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A dense set of sewings of two crumpled cubes yields S^3

by

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I. Introduction. In 1963 Hosay [8] and Lininger [9] proved that the space obtained by sewing a crumpled cube to a 3-cell with a homeomorphism between their boundaries is actually S^3. At the Wisconsin Topology Seminar in 1965 Lininger asked several questions about sewings of one crumpled cube with another. The primary result of this paper is Theorem 1 which answers Question 7 of [10]; this result shows that, given a sewing of two crumpled cubes, there is another sewing near the first (in the metric sense) which yields S^3.

Results by Harrold and Moise [7], Ball [1], and Martin [12] indicate that not every sewing of two crumpled cubes yields S^3. Neither Theorem 1 nor the techniques of its proof show which homeomorphisms do produce S^3. Section 3 contains some information about this problem in certain cases. The strongest result is Theorem 2, which shows that any sewing matching the wild points of one crumpled cube with points of a tame Sierpiński curve in the other yields S^3. Theorem 3 proves a necessary and sufficient condition that a sewing gives S^3 for special crumpled cubes.

A *crumpled cube* C is defined as a space homeomorphic to the closure of the interior of a topological 2-sphere in E^3. The *boundary* of C, denoted $\mathrm{Bd}\,C$, consists of the points where C fails to be a 3-manifold.

When two crumpled cubes K_1 and K_2 are sewn together by a homeomorphism h of $\mathrm{Bd}\,K_1$ to $\mathrm{Bd}\,K_2$, the resulting space S is obtained from the union (disjoint) of K_1 and K_2 by identifying each x in $\mathrm{Bd}\,K_1$ with $h(x)$ in $\mathrm{Bd}\,K_2$. The homeomorphism h is referred to as a *sewing* of K_1 and K_2, and S is called the *sum* of K_1 and K_2.

Suppose that C is a crumpled cube and p is a point in $\mathrm{Bd}\,C$. The statement that p is a *piercing point* of C means that there exists an embedding f of C in S^3 so that $f(\mathrm{Bd}\,C)$ can be pierced by a tame arc at $f(p)$. Similarly, a Sierpiński curve X on $\mathrm{Bd}\,C$ is *tame* if $f(X)$ is tame under an embedding f of C into S^3 so $\mathrm{Cl}(S^3-f(C))$ is a 3-cell. It follows from Theorem 11 of [11] that a Sierpiński curve X on $\mathrm{Bd}\,C$ is tame if and only if it is tame under some embedding of C in S^3.

The reader is referred to [2] for definition of other terms used in this paper.

Throughout this discussion the *standard* 3-*cell* will denote the set of points in E^3 whose norm is less than or equal to 1.

II. Existence of sewings which yield S^3. The following lemma is an easy consequence of the fact that any crumpled cube K can be embedded in S^3 so that $\mathrm{Cl}(S^3-K)$ is a 3-cell.

LEMMA 1. *Suppose that K is a crumpled cube in S^3; Y is a tame Sierpiński curve on $\mathrm{Bd}K$, B is the standard 3-cell, X is a Sierpiński curve on $\mathrm{Bd}B$; and f is a homeomorphism of Y onto X.*

Then there exist:

(1) *a null sequence of 3-cells $\{C_i\}$ in B such that $\mathrm{Bd}\,C_i\cap \mathrm{Bd}\,B$ is a 2-cell D_i with $\mathrm{Bd}\,D_i\subset X$ and $\mathrm{Int}\,D_i\subset(\mathrm{Bd}\,B-X)$,*

(2) *a homeomorphism $F\colon K\to \mathrm{Cl}(B-\bigcup C_i)$ such that $F|Y=f$.*

LEMMA 2. *If K_1 and K_2 are crumpled cubes, h is a homeomorphism of $\mathrm{Bd}K_1$ to $\mathrm{Bd}K_2$ and $\varepsilon>0$, then there are Sierpiński curves X and Y on $\mathrm{Bd}K_1$ and $\mathrm{Bd}K_2$, respectively, and a map $f\colon K_1\cup K_2\to S^3$ satisfying*

(1) *$f|K_j$ is an embedding $(j=1,2)$,*

(2) *$f(K_1)\cap f(K_2)=f(X)=f(Y)$,*

(3) *$f(X)$ is a tame Sierpiński curve,*

(4) *the diameter of each component of $\mathrm{Bd}K_1-X$ is less than ε,*

(5) *the diameter of each component of $\mathrm{Bd}K_2-h(X)$ is less than ε,*

(6) *for each x in X, $\varrho\big(\mathrm{Bd}K_2\cap f^{-1}f(x),\ h(x)\big)$ is less than ε, and*

(7) *$S^3-f(K_1\cup K_2)$ consists of a null sequence $\{B_i\}$ of components such that $\mathrm{Cl}B_i$ is a 3-cell.*

Proof. By [2] there exist tame Sierpiński curves X and Y in $\mathrm{Bd}K_1$ and $\mathrm{Bd}K_2$, respectively, and also a homeomorphism $g\colon h(X)\to Y$ such that (1) $\varrho(g,I)<\varepsilon$ and (2) each component of $\mathrm{Bd}K_1-X$ and of $\mathrm{Bd}K_2-Y$ has diameter less than ε. Let B denote the standard 3-cell in S^3, and let Z denote a Sierpiński curve on $\mathrm{Bd}B$.

We define homeomorphisms f_1 taking X onto Z and f_2 taking Y onto Z such that $f_1=f_2gh$. By Lemma 1, f_1 may be extended to an embedding of K_1 into B and f_2 may be extended to an embedding of K_2 into $\mathrm{Cl}(S^3-B)$. Then the required map f is obtained by piecing together the embeddings f_1 and f_2.

LEMMA 3. *Suppose that K_1 and K_2 are crumpled cubes in S^3 whose intersection is a tame Sierpiński curve X in the boundary of each and that $S^3-(K_1\cup K_2)$ consists of a null sequence $\{B_i\}$ of components such that each $C_i=\mathrm{Cl}B_i$ is a 3-cell. Given a neighborhood N of $(\bigcup C_i-X)$ and $\varepsilon>0$, there is a map f of S^3 onto S^3 satisfying*

(1) *$f|S^3-N=$ identity,*

(2) *$f|K_j$ is a homeomorphism $(j=1,2)$,*

(3) *the closure of each component of $S^3-\big(f(K_1)\cup f(K_2)\big)$ is a 3-cell of diameter less than ε,*

(4) *$f(K_1)\cap f(K_2)=f(\mathrm{Bd}K_1)\cap f(\mathrm{Bd}K_2)$ is a tame Sierpiński curve Y,*

(5) *each component of $(\mathrm{Bd}K_j)-f^{-1}(Y)$ has diameter less than ε $(j=1,2)$.*

Furthermore, if the diameter of each B_i is less than d, then f can be chosen so that $\varrho\big(x,f(x)\big)<d$ for each x in S^3.

Proof. In the following argument we work with each of the 3-cells C_i individually and only very near those of big diameter. For simplicity we assume that C_1 is the only 3-cell whose diameter is greater than ε. Let S denote the simple closed curve $S=\mathrm{Bd}\,C_1\cap K_1\cap K_2$. There exists a homeomorphism g of C_1 onto the standard 3-cell B in E^3 taking S onto the circle $\mathrm{Bd}B\cap\{(x,y,z)|\ x=0\}$. It follows from uniform continuity that there is a positive number α such that, for each α-subset M of B, $g^{-1}(M)$ has diameter less than $\varepsilon/3$; similarly, there is another positive number δ so that δ-subsets of C_1 are sent by g to $\alpha/24$-subsets of B.

There exist tame Sierpiński curves X_j in $\mathrm{Bd}\,C_1\cap\mathrm{Bd}K_j$ containing S such that each component of $(\mathrm{Bd}\,C_1\cap\mathrm{Bd}K_j)-X_j$ has diameter less than δ $(j=1,2)$. See Theorem 9.1 of [3]. With a slight adjustment of the homeomorphism g, we can push each accessible simple closed curve in $X_1\cup X_2$ to a round circle on $\mathrm{Bd}B$. To prevent further complications in epsilonics, we assume that if D is a component of $\mathrm{Bd}\,C_1-(X_1\cup X_2)$, then $g(\mathrm{Bd}D)$ is a geometric circle on $\mathrm{Bd}B$.

Let $\pi_0,\ldots,\pi_{2n+1}$ be horizontal planes in E^3 such that

(1) $\pi_0\cap\mathrm{Bd}B=(0,0,1)$,

(2) $\pi_{2n+1}\cap\mathrm{Bd}B=(0,0,-1)$, and

(3) $\varrho(\pi_i,\pi_{i+1})<\alpha/12$ for $i=0,\ldots,2n$.

Let J_i denote the simple closed curve where $\mathrm{Bd}B$ is intersected by a horizontal plane halfway between π_i and π_{i+1}. We also assume that the inaccessible part of $g(X_1\cup X_2)$ contains all the curves $\pi_i\cap\mathrm{Bd}B$ and J_i $(i=0,\ldots,2n)$.

Inside B is another 3-cell A which is obtained by removing a null-sequence of 3-cells from B. Each cell in the sequence is bounded by a component D of $\mathrm{Bd}B-g(X_1\cup X_2)$ and the 2-dimensional plane containing $\mathrm{Bd}D$. Observe that $g^{-1}(A)$ is a tame 3-cell in S^3 [3], so $g|g^{-1}(A)$ may be extended to a homeomorphism of S^3 onto itself. We also denote the extension by g. Note that each component of $C_1-g^{-1}(A)$ is a 3-cell of diameter less than $\varepsilon/3$.

Let A_i denote the 3-cell which is the slice of A between π_i and π_{i+1}, Let L denote $\mathrm{Bd}A\cap\{(x,y,z)|\ x\leqslant 0\}$ and R denote $\mathrm{Bd}A\cap\{(x,y,z)|\ x\geqslant 0\}$. We assume that $g(X_1)\subset L$ and $g(X_2)\subset R$. Let Y_i be a Sierpiński curve in $\pi_i \cap A$ containing $\pi_i \cap \operatorname{Bd} A$ such that each component of $(\pi_i \cap A) - Y_i$ has diameter less than $\alpha/12$.

We construct disjoint 3-cells $M_1, \ldots, M_{2n-1}$ satisfying:

(1) $M_i \cap A = \operatorname{Bd} A \cap \operatorname{Bd} M_i = L \cap \operatorname{Bd} A_i$ if i is odd,

(2) $M_i \cap A = \operatorname{Bd} A \cap \operatorname{Bd} M_i = R \cap \operatorname{Bd} A_i$ if i is even,

(3) $g^{-1}(M_i - S) \subset N$,

(4) the diameter of each component of $g\big(C_1 - g^{-1}(A)\big)$ intersected with M_i is less than $\alpha/12$, and

(5) M_i is so close to A that α-subsets of $A \cup (\bigcup M_i)$ go to $\varepsilon/3$-subsets of S^3 under g^{-1}.

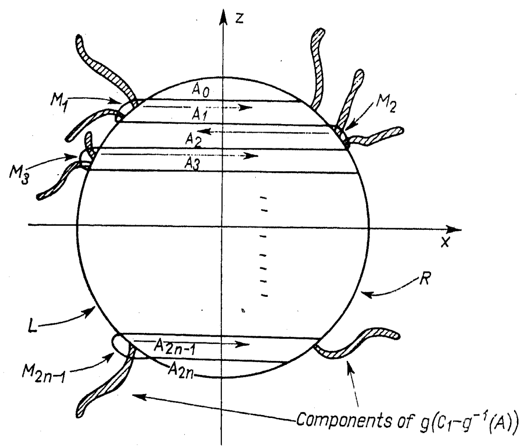

Fig. 1

It follows from Lemma 4 that there are homeomorphisms $f_i\colon M_i \to A_i \cup M_i$ such that

(6) $f_i|(\operatorname{Bd} M_i) - A_i = 1$,

(7) $f_i\big(g(X_1) \cap A_i\big) = Y_i \cup Y_{i+1} \cup \big(g(X_2) \cap A_i\big)$ if i is odd,

(8) $f_i\big(g(X_2) \cap A_i\big) = Y_i \cup Y_{i+1} \cup \big(g(X_1) \cap A_i\big)$ if i is even, and

(9) $\operatorname{Diam} f_i(Z \cap M_i) < \alpha/2$ if Z is a component of $g\big(C_1 - g^{-1}(A)\big)$.

A schematic diagram of the pushes f_i is given in Figure 1.

We construct a map h from $g(K_1 \cup K_2)$ to S^3 by defining h to be the identity on $g(K_1 \cup K_2) - \bigcup M_i$ and defining h to equal f_i on the points of M_i. Extending $g^{-1}hg$ so the domain is all of S^3 produces the required map f.

The proof of Lemma 3 is completed using Lemma 4 for the existence of the homeomorphisms f_i. For simplicity of description we assume that each cell A_i $(i = 1, \ldots, 2n-1)$ is a solid geometric cylinder.

Lemma 4. *Suppose $r > 0$, $h > 0$ and*

(1) $A = \{(x, y, z) \mid x^2 + y^2 \leqslant r^2,\ 0 \leqslant z \leqslant h\}$ *is a solid geometric cylinder,*

(2) $T = \{(x, y, z) \mid x^2 + y^2 \leqslant r^2,\ z = h\}$,

(3) $B = \{(x, y, z) \mid x^2 + y^2 \leqslant r^2,\ z = 0\}$,

(4) $D = \{(x, y, z) \mid x = \sqrt{r^2 - y^2},\ 0 \leqslant z \leqslant h\}$,

(5) $E = \operatorname{Bd} A - \operatorname{Int} D$,

(6) $J = \{(x, y, z) \mid x^2 + y^2 = r^2,\ z = \tfrac{1}{2}h\}$,

(7) *X is a Sierpiński curve in D containing* $\operatorname{Bd} D$ *and $J \cap D$ in its inaccessible part and the diameter of each component of $D - X$ is less than h,*

(8) *Y is a Sierpiński curve in E containing* $\operatorname{Bd} E$ *in its inaccessible part and the diameter of each component of $E - Y$ is less than h,*

(9) *$\{K_i\}$ is a null sequence of disjoint 3-cells such that $K_i \cap A = \operatorname{Bd} K_i \cap \operatorname{Bd} A$ is a disk whose interior is a component of $D - X$.*

Then, given a 3-cell M such that $M \cap A = D$ and $\operatorname{Diam}(M \cap K_i) < h$, *there exists a homeomorphism $f\colon M \to M \cup A$ such that*

(10) $f|\operatorname{Bd} M - \operatorname{Int} D = 1$,

(11) $f(X) = Y$, *and*

(12) $\operatorname{Diam} f(M \cap K_i) < 6h$ *for each cell K_i.*

Proof. Let α be a positive number such that

(13) $\alpha < r$ and the diameter of each of the disks

$$G_1 = \{(x, y, z) \mid z = 0,\ r - \alpha \leqslant y,\ x^2 + y^2 \leqslant r^2\},$$
$$G_2 = \{(x, y, z) \mid z = 0,\ y \leqslant -r + \alpha,\ x^2 + y^2 \leqslant r^2\}$$

is less than h.

Let $\{x_i\}_{i=0}^{n}$ be a finite decreasing sequence of points on the x-axis such that $x_0 = (r, 0, 0)$, $x_n = (-r, 0, 0)$ and $\varrho(x_i, x_{i+1}) < h/2$. Consider the circle containing x_i, $(0, r, 0)$ and $(0, -r, 0)$; let A_i be the arc on this circle that has an end point on each of the planes $y = r - \alpha$ and $y = -r + \alpha$, lies between these planes and contains the point x_i. Let F_i be the disk in the xy-plane whose boundary is a subset of $A_{i-1} \cup A_i \cup \{(x, y, z) \mid |y| = r - \alpha\}$. Let $C_i = \{(x, y, z) \mid 0 \leqslant z \leqslant h,\ (x, y, 0) \in F_i\}$.

We construct a 3-cell L such that $L \cap A = D$, $L \subset (\operatorname{Int} M) \cup D$, and straight lines parallel to the x-axis intersect $\operatorname{Bd} L$ in at most two points. Let $T_1 = L \cap \{(x, y, z) \mid |y| \leqslant r - \alpha\}$ and $E_1 = T_1 \cap D$. Since $\{K_i\}$ is a null sequence and $J \cap D$ lies in the inaccessible part of X, there is a finite sequence $\{T_i\}_{i=2}^{n}$ of 3-cells such that

(14) $T_i \cap A = \operatorname{Bd} T_i \cap \operatorname{Bd} A$ is a disk $E_i \subset D$ such that

$$\operatorname{Bd} E_i = \{(x, y, z)|\ \ |y| = r-a\,,\ \ |z-h/2| \leqslant t_i\,,\ \ x = \sqrt{r^2-(r-a)^2}\} \cup$$
$$\cup \{(x, y, z)|\ \ |z-h/2| = t_i\,,\ |y| \leqslant r-a\,,\ x = \sqrt{r^2-y^2}\}$$

where $h/2 = t_1 > t_2 > t_3 \ldots t_n > 0$.

(15) For $|t| \leqslant r-a$, $\{(x, y, z)|\ y = t\} \cap T_i$ is geometrically similar to the semi-circular disk $\{(x, y, z)|\ z = 0,\ x^2+y^2 \leqslant 1,\ x \leqslant 0\}$.

(16) $T_{i+1} \subset D \cup \operatorname{Int} T_i$.

(17) No K_j intersects more than two of the 3-cells $T_i - \operatorname{Int} T_{i+1}$ $(i = 1, 2, \ldots, n-1)$.

There exists a homeomorphism $g\colon M \to M \cup A$ such that

(18) g is the identity on $\operatorname{Bd} M - \operatorname{Int} D$ and outside of a small neighborhood of L,

(19) $g(D) = E$,

(20) $g(T_i - \operatorname{Int} T_{i+1}) = C_i$ $(i = 1, \ldots, n-1)$ and $g(T_n) = C_n$,

(21) $g(E_i - \operatorname{Int} E_{i+1}) = F_i \cup \{(x, y, z)|\ (x, y, 0) \in F_i,\ z = h\}$ $(i = 1, \ldots, n-1)$,

$$g(E_n) = F_n \cup \{(x, y, z)|\ (x, y, 0) \in F_n, z = h\} \cup$$
$$\cup \{(x, y, z)|\ |y| \leqslant r-a, 0 \leqslant z \leqslant h, x = -\sqrt{r^2-y^2}\}$$

(22) for $|t| \leqslant r-a$, g preserves the $y = t$ plane, and

(23) $g(L - \operatorname{Int} T_1) = A - \operatorname{Int}(\bigcup_{i=1}^{n} C_i)$.

The effect of the homeomorphism g in the $y = t$ plane for $|t| \leqslant r-a$ is illustrated in Figure 2.

It follows from (1), (17), (20) and (22) that $\operatorname{Diam} g(K_i \cap M) < 2h$. There is a homeomorphism $h\colon A \to A$ such that $h|D = 1$, $h\big(g(X)\big) = Y$ and $\varrho\big(x, h(x)\big) < 2h$. The required homeomorphism f equals hg.

THEOREM 1. *If K_1 and K_2 are crumpled cubes, h is a homeomorphism of $\operatorname{Bd} K_1$ to $\operatorname{Bd} K_2$ and $\varepsilon > 0$, then there is another homeomorphism g of $\operatorname{Bd} K_1$ to $\operatorname{Bd} K_2$ such that $\varrho(g, h) < \varepsilon$ and that the union of K_1 and K_2 sewn together by g is S^3.*

Proof. There exist tame Sierpiński curves X_0 and Y_0 on $\operatorname{Bd} K_1$ and $\operatorname{Bd} K_2$, respectively, and a map f_0 from $K_1 \cup K_2$ into S^3 satisfying the conclusions of Lemma 2.

Let $\varepsilon_1, \varepsilon_2, \ldots$ be a sequence of positive numbers with a finite sum. Using Lemma 3 we define inductively a sequence of maps $\{f_i\}$ from $K_1 \cup K_2$ into S^3 and sequences of tame Sierpiński curves $\{X_i\}$ on $\operatorname{Bd} K_1$ and $\{Y_i\}$ on $\operatorname{Bd} K_2$ such that

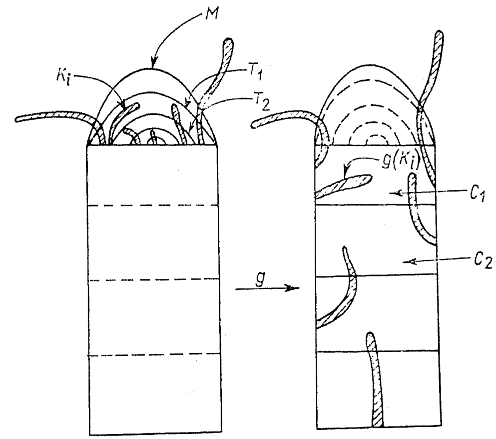

Fig. 2. $y = t$ plane for $|t| \leqslant r-a$

(1) $X_{i-1} \subset X_i$ and $Y_{i-1} \subset Y_i$,

(2) f_i takes K_j $(j = 1, 2)$ homeomorphically into S^3,

(3) $f_i(K_1) \cap f_i(K_2) = f_i(X_i) = f_i(Y_i)$,

(4) $f_i(X_i)$ is a tame Sierpiński curve,

(5) $\varrho(f_{i-1}, f_i) < \varepsilon_{i-1}$,

(6) $f_i|K_j - N(\operatorname{Bd} K_j, \varepsilon_i) = f_{i-1}|K_j - N(\operatorname{Bd} K_j, \varepsilon_i)$ $(j = 1, 2)$,

(7) the closure of each component of $S^3 - f_i(K_1 \cup K_2)$ is a 3-cell of diameter less than ε_i.

These Sierpiński curves are chosen, using condition 5 ot Lemma 3, so that $\operatorname{Bd} K_1 = \operatorname{Cl}(\bigcup X_i)$ and $\operatorname{Bd} K_2 = \operatorname{Cl}(\bigcup Y_i)$.

Let H_n designate the union of the closures of the components of $S^3 - f_n(K_1 \cup K_2)$. There exists a decreasing collection of open sets $N_1, N_2, \ldots$ satisfying

(8) N_i contains $H_{i-1} - f_{i-1}(X_{i-1})$,

(9) N_i is contained in the ε_i-neighborhood of H_{i-1}, and

(10) no two components of $H_{i-1} - f_{i-1}(X_{i-1})$ lie in the same component of N_i.

In addition, the maps f_i from $K_1 \cup K_2$ to S^3 are restricted so that

(11) $f_i|f_{i-1}^{-1}(S^3 - N_i) = f_{i-1}|f_{i-1}^{-1}(S^3 - N_i)$.

We define a map f of $K_1 \cup K_2$ into S^3 by $f(x) = \lim f_i(x)$. Clearly f is a continuous function. It follows from (6) that f is one to one on the domain $\operatorname{Int} K_1 \cup \operatorname{Int} K_2$, and it follows from (7) that f is onto. Furthermore, we have that

$$f(\operatorname{Bd} K_1) = f(\operatorname{Bd} K_2) \subset S^3 - f(\operatorname{Int} K_1 \cup \operatorname{Int} K_2) .$$

It can be verified using (11) that f takes $\operatorname{Bd} K_j$ $(j = 1, 2)$ homeomorphically into S^3. Therefore f embeds each of K_1 and K_2 in S^3.

We let g denote the homeomorphism of $\operatorname{Bd} K_1$ to $\operatorname{Bd} K_2$ satisfying $f(x) = fg(x)$ for each x in $\operatorname{Bd} K_1$. Then S^3 is the space obtained when K_1 is sewn to K_2 by g. It is easy to check that $\varrho(g, h) < 3\varepsilon$. With an appropriate change in the positive number employed, the proof is complete.

III. Other sewings which yield S^3.

THEOREM 2. *If K_1 and K_2 are crumpled cubes in S^3, W is the set of wild points of K_1, and h is a homeomorphism of $\operatorname{Bd} K_1$ to $\operatorname{Bd} K_2$ such that $h(W)$ lies in a tame Sierpiński curve X on $\operatorname{Bd} K_2$, then the union of K_1 and K_2 sewn together by h is S^3.*

Proof. As a consequence of Theorem 9.1 of [3], we may assume that $h(W)$ lies in the inaccessible part of X. We may also assume that K_1 is embedded in S^3 so $\operatorname{Cl}(S^3 - K_1)$ is a 3-cell B ([8], [9]). It follows from Lemma 1 that there is an embedding F of K_2 into B such that $F|X = h^{-1}|X$ and that $S^3 - (K_1 \cup F(K_2))$ consists of a null sequence of components $B_1, B_2, \ldots$ with $\operatorname{Cl} B_i$ a 3-cell C_i. Each cell C_i has the property that $K_1 \cap \operatorname{Bd} C_i$ is a disk D_i, where $\operatorname{Int} D_i$ is a component of $\operatorname{Bd} K_1 - h^{-1}(X)$. Note that each D_i is tame in S^3.

The sewing is completed by a map g of S^3 onto S^3 such that (a) g takes K_1 homeomorphically onto $\operatorname{Cl}(S^3 - F(K_2))$, (b) $g|F(K_2)$ is the identity and (c) for each positive integer i, $g|D_i = Fh|D_i$. This establishes the theorem.

DEFINITION. A crumpled cube C is *countably knotted* if there is an upper semi-continuous decomposition G of the standard 3-cell B in E^3 into points and at most a countable collection $\{A_i\}$ of wild arcs satisfying

(1) $A_i \cap \operatorname{Bd} B = \operatorname{Bd} A_i \cap \operatorname{Bd} B =$ one point and

(2) A_i is locally polyhedral $\operatorname{mod}(\operatorname{Bd} A_i - \operatorname{Bd} B)$,

such that C is homeomorphic to the decomposition space B/G. The *bad set* W of C is the image of the non-degenerate elements of the decomposition G.

Both the Fox Artin sphere and Martin's rigid sphere [13] produce countably knotted crumpled cubes. Unfortunately, many of the fiercest crumpled cubes are not countably knotted. Theorem 3 characterizes the homeomorphisms sewing two crumpled cubes together which give S^3, provided that one of these cubes is countably knotted.

THEOREM 3. *Suppose that W is the bad set of a countably knotted crumpled cube C_1, that C_2 is another crumpled cube, and that h is a homeomorphism of $\operatorname{Bd} C_1$ to $\operatorname{Bd} C_2$. Then S^3 is the space obtained by sewing C_1 and C_2 together by h if and only if, for each $w \in W$, $h(w)$ is a piercing point of C_2.*

Proof. Let G denote an upper semi-continuous decomposition of the standard 3-cell B, as in the definition of countably knotted, whose non-degenerate elements are the wild arcs $A_1, A_2, \ldots$, which has C_1 as its decomposition space.

Assume that for each w in W, $h(w)$ is a piercing point of C_2. There is an embedding f of B into S^3 such that (1) the closure of $S^3 - f(B)$ is C_2 and (2) for each $b \in \operatorname{Bd} B$, $h(b)$ is the point of C_2 corresponding to $f(b)$ ([8], [9]). Note that the space obtained by sewing C_1 to C_2 by h is homeomorphic to the decomposition space of S^3 where the non-degenerate elements in the decomposition are the arcs $f(A_i)$ in $f(B)$.

Each arc $f(A_i)$ is locally tame at its interior points; $f(A_i)$ is locally tame at $f(A_i \cap \operatorname{Bd} B)$, since $f(A_i \cap \operatorname{Bd} B)$ is a piercing point of C_2 (Lemma 2, 12). Therefore, $f(A_i)$ is cellular.

The decomposition of S^3 whose non-degenerate elements are the arcs in the collection $\{f(A_i)\}$ produces S^3 as its decomposition space, because each arc $f(A_i)$ is cellular and locally tame modulo one point [6]. Thus, sewing C_1 to C_2 by h yields S^3.

The converse implication may be proved with the same construction and an appeal to the results in [12].

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A note on inverse binary operation in abelian groups

by

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It is well known that in a group $\langle G, +, -, 0\rangle$, all the group operations can be expressed in terms of a single binary operation $a\ast b = a-b$. Thus, $0 = a\ast a$, $-a = 0\ast a$ and $a+b = a\ast(-b)$. It is not known whether the 'right subtraction', the 'left subtraction' and its transposes are the only binary operations in groups in terms of which all the other group operations can be expressed. However, in [1], Higman and Neumann have stated that, in the case of abelian groups, these are the only operations having the property and Professor Neumann says ([1]) that there exists no explicit publication of the proof so far. In this note we give a proof for the same.

Notations and definitions. A binary operation in a group $\langle G, +, -, 0\rangle$ is a *word in two symbols*, say a, b and in the group symbols $+$ and $-$. It is known that any word in a, b in an abelian group can be written in the form $ma+nb$ where m and n are integers (ma stands for '$a+a+\ldots m$ times'). If $f(a, b)$ is the word $ma+nb$, then the *length* of the word f is, as usual, the positive integer $|m|+|n|$, while the 'degree' of the word f is, by definition, the integer $m+n$.

THEOREM. *If $a\ast b$ is a binary operation in an abelian group $\langle G, +, -, 0\rangle$, in terms of which all the other group operations can be expressed, then $a\ast b = a-b$ or else $a\ast b = b-a$.*

Proof. Given that

$$a+b = g(a, b)\,, \quad \text{some word in the binary system } \langle G, \ast\rangle\,,$$
$$-a = h(a)\,, \quad \text{some word in the binary system } \langle G, \ast\rangle\,,$$

and so, (or even otherwise)

$$\begin{aligned} 0 &= a+(-a)\\ &= g\big(a, h(a)\big)\\ &= f(a)\,, \quad \text{some word in } \langle G, \ast\rangle\,. \end{aligned}$$

Moreover, we have $G+G = G$, i.e. given a in G, there exist elements b, c in G such that $a = b+c$, or $a = g(b, c) = u\ast v$, where u and v are words in b, c and the symbol $\ast$. So we have $G\ast G = G$.

([1]) In a private communication.