction of the following formulae:

(1) $$(X)\{\mathrm{Bord}(X) \to (EU)[\Phi(U) \,\&\, (X \precsim U)]\};$$

(2) $$(U)(Ea)[\Phi(U) \to \Psi(U, a)] .$$

THEOREM 1. *The following formula is provable in* T:

$$A \to (Ea)(X)\{\mathrm{Bord}(X) \to (EU)[\Psi(U, a) \,\&\, (X \precsim U)]\} .$$

Instead of carrying out a formal proof using axioms of T and rules of proof formulated in logic we shall sketch it in the everyday's language of the "working mathematician". We shall supply enough details to convince the reader that the proof can be transformed into a formal proof in T.

We assume A and the negation of the formula after the first arrow, i.e. the formula

(3) $$(a)(EX)\{\mathrm{Bord}(X) \,\&\, (U)[\Psi(U, a) \to \neg (X \precsim U)]\} .$$

Our aim is to derive a contradiction from these assumptions.

First we use the axiom of choice and derive from (3)

(4) $$(EY)(a)\{\mathrm{Bord}(Y^{(a)}) \,\&\, (U)[\Psi(U, a) \to \neg (Y^{(a)} \precsim U)]\} .$$

Let Y satisfy the condition stated above. From axiom IV we easily derive that there is a Z such that the following equivalence holds for arbitrary a, a′, n, n′:

(5) $$(a, n)Z(a', n') \equiv \{(a < a') \vee [(a \approx a') \,\&\, (n\, Y^{(a)} n')]\} .$$

We want to show that $Y^{(a)}$ can be imbedded into Z. The imbedding function is obviously the map $n \to \langle a, n\rangle$. Formally speaking, we define $F^{(a)}$ as $\big\{b : (En)\big(b \approx (n, (a, n))\big)\big\}$ and prove using D10 that $\mathrm{Fn}(F^{(a)})$. Since

$$n\, Y^{(a)}\, n' \equiv (a, n)Z(a, n') ,$$

we infer using D11 that $\mathrm{Imb}(\mathrm{F}^{(a)}, \mathrm{Y}^{(a)}, \mathrm{Z})$. Hence by D12

$$\mathrm{Y}^{(a)} \precsim \mathrm{Z}\,. \tag{6}$$

On the other hand, we can derive from (5) that $\mathrm{Bord}(\mathrm{Z})$ and hence, according to (1) and (2) that there is an a and a U such that $\Psi(\mathrm{U}, \mathrm{a})$ and $\mathrm{Z} \precsim \mathrm{U}$. Using (6) we obtain $\mathrm{Y}^{(a)} \precsim \mathrm{U}$ since the transitivity of $\precsim$ is provable in T. But now we have a contradiction since, according to (4) for no U such that $\Psi(\mathrm{U}, \mathrm{a})$ does the formula $\mathrm{Y}^{(a)} \precsim \mathrm{U}$ hold. Our theorem is thus proved.

We do not know whether this theorem remains valid when the axiom scheme V of choice is removed from the axioms of T.

We shall formulate theorem 1 in a semantical way. Let $\mathfrak{M}$ be a model of T and let $K = \{X \in S_{\mathfrak{M}}\colon \models_{\mathfrak{M}} \mathrm{Bord}[X]\}$. We shall say that a set $C \subseteq S_{\mathfrak{M}}$ is *unbounded* if for every X in K there is a U in C such that $X \precsim_{\mathfrak{M}} U$.

We say that a relation $D \subseteq N_{\mathfrak{M}} \times S_{\mathfrak{M}}$ covers C if for every element X in C there is an a in $N_{\mathfrak{M}}$ such that $\langle a, X\rangle \in D$. This can be expressed as $C \subseteq \bigcup \{D_a\colon a \in N_{\mathfrak{M}}\}$ where D_a is the set $\{X \in S_{\mathfrak{M}}\colon \langle a, X\rangle \in D\}$.

The pigeon hole principle in its semantical form is the following result:

THEOREM 2. *If $\mathfrak{M}$ is a model of T and D is a definable relation $\subseteq N_{\mathfrak{M}} \times S_{\mathfrak{M}}$ which covers an unbounded definable set $C \subseteq S_{\mathfrak{M}}$, then at least one D_a is unbounded.*

Proof. It is sufficient to take in theorem 1 for Φ a formula which defines C and for Ψ a formula which defines the relation D.

5. A theorem on β-models. In this section we shall use the pigeon hole principle in order to establish our main result.

THEOREM 3. *For any denumerable β-model $\mathfrak{M}$, there exists an ω-model which is an elementary extension of $\mathfrak{M}$ and is not a β-model.*

Proof. We introduce, as auxiliary symbols, the constant symbols Δ_m for every element m of $\mathfrak{M}$ and the constant symbol R. The language L augmented by those symbols is denoted by L_1.

The interpretation of the symbols of the language is determined by the structure $\mathfrak{M}$.

The value of R will in most cases be an element of $S_{\mathfrak{M}}$ which satisfies the formula $\mathrm{Bord}(\mathrm{X})$ in $\mathfrak{M}$.

In the relational systems of type L_1, which we shall consider, the constant Δ_m will always be interpreted as m. Hence the relational systems are determined by the value R of the constant R and can be denoted by $(\mathfrak{M}, R)$.

We shall assume that the arithmetical part of $\mathfrak{M}$ has been identified with $\mathfrak{A}_0$ (cf. p. 86) and elements of $S_{\mathfrak{M}}$ with sets of integers. We can and will interpret each element X of $S_{\mathfrak{M}}$ as a binary relation $\{\langle m, n\rangle\colon \models_{\mathfrak{M}} m X n\}$; in case this relation is many-one we can speak of X as being a function.

As in section 4 we denote by K the set of all X in $S_{\mathfrak{M}}$ for which $\models_{\mathfrak{M}} \mathrm{Bord}[X]$.

Let A be the set of all sentences of L_1 which do not contain symbol R and are true in the structure $\mathfrak{M}$. We can represent the set A as the union of an increasing sequence $\langle A_n\rangle_{n\in\omega}$ of finite sets of sentences for which the following condition (A) holds:

$$(\mathrm{Ev})\big(\mathrm{N}(\mathrm{v}) \,\&\, \Psi(\mathrm{v})\big) \in A_n \Rightarrow (\exists i)\big(\Psi(\Delta_i) \in A_n\big)\,.$$

Let us fix an enumeration $\langle \Phi_i\rangle_{i\in\omega}$ of all the sentences of the language L_1.

Let us say that R is in the class $D_S(i_0, \ldots, i_n)$ if the following conditions are satisfied:

I. *$R \in K$ and $S \in K$.*

II. *$i_n R i_{n-1} \ldots i_1 R i_0$ and $i_n \neq i_{n-1} \neq \ldots i_1 \neq i_0$.*

III. *There is a function in $\mathfrak{M}$ which maps the field of S order-isomorphically into the R-predecessors of i_n.*

It is obvious that D_S is extensional in the following sense: whenever S and S' are in $\mathfrak{M}$ and there is in $\mathfrak{M}$ a function which establishes an isomorphism between S and S', then $D_S(i_0, \ldots, i_n) = D_{S'}(i_0, \ldots, i_n)$. More generally, this equation holds for arbitrary S, S' in $S_{\mathfrak{M}}$ such that $S \precsim_{\mathfrak{M}} S'$ and $S' \precsim_{\mathfrak{M}} S$.

We define by induction a monotonically increasing sequence $\langle B_n\rangle_{n\in\omega}$ of finite sets of sentences of the language L_1 and a sequence of natural numbers $\langle i_n\rangle_{n\in\omega}$. These sequences are required to satisfy the following conditions $\langle C_n\rangle_{n\in\omega}$

(i) $\mathrm{Bord}(\mathrm{R})$ *is in* B_n,

(ii) $A_n \subseteq B_n$,

(iii) *if $n > 0$, then the sentences $\Delta_{i_n} \mathrm{R} \Delta_{i_{n-1}}$ and $\neg(\Delta_{i_n} \approx \Delta_{n-1})$ are in B_n,*

(iv) *for $j < n$, either Φ_j or $\neg\Phi_j$ is in B_n,*

(v) *if $j < n$, Φ_j is in B_n and Φ_j has the form $(\mathrm{Ev})\big(\mathrm{N}(\mathrm{v}) \,\&\, \Psi(\mathrm{v})\big)$, then $\Psi(\Delta_i)$ is in B_n for some i in ω,*

(vi) *for every S in the class K, there is an R such that $R \in D_S(i_0, \ldots, i_n)$ and $\models_{(\mathfrak{M},R)} B_n$.*

Construction of the sequences.

Step 0.

Determination of the number i_0. i_0 can be any natural number, say 0.

Determination of the set B_0. We take $A_0 \cup \{\mathrm{Bord}(\mathrm{R})\}$ as B_0.

Verification of the conditions C_0. Conditions (i) and (ii) are evident. Conditions (iii), (iv) and (v) are true vacuously. Finally, (vi) is satisfied, because for every S in K there is an R in K such that the last element of the field of R is i_0 and there is a function F in $\mathfrak{M}$ which maps the field of S order-isomorphically into the set of R-predecessors of i_0. If e.g. $i_0 = 0$, then it is sufficient to take $F(n) = n+1$ and define R as the set consisting of all pairs $(n, 0)_{\mathfrak{M}}$ and of pairs $\big(F(n), F(m)\big)_{\mathfrak{M}}$ with nSm.

Step $n+1$. We assume that we have already defined the sequences $\langle B_j\rangle_{0\leqslant j\leqslant n}$ and $\langle i_j\rangle_{0\leqslant j\leqslant n}$ which satisfy conditions $\langle C_j\rangle_{0\leqslant j\leqslant n}$. Let I_ε be the class of all S such that $S \in K$ and

$$(\exists R)\big(R \in D_S(i_0, \ldots, i_n) \;\&\; \models_{(\mathfrak{M},R)} B_n \;\&\; \models_{(\mathfrak{M},R)} \Phi_n^\varepsilon\big),$$

where $\varepsilon \in \{0, 1\}$ and $\Phi_n^0 = \Phi_n$ and $\Phi_n^1 = \neg\Phi_n$. Since the set B_n is finite, the set I_ε is definable in the structure $(\mathfrak{M}, R)$. We shall show that either I_0 or I_1 coincides with the class K of all well-orderings of ω in the structure $\mathfrak{M}$. Let us assume $S \notin I_0$ for some S in the class K. Hence

$$(\forall R)_{\mathfrak{M}}\big(R \in D_S(i_0, \ldots, i_n) \;\&\; \models_{(\mathfrak{M},R)} B_n \Rightarrow\; \models_{(\mathfrak{M},R)} \neg\,\Phi_n\big).$$

By our inductive assumption (vi), there is an R such that $R \in D_S(i_0, \ldots, i_n)$ and $\models_{(\mathfrak{M},R)} B_n$. S is therefore in I_1. Thus we proved that $I_0 \cup I_1 = K$. The set I_ε is monotone in the sense that if $S' \in I_\varepsilon$ and $S \lesssim_{\mathfrak{M}} S'$, then $S \in I_\varepsilon$. Since $I_0 \cup I_1 = K$, either I_0 or I_1 is cofinal with K. The set which is cofinal with K and is monotone must coincide with K. Hence either I_0 is cofinal with K and is monotone must coincide with K. Hence either I_0 or I_1 coincides with K. Let $\bar\varepsilon$ be the smallest ε such that $I_\varepsilon = K$.

Determination of the number i_{n+1}. Let S be in the set K_i if and only if

$$S \in K \;\&\; (\exists R)\big(R \in D_S(i_0, \ldots, i_n, i) \;\&\; \models_{(\mathfrak{M},R)} B_n \cup \{\Phi_n^{\bar\varepsilon}, \Delta_i \mathrm{R} \Delta_{i_n}, \neg(\Delta_i \approx \Delta_{i_n})\}\big).$$

Let S be in the set K and S^* be an element in K whose order type is the successor of that of S. By our choice of $\bar\varepsilon$, there is an $R \in D_{S^*}(i_0, \ldots, i_n)$ such that $\models_{(\mathfrak{M},R)} B_n \cup \{\Phi_n^{\bar\varepsilon}\}$. Let i^* be the greatest element in the ordering S^* and let i be the image of i^* by an order-preserving map in $\mathfrak{M}$ of the field of S^* into the R-predecessors of i_n. The conditions $R \in D_S(i_0, \ldots, i_n, i)$ and $\models_{(\mathfrak{M},R)} \{\Delta_i \mathrm{R} \Delta_{i_n}\} \cup \{\neg(\Delta_i \approx \Delta_{i_n})\}$ are satisfied. We have proved, therefore, $(\forall S)\big(S \in K \Rightarrow (\exists i)(S \in K_i)\big)$. Since the relation $S \in K_i$ is definable in $\mathfrak{M}$, we can apply the pigeon hole principle to prove

$$(\exists i)(\forall S)\big(S \in K \Rightarrow (\exists S')(S \lesssim_{\mathfrak{M}} S' \;\&\; S' \in K_i)\big).$$

Since the sets K_i are monotone, $(\exists i)(\forall S)(S \in K \Rightarrow S \in K_i)$. We take as i_{n+1} the least such i.

Determination of the set B_{n+1}.

Case 1. *$\Phi_n^{\bar\varepsilon}$ is already in B_n.* We take $B_n \cup A_{n+1} \cup \{\Delta_{i_{n+1}} \mathrm{R} \Delta_{i_n}\} \cup \{\neg(\Delta_{i_{n+1}} \approx \Delta_{i_n})\}$ as B_{n+1}.

Case 2. *$\Phi_n^{\bar\varepsilon}$ is not in B_n.*

Subcase 2.1. *$\bar\varepsilon = 1$, or $\bar\varepsilon = 0$ and Φ_n is not of the form* $(\mathrm{Ev})\big(\mathrm{N}(\mathrm{v}) \;\&\; \Psi(\mathrm{v})\big)$. We take $B_n \cup A_{n+1} \cup \{\Phi_n^{\bar\varepsilon}, \Delta_{i_{n+1}} \mathrm{R} \Delta_{i_n}, \neg(\Delta_{i_{n+1}} \approx \Delta_{i_n})\}$ as B_{n+1}.

Subcase 2.2. *$\bar\varepsilon = 0$ and Φ_n is of the form*

$$(\mathrm{Ex}_1)\big(\mathrm{N}(\mathrm{x}_1) \;\&\; \ldots \;\&\; (\mathrm{Ex}_s)\big(\mathrm{N}(\mathrm{x}_s) \;\&\; \Psi(\mathrm{x}_1, \ldots, \mathrm{x}_s)\big)\ldots\big)$$

where $\Psi(\mathrm{x}_1, \ldots, \mathrm{x}_s)$ is not of such a form.

Let $\tilde\Psi(\mathrm{a})$ be the formula

$$\Psi\big(\mathrm{pr}_1^s(\mathrm{a}), \ldots, \mathrm{pr}_s^s(\mathrm{a})\big)$$

and let $f_i(e)$ be the value of the term $\mathrm{pr}_i^s(\Delta_e)$ in $\mathfrak{M}$. Then we have the equivalence

$$\models_{(\mathfrak{M},R)} \tilde\Psi(\Delta_e) \iff\; \models_{(\mathfrak{M},R)} \Psi(\Delta_{f_1(e)}, \ldots, \Delta_{f_s(e)}).$$

Let S in the set C_e if and only if

$$S \in K \;\&\; (\exists R)(R \in D_S(i_0, \ldots, i_{n+1}) \;\&\; \models_{(\mathfrak{M},R)} B_n \cup A_{n+1} \cup \{\tilde\Psi(\Delta_e), \Delta_{i_{n+1}} \mathrm{R} \Delta_{i_n}, \neg(\Delta_{i_{n+1}} \approx \Delta_{i_n})\}.$$

We apply, once again, the pigeon hole principle to the sequence $\langle C_e\rangle_{e\in\omega}$ which is definable in $\mathfrak{M}$. By our choice of the numbers i_{n+1} and $\bar\varepsilon$, $(\forall S)\big(S \in K \Rightarrow (\exists e)(S \in C_e)\big)$. By applying the pigeon hole principle,

$$(\exists e)(\forall S)\big(S \in K \Rightarrow (\exists S')(S \lesssim_{\mathfrak{M}} S' \;\&\; S' \in C_e)\big).$$

Since the sets C_e are monotone, $(\exists e)(\forall S)(S \in K \Rightarrow S \in C_e)$. We take as $\bar e$ the least such e. We take as B_{n+1}, in this subcase,

$$B_n \cup A_{n+1} \cup \big\{\Psi(\Delta_{f_1(e)}, \ldots, \Delta_{f_s(e)}), (\mathrm{Ex}_s)\big(\mathrm{N}(\mathrm{x}_s) \;\&\; \Psi(\Delta_{f_1(e)}, \ldots, \Delta_{f_{s-1}(e)}, \mathrm{x}_s)\big), \ldots, \Phi_n\big\} \cup \{\Delta_{i_{n+1}} \mathrm{R} \Delta_{i_n}\} \cup \{\neg(\Delta_{i_{n+1}} \approx \Delta_{i_n})\}.$$

Verification of the conditions C_{n+1}. In all cases conditions (i)-(iv) are clearly satisfied. Conditions (v) and (vi) are satisfied in subcase 2.1 because of our choice of the numbers i_{n+1} and $\bar\varepsilon$ and because of property (A) of the sequence $\bigcup_n A_n = A$. Conditions (v) and (vi) are satisfied in the subcase 2.2 because of property (A) of the sequence $\bigcup_n A_n = A$ and because of the choice of the numbers $i_{n+1}, \bar\varepsilon, \bar e$.

Let us consider the set $B = \bigcup_n B_n$. This set is consistent since every finite subset of B has a model by condition (vi). The set B is ω-closed by condition (v). By the Henkin–Orey completeness theorem for ω-closed consistent theories (cf. [2], p. 231) there is an ω-standard model $\mathfrak{M}_1$ for the theory B. The structure $\mathfrak{M}_1$ is an ω-model since $\mathfrak{M}_1$ is ω-standard. The structure $\mathfrak{M}_1$ is an elementary extension of $\mathfrak{M}$ since the set A is included in the set B. Consider the value R_1 of the constant symbol R in $\mathfrak{M}_1$. By condition (i), $\models_{\mathfrak{M}_1} \mathrm{Bord}[R_1]$. By condition (iii), the sequence $\langle i_n \rangle_{n\in\omega}$ forms a descending chain with respect to the ordering R_1. The ω-model $\mathfrak{M}_1$ is not, therefore, a β-model. Thus the L-reduct of $\mathfrak{M}_1$ is the required model of T and our theorem is proved.

COROLLARY 1. *For any β-model $\mathfrak{M}$ of T, there is an elementarily equivalent ω-model $\mathfrak{M}_1$ which is not a β-model.*

Proof. Every β-model $\mathfrak{M}$ is an elementary extension of a denumerable β-model $\mathfrak{M}_1$ [1].

COROLLARY 2. *If there is an ω-model $\mathfrak{M}$ for a set of sentences A, then there is an ω-model $\mathfrak{M}_1$ for the set A which is not a β-model.*

6. An application to set theory. Using the construction carried out in section 5, we can construct a new family of non-standard models for set theory.

Let S be a consistent extension of ZF. A formula φ with one free variable is said to *define a cardinal* in S if the sentence $(\mathrm{E!v})\{\varphi(\mathrm{v}) \,\&\, (\mathrm{v})[\varphi(\mathrm{v}) \to \mathrm{Card}(\mathrm{v})]\}$ is provable in S. We denote by φ^+ the formula

$$\mathrm{Card}(\mathrm{v}_0) \,\&\, (\mathrm{v}_1)\big(\varphi(\mathrm{v}_1) \to \mathrm{v}_1 < \mathrm{v}_0\big) \,\&\, (\mathrm{v}_1)(\mathrm{v}_2)\big((\mathrm{v}_2 < \mathrm{v}_0)\, \mathrm{Card}(v_2) \,\&\, \varphi(\mathrm{v}_1) \to (\mathrm{v}_2 \leqslant \mathrm{v}_1)\big).$$

A model $\mathfrak{M}$ for S is called *φ-standard* if there is no infinite descending chain $a_0 \ni_{\mathfrak{M}} a_1 \ni_{\mathfrak{M}} \ldots$ of ordinals smaller than the cardinal $\aleph(\mathfrak{M}, \varphi)$ where $\aleph(\mathfrak{M}, \varphi)$ is the unique element of $\mathfrak{M}$ which satisfies the formula φ in $\mathfrak{M}$.

The existence of a φ-standard, φ^+-non-standard model is known in the case when φ is a formula defining the first infinite cardinal $\aleph_0$ (cf. [3]).

We shall prove the following

THEOREM 4. *For any denumerable φ-standard model $\mathfrak{M}$ for S there is an elementary extension $\mathfrak{M}_1$ or $\mathfrak{M}$ which is φ-standard but is not φ^+-standard.*

Proof. We introduce, as auxiliary symbols, the constant symbols Δ_m for every element m of $\mathfrak{M}$, the constant symbol R and an unary predicate symbol N. The interpretation of the symbols Δ_m of the extended language is the same as in the proof of theorem 3. The symbol N is interpreted as the set of ordinals smaller than $\aleph(\mathfrak{M}, \varphi)$. We can define the sequences $\langle B_n \rangle_{n\in\omega}$ and $\langle i_n \rangle_{n\in\omega}$ in almost the same way as above. The pigeon hole principle which played a crucial role in the previous construction can be used in the present situation. To see this we merely notice that $\aleph(\mathfrak{M}, \varphi)$ and $\aleph(\mathfrak{M}, \varphi^+)$ are different cardinals of $\mathfrak{M}$ and, since $\mathfrak{M}$ is a model ZF, the pigeon hole principle holds in $\mathfrak{M}$ for these cardinals.

We can also prove the following corollaries:

COROLLARY 3. *For any φ-standard model $\mathfrak{M}$ of S there is an elementarily equivalent structure $\mathfrak{M}_1$ which is φ-standard but not φ^+-standard.*

COROLLARY 4. *For any set of sentences A, if there is a φ-standard model $\mathfrak{M}$ of A, then there is a φ-standard, φ^+-non-standard model of A.*

Note added on June 20, 1968. Several weeks after the present paper was accepted for publication we saw a paper: H. J. Keisler and M. Morley, *Elementary Extensions of Models of Set Theory* (Israel Jour. Math. 6 (1968), pp. 49-65) which appeared in March 1968. From the strictly logical point of view the results contained in Sections1-5 of our paper are independent from results established by Keisler and Morley. However, the methods used by these authors are the same as those which were used by us. The results of our Section 6 are weaker than those established by Keisler and Morley.

After some deliberations we decided not to withdraw our paper because we believe that the readers who will compare both papers will get useful insights into the close relationship which exists between the meta-mathematics of set theory and that of the second order arithmetic.

Note added on March 19, 1969. Results of our section 6 were also obtained by K. Hrbaček who used a completely different method.

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