A generalized version of the Vietoris-Begle Theorem *

by

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1. Introduction. In 1961 a paper by A. D. Wallace [10] appeared in which he showed that a topological space which supports a relation with certain properties is acyclic; thus he was able to interrelate the notions of order and algebraic topology. The purpose of this paper is to further the study of the relationship between order and algebraic topology by presenting a generalized version of the Vietoris-Begle Theorem (see [1], [2], [4], and [7]) in an order theoretic setting. Many of the techniques employed are reminiscent of those used by Wallace. Other generalizations of the Vietoris-Begle Theorem have been given by A. Białynicki-Birula [3] and E. G. Skljarenko [5]. The theorem of Białynicki-Birula is a special case of Theorem 3.6 in which R is an equivalence relation.

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2. Preliminaries. Let G_1 and G_2 be groups. If f is a homomorphism from G_1 into G_2, then f is a *monomorphism* if f is one-to-one, f is an *epimorphism* if $f(G_1) = G_2$, and f is an *isomorphism* if f is both an epimorphism and a monomorphism.

Throughout the remainder of this section let G denote a fixed commutative group. We denote the nth Alexander-Wallace-Spanier cohomology group of a topological space X relative to a subset A with G as coefficient group by $H^n(X, A)$. Basic theorems and notation of the Alexander-Wallace-Spanier cohomology theory may be found in [6].

Let A and B be subsets of a topological space X such that $B \subseteq A$; let i be the injection of B into A (denoted by $i\colon B \subseteq A$). If $e \in H^n(A)$, we denote $i^*(e) \in H^n(B)$ by $e|B$.

For a compact, Hausdorff space X and a closed subset A, Wallace [9] has proved the following two results.

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2.1. The Extension Theorem. *If $e \in H^n(A)$, then there exists an open set P which contains A such that e can be extended to $H^n(P^*)$, i.e., if $j\colon A \subseteq P^*$, then $e \in j^*(H^n(P^*))$.*

2.2. The Reduction Theorem. *If $e \in H^n(X)$ and $e|A = 0$, then there exists an open set Q containing A such that $e|Q^* = 0$.*

The Mayer-Vietoris Theorem may be found in Spanier's book [7]. The following formulation is due to Wallace [8].

2.3. The Mayer-Vietoris Theorem. *Let X be a compact, Hausdorff space with closed subsets X_1 and X_2 such that $X = X_1 \cup X_2$. Then there exists an exact sequence*

$$\ldots H^n(X) \xrightarrow{J} H^n(X_1)\times H^n(X_2) \xrightarrow{I} H^n(X_1 \cap X_2) \xrightarrow{\Delta} H^{n+1}(X)\ldots$$

If X is a topological space, a *closed relation* on X is a closed subset of $X \times X$. If R is a closed relation on X and $A \subseteq X$, then

$$L(A) = \{x \in X\colon (x, a) \in R \text{ for some } a \in A\}$$
$$M(A) = \{y \in X\colon (a, y) \in R \text{ for some } a \in A\}$$

2.4. Lemma. *Let R be a closed relation on a topological space X.*

(i) *If A and B are subsets of X, then*

$$L(A) \cup L(B) = L(A \cup B)\,.$$

(ii) *If T is a compact subset of X, then $L(T)$ is closed.*

(iii) *If $\{A_\gamma\}_{\gamma\in\Gamma}$ is a tower of compact subsets, then*

$$L\Big(\bigcap_{\gamma\in\Gamma} A_\gamma\Big) = \bigcap_{\gamma\in\Gamma} L(A_\gamma)\,.$$

Proof. Part (i) is an easy consequence of the definition of the L operator. For part (ii), let $\{x_a\}_{a\in D}$ be a net in $L(T)$ which converges to x. For each $a \in D$ there exists $t_a \in T$ such that $(x_a, t_a) \in R$. Since T is compact, the net $\{t_a\}_{a\in D}$ clusters to some $t \in T$. Then $\{(x_a, t_a)\}_{a\in D}$ clusters to (x, t); since R is closed, we have $(x, t) \in R$. Hence $x \in L(T)$.

For part (iii) let $P = \bigcap_{\gamma\in\Gamma} A_\gamma$. The inclusion $L(P) \subseteq \bigcap_{\gamma\in\Gamma} L(A_\gamma)$ is immediate. Let $y \in \bigcap_{\gamma\in\Gamma} L(A_\gamma)$. By the dual of part (ii), $M(y)$ is a closed set; hence $\{M(y) \cap A_\gamma\}_{\gamma\in\Gamma}$ is a tower of non-empty compact sets. Thus there exists $a \in \bigcap_{\gamma\in\Gamma} (M(y) \cap A_\gamma)$; then $a \in P$ and $y \in L(a)$.

3. The Generalized Vietoris-Begle Theorem.

We shall have need of the following purely algebraic lemma. The proof is only a slight modification of the proof of the five lemma given by Spanier [7].

3.1. Lemma. *Suppose that in the following diagram the Latin letters represent abelian groups and that the arrows represent homomorphisms:*

$$\begin{array}{ccccccc}
A & \longrightarrow & B & \longrightarrow & C & \xrightarrow{\tau} & D \\
\downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\lambda} \\
E & \longrightarrow & F & \xrightarrow{\sigma} & G & \longrightarrow & H
\end{array}$$

Suppose further that the rows are exact, the squares commute, and that α is an epimorphism.

(i) *If λ is a monomorphism and $e \in F$ such that $\sigma(e) \in \gamma(C)$, then $e \in \beta(B)$.*

(ii) *If β is a monomorphism and if $d \in (\text{kernel } \tau \cap \text{kernel } \gamma)$, then $d = 0$.*

We now introduce some notation that will be employed throughout the remainder of this paper. Let f be a continuous function from X into Y, and let Y_n be a subset of Y. Then we denote by X_n the set $f^{-1}(Y_n)$ and by f_n the restriction of f from X_n into Y_n, i.e., $f_n = f|(f^{-1}(Y_n))$. If Y_1 and Y_2 are subsets of Y, we define the homomorphism $f_1^*\times f_2^*$ from $H^p(Y_1)\times H^p(Y_2)$ into $H^p(X_1)\times H^p(X_2)$ by

$$(f_1^*\times f_2^*)(g, h) = \big(f_1^*(g), f_2^*(h)\big) \quad \text{for} \quad (g, h) \in H^p(Y_1)\times H^p(Y_2)\,.$$

3.2. Lemma. *Let Y be a compact space, A a closed subset of Y, and $h \in H^p(A)$. If $\mathbf{M}$ is a non-empty tower of closed subsets of A such that $h|M \neq 0$ for every $M \in \mathbf{M}$, then $h|\bigcap_{M\in\mathbf{M}} M \neq 0$.*

The preceding lemma is an easy consequence of the reduction theorem.

3.3. Lemma. *Let X and Y be compact, Hausdorff spaces, and let f be a continuous function from X onto Y. Let Y_1 be a closed subset of Y and $h \in H^p(X)$. If $h|X_1 \in f_1^*(H^p(Y_1))$, then there exists Y_2, a closed subset of Y, such that $Y_1 \subseteq Y_2^0$ and $h|X_2 \in f_2^*(H^p(Y_2))$.*

Proof. Since $h|X_1 \in f_1^*(H^p(Y_1))$, there exists $g \in H^p(Y_1)$ such that $f_1^*(g) = h|X_1$. By the extension theorem there exists an open set U such that $Y_1 \subseteq U$ and g can be extended to U^*. We set $Y_3 = U^*$. Let $i\colon Y_1 \subseteq Y_3$ and $j\colon X_1 \subseteq X_3$. Since $f_3 j = i f_1$, we have $j^* f_3^* = f_1^* i^*$.

Since $g \in i^*(H^p(Y_3))$, there exists $g_1 \in H^p(Y_3)$ such that $i^*(g_1) = g$. If $h_1 = f_3^*(g_1)$, then

$$\begin{aligned} j^*(h_1 - h|X_3) &= j^*(h_1) - j^*(h|X_3) = j^* f_3^*(g_1) - h|X_1 \\ &= f_1^* i^*(g_1) - h|X_1 = f_1^*(g) - h|X_1 = 0\,. \end{aligned}$$

We set $a = h_1 - h|X_3$. Since we have just shown that $a|X_1 = 0$, by the reduction theorem there exists a set W open in X_3 such that $X_1 \subseteq W$ and $a|W^* = 0$. Then $W \cap f^{-1}(U)$ is open in X since $f^{-1}(U) \subseteq X_3$. Since f is closed, there exists an open set T in Y such that $Y_1 \subseteq T \subseteq U$ and $f^{-1}(T) \subseteq W \cap f^{-1}(U)$, e.g., we may take $T = Y \setminus f\big(X \setminus W \cap f^{-1}(U)\big)$.

Since Y is normal, there exists an open set V such that $Y_1 \subseteq V$ and $V^* \subseteq T$; then $f^{-1}(V^*) \subseteq f^{-1}(T) \subseteq W^*$. Since $a|W^* = 0$, we have $a|f^{-1}(V^*) = 0$.

We set $Y_2 = V^*$. Let $g\colon X_2 \subseteq X_3$ and $\gamma\colon Y_2 \subseteq Y_3$. The following diagram is commutative:

$$\begin{array}{ccc} H^p(X_3) & \xrightarrow{g^*} & H^p(X_2) \\ \uparrow{\scriptstyle f_3^*} & & \uparrow{\scriptstyle f_2^*} \\ H^p(Y_3) & \xrightarrow{\gamma^*} & H^p(Y_2)\,. \end{array}$$

We thus have

$$\begin{aligned} 0 = a|X_2 &= g^*(a) = g^*(h_1 - h|X_3) = g^*(h_1) - g^*(h|X_3) \\ &= g^* f_3^*(g_1) - h|X_2 = f_2^*\gamma^*(g_1) - h|X_2\,. \end{aligned}$$

Hence we have $h|X_2 \in f_2^*\big(H^p(Y_2)\big)$.

3.4. Lemma. *Let X and Y be compact, Hausdorff spaces, and f a continuous mapping from X onto Y. Let $h \in H^p(X)$ and let $\{Y_\gamma\}_{\gamma\in\Gamma}$ be a non-empty tower of closed subsets of Y such that $h|X_\gamma \notin f_\gamma^*\big(H^p(Y_\gamma)\big)$ for all $\gamma \in \Gamma$. If $Y_1 = \bigcap_{\gamma\in\Gamma} Y_\gamma$, then $h|X_1 \notin f_1^*\big(H^p(Y_1)\big)$.*

Proof. We suppose that $h|X_1 \in f_1^*\big(H^p(Y_1)\big)$ and show this assumption leads to a contradiction. Let $g \in H^p(Y_1)$ such that $f_1^*(g) = h|X_1$. By Lemma 3.3 there exists a closed subset Y_2 of Y such that $Y_1 \subseteq Y_2^0$ and $h|X_2 \in f_2^*\big(H^p(Y_2)\big)$. Since $Y_1 = \bigcap_{\gamma\in\Gamma} Y_\gamma$ there exists $\alpha \in \Gamma$ such that $Y_\alpha \subseteq Y_2^0$. Let $i\colon X_\alpha \subseteq X_2$ and $j\colon Y_\alpha \subseteq Y_2$. The following diagram is commutative:

$$\begin{array}{ccc} H^p(Y_2) & \xrightarrow{j^*} & H^p(Y_\alpha) \\ \downarrow{\scriptstyle f_2^*} & & \downarrow{\scriptstyle f_\alpha^*} \\ H^p(X_2) & \xrightarrow{i^*} & H^p(X_\alpha) \end{array}$$

Since $h|X_2 \in f_2^*\big(H^p(Y_2)\big)$, there exists $e \in H^p(Y_2)$ such that $f_2^*(e) = h|X_2$. Then we have $h|X_\alpha = i^*(h|X_2) = i^* f_2^*(e) = f_\alpha^* j^*(e) = f_\alpha^*\big(j^*(e)\big)$. Hence $h|X_\alpha \in f_\alpha^*\big(H^p(Y_\alpha)\big)$; thus we have reached a contradiction.

We now marshall our forces and proceed toward the main theorem. First, however, we introduce some additional notation. Let f be a continuous function from X onto Y, R a closed relation on Y, and C a closed subset of Y. Then f_C will denote the restriction of f to $f^{-1}\big(L(C)\big)$ with range $L(C)$. This notation differs from that employed for subscripted subsets of Y. In the following theorem both conventions are employed.

3.5. Theorem. *Let f, X, Y, and R satisfy the following hypotheses:*

(i) *X and Y are compact and Hausdorff, and f is a mapping from X onto Y.*

(ii) *R is a closed relation on Y.*

(iii) *If A and B are closed subsets of Y, there exists a closed subset C of Y such that $L(A) \cap L(B) = L(C)$.*

(iv) *If $y \in Y$, then $f_y^*\colon H^p\big(L(y)\big) \to H^p\big(f^{-1}(L(y))\big)$ is an isomorphism for $p = 0, \dots, n$.*

Then for any closed subset D of Y the induced homomorphism $f_D^*\colon H^p\big(L(D)\big) \to H^p\big(f^{-1}(L(D))\big)$ is an isomorphism for $p = 0, \dots, n$.

Proof. We suppose that for some non-negative integer $m \leqslant n$ the conclusion is false for $p = m$. We may assume that m is the least such integer.

If m is greater than 0, then for any closed subset C of Y the homomorphism $f_C^*\colon H^p\big(L(C)\big) \to H^p\big(f^{-1}(L(C))\big)$ is an isomorphism for $p = 0, \dots, m-1$; otherwise, m would not be the least integer for which the conclusion fails. We reach a contradiction by showing that f_C^* is also an isomorphism for $p = m$.

We first show that for any closed subset C of Y the homomorphism f_C^* is a monomorphism in dimension m. If it is not, then there exists a non-zero $h \in H^m\big(L(C)\big)$ such that $f_C^*(h) = 0$. Let $\boldsymbol{M}$ be a maximal tower of closed subsets of C such that $h|L(M) \neq 0$ for each $M \in \boldsymbol{M}$. The tower is non-empty since $C \in \boldsymbol{M}$. We set $B = \bigcap_{M\in\boldsymbol{M}} M$. By Lemma 2.4, $L(B) = \bigcap_{M\in\boldsymbol{M}} L(M)$. By Lemma 3.2 we have $h|L(B) \neq 0$. By the maximality of the tower $\boldsymbol{M}$, we have $B \in \boldsymbol{M}$ and B is a minimal element for $\boldsymbol{M}$.

We set $Y_1 = L(B)$ and $g = h|L(B)$. Let $t\colon X_1 \subseteq f^{-1}\big(L(C)\big)$ and $\tau\colon Y_1 \subseteq L(C)$. Then $g = \tau^*(h)$. Since $\tau f_1 = f_C t$, we have $f_1^*(g) = f_1^*\tau^*(h) = t^* f_C^*(h) = t^*(0) = 0$. Hence f_1^* is not a monomorphism. By hypothesis (iv) B is not a singleton.

Since B is not a singleton, there exist proper closed subsets M and N of B such that $B = M \cup N$. We set $Y_2 = L(M)$, $Y_3 = L(N)$, and $Y_4 = Y_2 \cap Y_3$. By Lemma 2.4, $Y_1 = Y_2 \cup Y_3$; by hypothesis (iii) there exists a closed subset P such that $Y_4 = L(P)$. Since m is the least integer for which the theorem fails, we have that $f_i^*\colon H^{m-1}(Y_i) \to H^{m-1}(X_i)$ is an isomorphism for $i = 2, 3, 4$ (with the convention if $m = 0$, then we define the -1 groups to be trivial).

If m is greater than 0, then by the Mayer-Vietoris Theorem the rows in the following diagram are exact:

$$\begin{array}{ccccccc} H^{m-1}(Y_2) \times H^{m-1}(Y_3) & \xrightarrow{I} & H^{m-1}(Y_4) & \xrightarrow{\Delta} & H^m(Y_1) & \xrightarrow{J} & H^m(Y_2) \times H^m(Y_3) \\ \downarrow{\scriptstyle f_2^*\times f_3^*} & & \downarrow{\scriptstyle f_4^*} & & \downarrow{\scriptstyle f_1^*} & & \downarrow{\scriptstyle f_2^*\times f_3^*} \\ H^{m-1}(X_2) \times H^{m-1}(X_3) & \xrightarrow{I} & H^{m-1}(X_4) & \xrightarrow{\Delta} & H^m(X_1) & \xrightarrow{J} & H^m(X_2) \times H^m(X_3)\,. \end{array}$$

The definitions of I, J, and Δ (see [8]) and some straightforward calculations yield that the squares in the preceding diagram are commutative. With the preceding convention that negative groups are trivial, the rows are still exact if $m = 0$ since J is a monomorphism in dimension 0.

Each square remains commutative since the only new homomorphisms introduced are trivial.

Since f_2^* and f_3^* are isomorphisms in dimension $m-1$, so also is $f_2^*\times f_3^*$. Let $\pi_i\colon Y_i\subseteq Y_1$ for $i=2,3$. Since M and N are proper subsets of B and B is minimal in $\boldsymbol{M}$, we have $h|Y_2=0$ and $h|Y_3=0$, i.e., $\pi_2^*(g)=0$ and $\pi_3^*(g)=0$. Hence we have $J(g)=\big(\pi_2^*(g),\pi_3^*(g)\big)=(0,0)$. By part (ii) of Lemma 3.1, we conclude that $g=0$. Since this last conclusion is impossible, we have that f_C^* is a monomorphism in dimension m for each closed subset C of Y.

Since we have assumed the theorem fails for m and we have just shown each f_C is a monomorphism in dimension m for each closed set C, there exists a closed subset A of Y such that $f_A^*\colon H^m\big(L(A)\big)\to H^m\big(f^{-1}(L(A))\big)$ is not a epimorphism. Thus there exists $e\in H^m\big(f^{-1}(L(A))\big)$ such that $e\notin f_A^*\big(H^m(L(A))\big)$. We set

$$\boldsymbol{B}=\Big\{K\subseteq A\colon K\text{ is closed, } e|f^{-1}\big(L(K)\big)\notin f_K^*\big(H^m(L(K))\big)\Big\}.$$

Let $\boldsymbol{D}$ be a maximal tower in $\boldsymbol{B}$. Since $A\in\boldsymbol{D}$, we have that $\boldsymbol{D}\neq\square$. We set $F=\bigcap_{D\in\boldsymbol{D}}D$; by Lemma 2.4 $L(F)=\bigcap_{D\in\boldsymbol{D}}L(D)$. Applying Lemma 3.4 to f_A, $L(A)$, and $f^{-1}\big(L(A)\big)$, we conclude that $e|f^{-1}\big(L(F)\big)\notin f_F^*\big(H^m(L(F))\big)$. Since $\boldsymbol{D}$ is a maximal tower in $\boldsymbol{B}$, we have that $F\in\boldsymbol{D}$ and F is a minimal element in $\boldsymbol{B}$.

Since f_F^* is not an epimorphism in dimension m, by part (iv) of the hypothesis F is not a singleton set. Hence there exist proper closed subset R and S of F such that $F=R\cup S$. We set $Y_1=L(F)$, $Y_2=L(R)$, $Y_3=L(S)$, and $Y_4=L(R)\cap L(S)$. By part (iii) of the hypothesis there exists a closed subset T of Y such that $Y_4=L(T)$. As we mentioned previously f_4^* is an isomorphism from $H^{m-1}(Y_4)$ onto $H^{m-1}(X_4)$. By the first part of the proof $f_4\colon H^m(Y_4)\to H^m(X_4)$ is a monomorphism.

The rows in the following diagram are exact since they form part of the Mayer-Vietoris exact sequence:

$$\begin{array}{ccccccc}
H^{m-1}(Y_4) & \xrightarrow{\Delta} & H^m(Y_1) & \xrightarrow{J} & H^m(Y_2)\times H^m(Y_3) & \xrightarrow{I} & H^m(Y_4)\\
\downarrow{\scriptstyle f_4^*} & & \downarrow{\scriptstyle f_1^*} & & \downarrow{\scriptstyle f_2^*\times f_3^*} & & \downarrow{\scriptstyle f_4^*}\\
H^{m-1}(X_4) & \xrightarrow{\Delta} & H^m(X_1) & \xrightarrow{J} & H^m(X_2)\times H^m(X_3) & \xrightarrow{I} & H^m(X_4).
\end{array}$$

The squares are commutative as before. Special care must be taken if $m=0$; we omit the details, however, since they resemble those in the first part of the proof. Since R and S are proper subsets of F and F is minimal in $\boldsymbol{B}$, we have $e|X_i\in f_i^*\big(H^m(Y_i)\big)$ for $i=2,3$, i.e., there exists $g_i\in H^m(Y_i)$ such that $f_i^*(g_i)=e|X_i$ for $i=2,3$. We then have that $J(e|X_1)=(e|X_2,\ e|X_3)=\big(f_2^*(g_2),f_3^*(g_3)\big)=(f_2^*\times f_3^*)(g_2,g_3)$. Hence we conclude that $J(e|X_1)\in(f_2^*\times f_3^*)\big(H^m(Y_2)\times H^m(Y_3)\big)$. From this paragraph and the preceding one we see that the hypotheses of part (i) of Lemma 3.1 are satisfied. Thus we have that $e|X_1\in f_1^*\big(H^m(Y_1)\big)$, i.e., $e|f^{-1}\big(L(F)\big)\in f_F^*\big(H^m(L(F))\big)$. This last statement is impossible since $F\in\boldsymbol{B}$. Whence $f_A^*\colon H^m\big(L(A)\big)\to H^m\big(f^{-1}(L(A))\big)$ is an epimorphism.

3.6. Theorem. *Let X, Y, f and R satisfy the hypotheses of the preceding theorem except that part* (iv) *of the hypotheses is replaced by the following*:

(iv') *For each $y\in Y$ the homomorphism $f_y^*\colon H^p\big(L(y)\big)\to H^p\big(f^{-1}(L(y))\big)$ is an isomorphism for $p=0,\dots,n-1$ and a monomorphism for $p=n$.*

Then for any closed subset A of Y, $f_A^*\colon H^p\big(L(A)\big)\to H^p\big(f^{-1}(L(A))\big)$ is an isomorphism for $p=0,\dots,n-1$ and a monomorphism for $p=n$.

Proof. That f_A^* is an isomorphism in dimensions 0 through $n-1$ follows immediately from Theorem 3.5. To show that f_A^* is a monomorphism in dimension n, we simply repeat the first part of the proof of Theorem 3.5.

3.7. Corollary. (Vietoris-Begle Mapping Theorem). *Let X and Y be compact and Hausdorff, and let $f\colon X\to Y$ be continuous and onto. If $\big(f|f^{-1}(y)\big)^*\colon H^p(\{y\})\to H^p\big(f^{-1}(y)\big)$ is an isomorphism for each $y\in Y$ for $p=0,\dots,n-1$, then $\big(f|f^{-1}(A)\big)^*\colon H^p(A)\to H^p\big(f^{-1}(A)\big)$ is an isomorphism for each closed subset A of Y for $p=0,\dots,n-1$ and a monomorphism for $p=n$.*

Proof. We define a relation $R=\{(y,y)\colon y\in Y\}$. This relation is easily seen to be closed and to satisfy the condition $L(A)\cap L(B)=L(C)$ for closed sets A and B of Y. Since $L(y)=\{y\}$, $f|\big(f^{-1}(L(y))\big)$ induces an isomorphism in dimensions $0,\dots,n-1$ and a monomorphism in dimension n. Since $L(A)=A$ for each closed subset A of Y, the conclusion follows from Theorem 3.6.

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Open and closed mappings and compactification

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Introduction. In this paper we study the extension of open and closed mappings to compactifications such that the extension is open. It is shown that this can be done in such a way that the compactification has the same weight and dimension as the original space. A characterization of open and closed mappings in terms of the ring of bounded real valued continuous functions of a space is given which facilitates the study of the extension of such mappings to compactifications. Also a sufficient condition is given for the extension of a mapping to a compactification to be open. These results should be of interest in themselves. Among those who have studied the extension of mappings to compactifications have been R. Engelking [2], R. Engelking and E. Skljarenko [3], A. B. Forge [4], H. de Vries [13], and A. Zarelua [15]. J. de Groot and R. McDowell have studied the extension of mappings on metric spaces to completions [6].

The last section of the paper deals with finite to one open and closed mappings and dimension. Dimension and finite to one open mappings have been studied by K. Nagami [11] for domain and range paracompact. The author has studied the case with domain and range metrizable [8]. The theorems of this section are an attempt to generalize these results to more general spaces. A. Arhangelskiĭ has studied finite to one open and closed mappings and metrization [1].

The paper has three sections. The first deals with the preliminaries and reviews the relation between $C^*(X)$ and compactification. The second characterizes open and closed mappings on normal spaces and proves the results dealing with extending such mappings to compactifications. The last section deals with finite to one open and closed mappings and dimension.

Notation. Throughout the paper all spaces are assumed completely regular. By *mapping* is meant a continuous function. By $B(X)$ is meant the *ring of bounded real valued functions on* X. The set $C^*(X)$ is the subset of $B(X)$ consisting of those functions which are also continuous. The *modified Lebesgue covering dimension* of the space X is denoted by $\dim X$.