[8] A. D. Wallace, *An outline for algebraic topology I*, New Orleans, Tulane University 1949.
[9] — *The Map Excision Theorem*, Duke Math. Journ. 19 (1952), pp. 177-182.
[10] — *A Theorem on Acyclicity*, Bull. Amer. Math. Soc. 67 (1961), pp. 123-124.

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# Open and closed mappings and compactification

by

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**Introduction.** In this paper we study the extension of open and closed mappings to compactifications such that the extension is open. It is shown that this can be done in such a way that the compactification has the same weight and dimension as the original space. A characterization of open and closed mappings in terms of the ring of bounded real valued continuous functions of a space is given which facilitates the study of the extension of such mappings to compactifications. Also a sufficient condition is given for the extension of a mapping to a compactification to be open. These results should be of interest in themselves. Among those who have studied the extension of mappings to compactifications have been R. Engelking [2], R. Engelking and E. Skljarenko [3], A. B. Forge [4], H. de Vries [13], and A. Zarelua [15]. J. de Groot and R. McDowell have studied the extension of mappings on metric spaces to completions [6].

The last section of the paper deals with finite to one open and closed mappings and dimension. Dimension and finite to one open mappings have been studied by K. Nagami [11] for domain and range paracompact. The author has studied the case with domain and range metrizable [8]. The theorems of this section are an attempt to generalize these results to more general spaces. A. Arhangelskiĭ has studied finite to one open and closed mappings and metrization [1].

The paper has three sections. The first deals with the preliminaries and reviews the relation between $C^*(X)$ and compactification. The second characterizes open and closed mappings on normal spaces and proves the results dealing with extending such mappings to compactifications. The last section deals with finite to one open and closed mappings and dimension.

**Notation.** Throughout the paper all spaces are assumed completely regular. By *mapping* is meant a continuous function. By $B(X)$ is meant the *ring of bounded real valued functions on* $X$. The set $C^*(X)$ is the subset of $B(X)$ consisting of those functions which are also continuous. The *modified Lebesgue covering dimension* of the space $X$ is denoted by $\dim X$.

For a discussion of this dimension function see Gillman and Jerison [5], p. 243 or Isbell [7], p. 97. A *compactification* of the space $X$ is a pair $gZ$ such that $g$ is an embedding of $X$ onto a dense subset of $Z$ which is compact. If $gZ$ and $hY$ are compactifications of $X$, then $gZ \geqslant hY$ whenever there is a mapping $f(Z) = Y$ such that $h = f \circ g$. If $gZ \geqslant hY$ and $hY \geqslant gZ$, then $f$ must be a homeomorphism and we write $gZ \sim hY$. By $\beta X$ is meant the *Stone-Čech compactification* of $X$. If $f\colon X \to Y$ is a mapping, then the *Stone extension* of $f$ is denoted by $\beta f\colon \beta X \to \beta Y$.

A *compaction* of the space $X$ is a pair $gZ$ such that $g$ is a mapping of $X$ onto a dense subset of $Z$ which is compact. We have the same quasi-ordering and equivalence for compactions as for compactifications. For terminology see H. de Vries [13].

The *weight* of a space $X$ is the least cardinality of a basis for $X$ and is denoted by $w(X)$. We will use the standard result that considering $C^*(X)$ as a metric space with the metric induced by the uniform norm $w\big(C^*(X)\big) = w(X)$ whenever $X$ is compact and $w(X)$ is infinite.

**Part I. Preliminaries.** The results of this section seem to be entrenched in the folklore of compactification, but seldom referred to in the literature. Authors tend to use the relation of compactifications to precompact uniformities on the space. Such an approach will not suit our purpose and because of the scarcity of detailed discussion in the literature, a presentation must be made of the precise relation between the closed subrings of $C^*(X)$ and compactions of $X$. The result needed is basically stated in Zarelua [14], but without proof and without the precision needed here.

I.1. DEFINITION. Let $gZ$ be a compaction for $X$. Define $F_Z = \{h \circ g\colon h \in C^*(Z)\}$. Then $F_Z \subset C^*(X)$.

I.2. LEMMA. *If $gZ$ is a compaction for $X$, then $F_Z$ is a closed subring of $C^*(X)$ containing the constant functions.*

Proof. Trivial. By *closed* is meant closed in the metric induced by the uniform norm.

I.3. DEFINITION. If $F$ is a closed subring of $C^*(X)$ which contains the constant functions, then by $\beta F$ is meant the *maximal ideal space of $F$ with the hull-kernel topology*. For a commutative ring with unit it is known that this space is compact, but possibly not Hausdorff. Our purpose is to indicate that $\beta F$ is Hausdorff and in a natural way a compaction for $X$. Let $e_F\colon X \to \beta F$ be defined by $e_F(x) = M_x$ where $M_x = \{f \in F\colon f(x) = 0\}$. Then $M_x$ can be shown to be a maximal ideal in $F$ and thus an element of $\beta F$.

I.4. THEOREM. *With the notation above $e_F \beta F$ is a compaction for $X$ whenever $F$ is a closed subring of $C^*(X)$ containing the constant functions.*



Proof. The proof will only be sketched. $F$ can be shown to be a sub-lattice of $C^*(X)$ and all maximal ideals in $F$ can be shown to be absolutely convex in $F$. Thus $F/M$ can be ordered in the way indicated in Chapter 5 of Gillman and Jerison [5]. $F/M$ can also be shown to be totally ordered and Archimedean. Therefore $F/M$ is isomorphic to the real numbers. If $g \in F$, then if we let $\hat{g}(M) = M + g$ in $F/M$, the function $\hat{g}$ will be continuous and an extension of $g$ to $\beta F$. The function $\hat{}\colon F \to C^*(\beta F)$ can be shown to be a ring isomorphism. Showing that $\hat{}$ is onto involves use of a form of the Stone–Weierstrass theorem. It follows that $\beta F$ must be a Hausdorff space since points are separated by real valued continuous functions. All that is needed now is a discussion of the continuity of $e_F$. If $g \in C^*(\beta F)$, then $g = \hat{h}$ for some $h \in F$. Thus $g \circ e_F = h$ is continuous. Since $\beta F$ is completely regular, this implies the continuity of $e_F$.

The isomorphism $\hat{}$ identifies $F$ with $C^*(\beta F)$. It will sometimes be convenient to consider this identification without specific reference to the function $\hat{}$. The next theorem is a consequence of I.2 and I.4.

I.5. THEOREM. *If $gZ$ is a compaction for $X$, then $\beta F_Z$ with $e_{F_Z}$ is an equivalent one. If $F$ is a closed subring of $C^*(X)$ containing the constants, then $F = F_{\beta F}$. Furthermore, if $gZ$ and $hY$ are compactions for $X$, then $gZ \geqslant hY$ if and only if $F_Z \supset F_Y$.*

Proof. It will be sufficient to prove the last part of the theorem. If $gZ \geqslant hY$, then clearly $F_Z \supset F_Y$. On the other hand, if $F = F_Z \supset F_Y = G$, then if we consider $e_F \beta F$ and $e_G \beta G$ and define $h(M) = M \cap G$ for all $M \in \beta F$, then using the Stone-Weierstrass theorem again we get that $h(M) \in \beta G$. It can be shown that $h$ is continuous and that $e_G = h \circ e_F$. Thus $e_F \beta F \geqslant e_G \beta G$. However, $gZ \sim e_F \beta F$ and $hY \sim e_G \beta G$. Therefore $gZ \geqslant hY$.

As a result of I.5 we have that $e_{C^*(X)} \beta C^*(X) \sim \beta X$.

I.6. THEOREM. *If $F$ is a closed subring of $C^*(X)$ containing the constants, then $e_F \beta F$ is a compactification of $X$ if and only if $F$ induces the topology of $X$.*

Proof. If $K$ is a closed subset of $X$ and $I_K = \{g \in F\colon g(K) = 0\}$, then the closure of $e_F(K)$ in $\beta F$ can be shown to be the set of all $M \in \beta F$ such that $M$ contains $I_K$. If $F$ induces the topology of $X$, then $e_F(K)$ contains all of the points in the closure of $e_F(K)$ which are also in $e_F(X)$. Thus $e_F$ is an embedding and $e_F \beta F$ is a compactification. The converse is trivial.

If $f\colon X \to Y$ is a mapping onto a dense subset of $Y$ and $gZ$ is a compactification for $X$ and $hW$ is a compaction for $Y$, then the following theorem holds.

I.7. THEOREM. *There is a mapping $m\colon Z \to W$ such that $m \circ g = h \circ f$ if and only if $h \circ fW \leqslant gZ$ as compactions of $X$.*

The theorem is trivial, but a useful rephrasing is the following: *defining* $f^*$: $C^*(Y)\to C^*(X)$ *by* $f^*(q) = q\circ f$ *for* $q\in C^*(Y)$, *then the mapping* $m$ *exists in* I.7 *if and only if* $(h\circ f)^*(F_W)\subset F_Z$.

The relation of $\dim X$ to $C^*(X)$ has received considerable attention. The next theorems are headed toward I.11 which will be a characterization of $\dim X$ in terms of $C^*(X)$ which will be of considerable usefulness later. To reduce terminology we define the following algebraic closure operation.

I.8. DEFINITION. If $L\subset C^*(X)$, then $L^c$ denotes the *smallest closed subring of* $C^*(X)$ *containing* $L$ *and the constants.*

The following result is essentially due to Mardešič [9]:

I.9. THEOREM. *The space* $X$ *has* $\dim X\leqslant n$ *if and only if whenever* $gZ$ *is a compaction for* $X$ *with* $Z$ *metrizable and* $\dim Z<\infty$, *then there is a compaction* $hY$ *for* $X$ *with* $Y$ *metrizable such that* $\dim Y\leqslant n$ *and* $hY\geqslant gZ$.

A rephrasing of I.9 is the following.

I.10. THEOREM. *For a space* $X$, $\dim X\leqslant n$ *if and only if whenever* $F=\{g_i\colon i=1,\ldots,p\}$ *is a subset of* $C^*(X)$, *then there is a set* $G=\{g_i\colon i=1,\ldots,k\}$ *in* $C^*(X)$ *containing* $F$ *such that* $\dim\beta G^c\leqslant n$.

Note that if $Y$ is compact metric (or just separable metric), then there are a finite number of real valued mappings which generate the topology of $Y$ if and only if $\dim Y<\infty$.

The next is just a sharpening of I.10.

I.11. THEOREM. *For a space* $X$ $\dim X\leqslant n$ *if and only if there is a dense subring* $F\subset C^*(X)$, $F=\{g_a\colon a\in A\}$, *such that if* $\{g_{a_i}\colon i=1,\ldots,k\}\subset F$, *then there are functions* $\{g_{a_i}\colon i=k+1,\ldots,p\}$ *such that if* $G=\{g_{a_i}\colon i=1,\ldots,p\}^c$, *then* $\dim\beta G\leqslant n$.

This is the last preliminary.

I.12. THEOREM. *If* $X$ *is compact and* $w(X)$ *is infinite, then* $w(X) = w(C^*(X))$. *If* $X$ *is not necessarily compact and there are* $a$ *functions in* $C^*(X)$ *which generate the topology of* $X$, $a$ *infinite, then* $w(X)\leqslant a$. *Again if* $w(X)$ *is infinite, then* $w(X)$ *is the least such* $a$.

*Equivalently, if* $w(X)$ *is infinite, then there is a closed subring* $F$ *in* $C^*(X)$ *containing the constants and generating the topology of* $X$ *such that* $w(F)=w(X)$.

Note that $w(F)=w(\beta F)$ by the first part of I.12.

**Part II. Open and closed mappings.** In the following let $f(X)=Y$ be a mapping.

II.1. DEFINITION. Let $f_+\colon B(X)\to B(Y)$ be defined by $f_+(g)(y) = \sup g(f^{-1}(y))$ for all $y\in Y$ for $g\in B(X)$.

II.2. LEMMA. $f_+$ *is a continuous function onto* $B(Y)$. *Also* $f_+(C^*(X))\supset C^*(Y)$.



Proof. If $g$ and $h$ are elements of $B(X)$ and $\delta$ is a positive number, then if $g$ and $h$ are within $\delta$ in uniform norm, then $f_+(g)$ and $f_+(h)$ are within $\delta$ in the uniform norm of $B(Y)$. Let $g\in B(Y)$. Then $g\circ f\in B(X)$. But $f_+(g\circ f)=g$ and therefore $f_+$ is onto. If $g\in C^*(Y)$, then $g\circ f\in C^*(X)$. Since $f_+(g\circ f)=g$ we have that $f_+(C^*(X))\supset C^*(Y)$.

II.3. THEOREM. *If* $f$ *is open and closed, then* $f_+(C^*(X))\subset C^*(Y)$.

Proof. Let $2^X$ be the space of non-empty closed subsets of $X$ with the finite topology as defined in Michael [10]. The function $F\colon Y\to 2^X$ defined by $F(y)=f^{-1}(y)$ is continuous by Theorem 5.10 in Michael [10]. Now let $g\in C^*(X)$. Then by Proposition 4.7 in Michael [10], the function $G\colon 2^X\to R$, defined by $G(K)=\sup g(K)$ for $K$ closed in $X$, is continuous. But $G\circ F(y)=\sup g(f^{-1}(y)) = f_+(g)(y)$. Therefore $G\circ F=f_+(g)\in C^*(Y)$.

II.4. THEOREM. *If* $X$ *is normal, then* $f$ *is open and closed if and only if* $f_+(C^*(X))\subset C^*(Y)$.

Proof. We need only show that $f$ is open and closed if $f_+$ has the stated property. Suppose that $x\in U$ with $U$ an open set in $X$. Let $g\in C^*(X)$ such that $g\colon X\to[0,1]$ with $g(x)=1$ and $g(X-U)=0$. Then $f_+(g)(f(x)) = 1$ and $f(U)\supset\{y\in Y\colon f_+(g)(y)\neq 0\}$. Since we are assuming that $f_+(g)$ is continuous on $Y$, we get that this last set is open and contains $f(x)$. Therefore $f(U)$ is open. Now suppose that $K$ is closed in $X$ and that $y\in Y-f(K)$. Then $f^{-1}(y)\cap K=\emptyset$. Let $g\colon X\to[0,1]$ be continuous so that $g(K)=1$ and $g(f^{-1}(y))=0$. Then let $V=\{y\in Y\colon f_+(g)<1\}$. Then $V$ is open, $y\in V$ and $V\cap f(K)=\emptyset$. This implies that $f(K)$ is closed.

II.5. Note. The proof indicates that even if $X$ is not assumed normal, if $f_+(C^*(X))\subset C^*(Y)$, then $f$ must be open. If $f$ has compact point inverses, then $f$ is open and closed if and only if $f_+(C^*(X))\subset C^*(Y)$.

Theorem II.4 will be our basic tool in showing the existence of open continuous extensions of open and closed mappings to compactifications.

II.6. THEOREM. *Let* $f(X_1)=X_2$ *be a mapping. Then if* $F_i=F_i^c$ *in* $C^*(X_i)$ *such that* $F_i$ *generates the topology of* $X_i$ *and such that* (1) $f^*(F_2)\subset F_1$ *and* (2) $f_+(F_1)\subset F_2$, *then there is an extension* $f'$ *of* $f$ *to* $\beta F_1$ *which maps onto* $\beta F_2$ *and this extension is open.*

Proof. The existence of the extension follows from the fact that $f^*(F_2)\subset F_1$. To demonstrate the openness of $f'$ let $g\in F_1$ and $g\colon X\to[0,1]$. Let $U=\{M\in\beta F_1\colon \hat{g}(M)>0\}$. Define $f_+(g)\hat{}$ to be the extension of $f_+(g)$ to $\beta F_2$ and let $V=\{M\in\beta F_2\colon f_+(g)\hat{}(M)>0\}$. We now want to show that $f'(U)=V$. Let $M\in U$. Since $f^*f_+(g)\geqslant g$ and $f^*f_+(g)$ has an extension to $\beta F_1$ we have $f^*f_+(g)\hat{}(M)\geqslant\hat{g}(M)$. It can be shown that $f^*f_+(g)\hat{}(M) = f_+(g)\hat{}(f'(M))$. Therefore $f_+(g)\hat{}(f'(M))>0$ and $f'(M)\in V$. Now let $M\in V$ and let $f_+(g)\hat{}(M)=\delta>0$. Let $S=\{Q\in\beta F_1\colon \hat{g}(Q)\geqslant\delta/2\}$. Then $S$ is closed in $\beta F_1$. Therefore $f'(S)$ is closed in $\beta F_2$. Now $f'(S)\supset\{x\in X_2\colon$

$f_+(g)(x) > \delta/2\} = A$. Since $M$ is in the closure of $A$, $M \in f'(S)$. Therefore $f'^{-1}(M) \cap S \neq \emptyset$. Since $S \subset U$, this implies that $M \in f'(U)$. Therefore $f'(U) = V$. We have now shown that $f'$ takes cozero sets to cozero sets. This implies the openness of $f'$.

It would be quite nice if II.6 had a converse and that properties (1) and (2) characterized the open extensions of $f$ to compactifications for an arbitrary mapping $f$. This is not the case, however, as the next example shows.

II.7. Example. Let $X = \{(x, y) \in R^2\colon 0 < x \leqslant 1,\ 0 \leqslant y \leqslant 1$ or $x = 0$ and $y = 0\}$. Let $Y = \{(x, y) \in R^2\colon y = 0$ and $0 \leqslant x \leqslant 1\}$. Then let $f(X) = Y$ be the projection $f(x, y) = (x, 0)$. Then let $X' = \{(x, y) \in R^2\colon 0 \leqslant x \leqslant 1$ and $0 \leqslant y \leqslant 1\}$. Then $X'$ is a compactification for $X$ and the extension of $f$ to $f'\colon X' \to Y$ is open. If we let $F = F^c \subset C^*(X)$ be the subring of $C^*(X)$ associated with $X'$, then the mapping $g\colon X \to [0, 1]$ defined by $g(x, y) = y$ is an element of $F$. However, $f_+(g)(0, 0) = 0$ and $f_+(g)(x, 0) = 1$ for $x > 0$. Therefore $f_+(g) \notin C^*(Y)$. Note that $f$ is open but not closed.

We now give two applications of our theory of open and closed mappings. The first has to do with constructing compactifications satisfying certain conditions and allowing open extensions of a collection of open and closed mappings. The second application is in Part III of the paper.

II.8. Theorem. *Suppose that $f_a(X) = Y_a$ is an open and closed mapping for all $a \in A$. Then if $\tau$ is an infinite cardinal with $|A| \leqslant \tau$ and $w(X) \leqslant \tau$, then there are compactifications $Y'_a$ for $Y_a$ for each $a$ and $X'$ for $X$ such that:*

(1) *$f_a$ has an extension $f'_a(X') = Y'_a$ with $f'_a$ and open mapping;*

(2) *$w(X') \leqslant \tau$, $w(Y'_a) \leqslant \tau$ for all $a \in A$; and*

(3) *$\dim X' \leqslant \dim X$, $\dim Y'_a \leqslant \dim Y_a$ for all $a \in A$.*

Proof. Note that $w(Y_a) \leqslant \tau$ for all $a$ by the openness of $f_a$. Now let $L = L^c \subset C^*(X)$ be any subring generating the topology of $X$ with $w(L) \leqslant \tau$. Let $\{g_\beta\colon \beta \in B_0\}$ be a dense subring of $L$ such that $|B_0| \leqslant \tau$. By induction it is possible to obtain collections $\{B_i\}_{i=0}^\infty$ and $\{C_i^a\}_{i=1}^\infty$ such that (1) $F_1 = \{g_\beta\colon \beta \in B_i\}$ is a subring of $C^*(X)$; (2) $G_i^a = \{g_\gamma\colon \gamma \in C_i^a\}$ is a subring of $C^*(Y_a)$; (3) $B_i \subset B_{i+1}$; (4) $C_i^a \subset C_{i+1}^a$ for each $a$; (5) $f_{a+}(F_i) \subset G_i^a$; (6) $f_a^*(G_i^a) \subset F_{i+1}$; (7) $|B_i| \leqslant \tau$; (8) $|C_i^a| \leqslant \tau$; (9) if $\{\beta_1, \ldots, \beta_k\} \subset B_i$, then there is a set $\{\beta_{k+1}, \ldots, \beta_p\} \subset B_{i+1}$ such that $\dim\beta\{g_{\beta_1}, \ldots, g_{\beta_p}\}^c \leqslant \dim X$; and (10) if $\{\gamma_1, \ldots, \gamma_k\} \subset C_i^a$, then there is a set $\{\gamma_{k+1}, \ldots, \gamma_p\} \subset C_{i+1}^a$ such that $\dim\beta\{g_{\gamma_1}, \ldots, g_{\gamma_p}\}^c \leqslant \dim Y_a$. The construction of these collections $\{B_i\}$ and $\{C_i^a\}$ is rather tedious. Properties (9) and (10) of the collections can be satisfied using I.10 of the preliminaries. The construction will not be belabored here.

Now let $F = (\bigcup_{i=1}^\infty F_i)^c$ and $G_a = (\bigcup_{i=1}^\infty G_i^a)^c$. Then $G_a$ and $F$ are just the closures of $\bigcup_{i=1}^\infty F_i$ and $\bigcup_{i=1}^\infty G_i^a$, respectively. By the continuity of $f_{a+}\colon C^*(X) \to C^*(Y_a)$, $f_{a+}(F) \subset G_a$ using properties (3), (4), and (5) above. By the continuity of $f_a^*\colon C^*(Y_a) \to C^*(X)$, $f_a^*(G_a) \subset F$ using properties (3), (4), and (6) above. By Theorem II.6 each $f_a$ has an extension to $\beta F$ which maps onto $\beta G_a$ with the extension being open. Now $\dim\beta F \leqslant \dim X$ by I.11 and (9) above. Similarly $\dim\beta G_a \leqslant \dim Y_a$ by I.11 and (10) above. Using I.12 we get that $w(\beta F) \leqslant w(F)$ and $w(\beta G_a) \leqslant w(G_a)$. But by (7), $w(F) \leqslant \tau$, and by (8), $w(G_a) \leqslant \tau$.

II.9. Note. We can require $\dim X' = \dim X$ and $\dim Y'_a = \dim Y_a$ in II.8. This can be done by requiring $F_0$ to contain a finite collection of functions $\{g_1, \ldots, g_k\}$ such that if $F = F^c \subset C^*(X)$ and $\{g_1, \ldots, g_k\} \subset F$, then $\dim\beta F \geqslant \dim X$. If $\dim X$ is infinite, we may be required to use a countable collection $\{g_i\}_{i=1}^\infty$. In any case this requirement can be satisfied without altering the properties (1), ..., (10) in the construction. But then we would have $\dim\beta F \geqslant \dim X$ and thus $\dim\beta F = \dim X$. In a similar manner we can require $\dim Y_a = \dim\beta G_a$ for all $a$ simultaneously.

II.10. Corollary. *If $f_i(X) = Y_i$ is a countable collection of open and closed mappings with $X$ separable metrizable, then there are metric compactifications $X'$ and $Y'_i$ for $X$ and $Y_i$, respectively, such that:*

(1) *$f_i$ has an extension $f'_i(X') = Y'_i$ which is open and*

(2) *$\dim X' = \dim X$, $\dim Y'_i = \dim Y_i$.*

Proof. This is nothing more than the case that $\tau$ is the first infinite cardinal in II.8.

II.11. Example. Let $f(Z) = Y$ be an open mapping such that $Z$ and $Y$ are separable metrizable and $\dim Z = 0$. Such mappings exist with $\dim Y > 0$. But if $\dim Y > 0$, then there is no compactification $Z'$ of $Z$ allowing an open extension of $f$ to $Z'$ with $\dim Z' = 0$ onto a compactification $Y'$ of $Y$ even if $Z'$ is not required to be metrizable. Therefore the requirement that each $f_a$ be closed in addition to being open was not superfluous in either II.8 or II.10.

**Part III. Open and closed finite to one mappings.** In then following let $f(X) = Y$ be an open and closed mapping which is finite to one.

III.1. Theorem. $\dim X = \dim Y$.

Proof. We use the standard result that if $X \subset X' \subset \beta X$, then $\dim X = \dim X'$. Let $\beta f(\beta X) = \beta Y$ be the Stone extension of $f$. Then $\beta f$ is open by II.3 and II.6. Let $Y_k = \{y \in \beta Y\colon \beta f^{-1}(y)$ has at most $k$ points$\}$ and $X_k = \beta f^{-1}(Y_k)$. Since $f$ is a proper mapping $\beta f^{-1}(y) = f^{-1}(y)$ for all $y \in Y$. Therefore if we let $Y' = \bigcup_{i=1}^\infty Y_i$ and $X' = \bigcup_{i=1}^\infty X_i$, then $Y \subset Y' \subset \beta Y$ and

$X \subset X' \subset \beta X$. Since $\beta f$ is open, $Y_k$ and $X_k$ are each closed in $\beta Y$ and $\beta X$, respectively. Therefore $X'$ and $Y'$ are both $\sigma$-compact, Lindelöf, and paracompact. The mapping $\beta f|X'$ is open onto $Y'$ and thus $\dim X' = \dim Y'$ by Theorem 4.1 of Nagami [11] or by III.2 below. Therefore $\dim X = \dim Y$.

III.2. THEOREM. *If $X$ is normal, then for every closed set $K \subset X$ we get that* $\dim K = \dim f(K)$.

Let us prove a special case first.

III.3. LEMMA. *If $X$ is compact and the number of points in a point inverse of $f$ is bounded by some natural number $n$, then* III.2 *holds.*

Proof. The proof is by induction on $n$. In case $n = 1$, then $f$ is a homeomorphism on $X$ and $\dim K = \dim f(K)$ holds for every subset $K$ of $X$. Now suppose that it is true for all lesser values of $n$ and let $n > 1$. Then let $K \subset X$ be closed. Then if $Y_{n-1} = \{y \in Y\colon f^{-1}(y)$ has at most $n-1$ points$\}$ and $X_{n-1} = f^{-1}(Y_{n-1})$, then $Y_{n-1}$ and $X_{n-1}$ are compact and $f|X_{n-1}\colon X_{n-1} \to Y_{n-1}$ is open with $|f^{-1}(y)| \leqslant n-1$ for all $y \in Y_{n-1}$. By the induction assumption $\dim K \cap X_{n-1} = \dim f(K \cap X_{n-1})$. Now let $L$ be any compact subset of $K \cap (X_n - X_{n-1})$, that is, any closed subset of $K$ which is separated from $K \cap X_{n-1}$. If we can show that $\dim L = \dim f(L)$ for all such $L$, then we will have $\dim K = \dim f(K)$ by Theorem 6, p. 79 of Isbell [7]. But $f|(X_n - X_{n-1})$ is an open mapping which is exactly $n$ to one, and thus a local homeomorphism onto $Y_n - Y_{n-1}$. Thus we can find a finite closed cover $\{L_i\colon i = 1, \ldots, k\}$ of $L$ such that $\bigcup_{i=1}^{k} L_i = L$ and $f|L_i$ is a homeomorphism onto $f(L_i)$. But then $\dim L_i = \dim f(L_i)$. By the finite sum theorem for uniform spaces, or the general sum theorem for normal spaces (Isbell [7], p. 80, Corollary 8 or Nagata [12], p. 193, Theorem VII.2) we get that $\dim L = \dim f(L)$. Therefore $\dim K = \dim f(K)$.

Proof of III.2. First note that $Y$ must also be normal since $f$ is closed. Let $Y_k = \{y \in \beta Y\colon \beta f^{-1}(y)$ has at most $k$ elements$\}$ and $X_k = \beta f^{-1}(Y_k)$. Let $X' = \bigcup_{k=1}^{\infty} X_k$ and $Y' = \bigcup_{k=1}^{\infty} Y_k$. As in the proof of III.1, $X \subset X'$ and $Y \subset Y'$. Since $K$ is closed in $X$ and $f(K)$ is closed in $Y$, $\beta K$ and $\beta(f(K))$ are in a natural way just the closure of $K$ (resp., $f(K)$) in $\beta X$ (resp., in $\beta Y$). Let $K'$ be the closure of $K$ in $X'$. Then $K \subset K' \subset \beta K$ and $f(K) \subset \beta f(K') \subset \beta(f(K))$. Therefore it will be sufficient to show that $\dim K' = \dim \beta f(K')$. If we let $K_i = K' \subset X_i$, then $\dim K_i = \dim \beta f(K_i)$ by III.3. Again note that $K'$ and $\beta f(K')$ are $\sigma$-compact and thus normal. So we can apply Theorem VII.2 p. 193 of Nagata [12] to get that $\dim K' = \sup \dim K_i = \sup \dim \beta f(K_i) = \dim \beta f(K')$. Thus $\dim K = \dim f(K)$.

The author has shown that III.2 holds for finite to one open mappings between metric spaces without assuming $f$ to be closed [8].



**References**

[1] A. Arhangelskii, *A theorem on the metrizability of the inverse image of a metric space under an open-closed finite-to-one mapping: Example and unsolved problems*, (in Russian) Dokl. Akad. Nauk SSSR 170 (1966), pp. 1258-1262.

[2] R. Engelking, *Sur la compactification des espaces métriques*, Fund. Math. 48 (1960), pp. 321-324.

[3] — and E. G. Skljarenko, *On compactifications allowing extensions of mappings*, Fund. Math. 53 (1963), pp. 65-79.

[4] A. B. Forge, *Dimension preserving compactifications allowing extensions of continuous functions*, Duke Math. J. 28 (1961), pp. 625-627.

[5] L. Gillman and M. Jerison, *Rings of continuous functions*, Van Nostrand, 1960.

[6] J. de Groot and R. H. McDowell, *Extensions of mappings on metric spaces*, Fund. Math. 48 (1960), pp. 251-263.

[7] J. R. Isbell, *Uniform spaces*, Math. Surveys No. 12, Amer. Math. Soc., 1964.

[8] J. Keesling, *Mappings and dimension in general metric spaces*, Pac. J. Math. 25 (1968), pp. 277-288.

[9] S. Mardešić, *On covering dimension and inverse limits of compact spaces*, Illinois J. Math. 4 (1960), pp. 278-291.

[10] E. Michael, *Topologies on spaces of subsets*, Trans. Amer. Math. Soc. 71 (1951), pp. 152-182.

[11] K. Nagami, *Mappings of finite order and dimension theory*, Japan J. Math. 30 (1950), pp. 25-54.

[12] J. Nagata, *Modern dimension theory*, John Wiley, 1965.

[13] H. de Vries, *Compact spaces and compactification*, Van Gorcum, 1963.

[14] A. Zarelua, *A universal bicompactum of given weight and dimension*, (in Russian) Dokl. Akad. Nauk SSSR 154 (1964), pp. 1015-1018.

[15] — *The continuation of mappings to extensions with certain special properties*, (in Russian) Sibirsk. Math. Z. 5 (1964), pp. 532-548.

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