please to any desired position. S, as well as its multiples $\theta^k S$, is measure inducing since $\lim_t(\log_\beta(t)/t) = 0$. Hence $\mu(\theta^k S \cap N) = \theta^k\mu(\theta^k S \cap \theta^k N) = \theta^k\mu(\theta^k(S \cap N)) = \mu(S \cap N)$, the final equality following because $\mu \in S_0$. Let $P^{(k)} = \theta^k S \cap N$, a set of integers which corresponds to the arc $U^{(k)}$, all $P^{(k)}$ having the same measure with respect to μ. Now consider $p, q \in N$ such that $(p/q) < \lambda(U)$. By a simple geometric argument this implies that we can find q arcs $U^{(k)}$, $k \in A$, such that every point of C belongs to at least p of them. Correspondingly every $m \in N$ belongs to at least p of the q sets $P^{(k)}$, $k \in A$. For $n \in N$ let B_n be the set of $m \in N$ which belong to exactly n of the $P^{(k)}$, $k \in A$. Then we have

$$q\mu(P^{(0)}) = \sum_{k \in A} \mu(P^{(k)}) = \sum_{n \geqslant p} n\mu(B_n) \geqslant p .$$

Thus $(p/q) \leqslant \mu(P^{(0)})$. Proceeding in exactly the same way we can show that if $\lambda(U) < (p/q)$, then $\mu(P^{(0)}) \leqslant (p/q)$. If we combine these results, we see that $\mu(S \cap N) = \mu(P^{(0)}) = \lambda(U)$. q.e.d.

COROLLARY. *If P_n is the set of natural numbers whose first significant digit lies between* 1 *and* n, $1 \leqslant n \leqslant 9$, *and* $\mu \in S$ (*in fact to any* S_0 *where* $\log_{10}\theta$ *is irrational*), *then* $\mu(P_n) = \log_{10}(n+1)$.

Proof. For we can describe $P_n = S_n \cap N$ where S_n is the set of all $x \in R^+$ such that $0 \leqslant e(\log_{10}x) \leqslant \log_{10}(n+1)$. q.e.d.

References

[1] N. Dunford and J. Schwartz, *Linear Operators*. I, Interscience, New York, 1958.

[2] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, Oxford, 1938.

[3] M. Jerison, *The set of all generalized limits of bounded sequences*, Canad. Journ. Math. 9 (1957), pp. 79-89.

[4] R. S. Pinkham, *On the distribution of first significant digits*, Ann. of Math. Stat. 32 (1961), pp. 1223-1230.

[5] A. Tarski, *Une contribution à la théorie de la mesure*, Fund. Math. 15 (1930), pp. 42-50.

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Metrizability of trees *

by

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Introduction. It is a well-known result that dendrites (acyclic Peano continua) can be alternatively defined as metrizable continua in which each pair of points can be separated by a third point. L. E. Ward, in [6], generalized the notion of dendrite by removing the metrizability condition in the second definition, and called such objects trees. He then showed that many properties of dendrites carry over to trees. In this paper we shall be concerned with establishing properties of trees which yield metrizability theorems. The principal results in this connection are I.6, III.1, III.2, and III.5.

I. Separable trees are metrizable. By a *continuum* we mean a compact connected Hausdorff space. A continuum is *hereditarily unicoherent* provided the intersection of any two of its subcontinua is connected. A *tree* is a locally connected hereditarily unicoherent continuum. An *arc* is a continuum with precisely two non-cutpoints.

In Whyburn [7], pp. 88-89, several properties of metric trees (= dendrites) are established. L. E. Ward showed in [6] that a number of these properties carry over to the nonmetric case.

For the rest of this section X will denote a tree. Proposition I.1 is due to Ward.

I.1. PROPOSITION. *For each* x *and* y *in* X, $[x, y] = \bigcap\{C \mid x, y \in C$ *and* C *is a subcontinuum of* $X\}$ *is an arc with endpoints* x *and* y.

Proof. It follows from the hereditary unicoherence of X that $[x, y]$ is the only subcontinuum of X irreducible between x and y. Suppose $z \in (x, y) = [x, y]\setminus\{x, y\}$. If $[x, y]\setminus z$ were connected, then x and y would lie in the same component of $X\setminus z$. But this is impossible since the components of open sets in locally connected continua are continuum-wise connected ([1], p. 110). Hence $[x, y]$ is an arc.

I.2. PROPOSITION. *If C is a component of* $X\setminus p$, *then* $[x, p) = [x, p]\setminus p \subset C$ *for each* x *in* C.

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Proof. This follows from the fact that C^*, the closure of C, is $C \cup \{p\}$.

A point p in X is called a *branch point* of X provided $X \setminus p$ has more than two components. It follows readily from I.2 that p is a branchpoint of X iff there are at least three arcs in X with p as a common endpoint which are pairwise disjoint except for p. Let B denote the branchpoints of X.

I.3. Proposition. *Suppose D that is a dense subset of X which contains B. Let $\mathcal{C}$ be the collection of all components of $X \setminus d$ as d ranges over D. Then $\mathcal{C}$ is a subbasis for the topology of X.*

Proof. Let $x \in X$ and U a connected open set containing x. Let $\mathcal{V}$ be the collection of all components of $X \setminus d$ which fail to contain x as d ranges over $(U \cap D) \setminus x$. We show that $\mathcal{V}$ covers $X \setminus U$. Let $y \in X \setminus U$ and choose $z \in (x, y) \cap U$. Then $[y, z] \subset U$. Let $p \in (z, x)$ and V a connected open set containing p which does not contain x and y. Suppose $q \in V \setminus (x, y)$. Then $[p, q] \cap [x, z]$ is an arc $[p, r]$ lying in (x, y) and the point r is a branchpoint of X. Hence (x, z) contains a branchpoint of X or (x, z) is open. In both cases, we conclude that $(x, z) \cap D \neq \emptyset$. Let $d \in (x, z) \cap D$. Then the component of $X \setminus d$ which contains y does not contain x. We conclude that $\mathcal{V}$ covers $X \setminus U$. Consequently there exist $d_1, d_2, \ldots, d_n$ in $(U \cap D) \setminus x$ such that every point of $X \setminus U$ lies in a component of $X \setminus d_i$ not containing x for some i. Let C_i be the component of $X \setminus d_i$ which contains x. Then $W = \bigcap_{i=1}^{n} C_i$ is an open set containing x. Furthermore, if $y \in X \setminus U$, then for some i, y lies in a component of $X \setminus d_i$ which does not contain x and hence $y \notin C_i$. Thus $W \subset U$ and the proof is complete.

I.4. Proposition. *Suppose that D is a dense subset of X. Then* $\operatorname{card} B \leqslant \operatorname{card} D$, *and hence* $\operatorname{card} B \cup D = \operatorname{card} D$.

Proof. Fix $b \in B$. Define a function $f\colon D \times D \to B$ by $f(x, y) = b$ if $y \in [x, b]$ or $x \in [y, b]$ or $[x, b] \cap [y, b] = b$ and $f(x, y) = b'$ if $[x, b] \cap [y, b] = [b', b]$, where $b' \neq b$. If $b' \in B$, then there are two components C_1 and C_2 of $X \setminus b'$ which do not contain b. Since C_1 and C_2 are open, they must contain points d_1 and $d_2 \in D$ respectively. Hence $f(d_1, d_2) = b'$ by I.2. We conclude that $\operatorname{card} D = \operatorname{card} D \times D \geqslant \operatorname{card} B$.

The *weight* of a space is the smallest cardinal such that there is a basis for the space with that cardinal. The *density* of a space is the smallest cardinal such that there is a dense subset of the space with that cardinal ([4], p. 50).

I.5. Theorem. *The weight of X equals the density of X.*

Proof. Let D be a dense subset of X whose cardinal is the density of X. By I.4, we may assume that $B \subset D$. Let $\mathcal{C}$ be the collection of components of $X \setminus d$ as d ranges over D. For each $d \in D$, let $\mathcal{C}_d$ be the collection of components of $X \setminus d$. Then $\operatorname{card} \mathcal{C}_d \leqslant \operatorname{card} D$, because $\mathcal{C}_d$ is a collection of pairwise disjoint open sets each of which must contain points of D. Since $\mathcal{C} = \bigcup \mathcal{C}_d$, we conclude that $\operatorname{card} \mathcal{C} \leqslant \operatorname{card} D$. By I.3, $\mathcal{C}$ is a subbasis for the topology on X. Let $\mathcal{B}$ be the basis generated by $\mathcal{C}$. Then $\operatorname{card} \mathcal{B} = \operatorname{card} \mathcal{C} \leqslant \operatorname{card} D$. Consequently the weight of X is less than or equal the density of X. Since the reverse inequality is true for arbitrary spaces, the theorem is proved.

The following corollary can also be obtained from 6.6 of [2].

I.6. Corollary. *X is metrizable iff X is separable.*

Proof. A compact Hausdorff space is metrizable iff it has a countable base.

II. G_δ and F_σ sets of endpoints.

As in section I, X will denote a tree. A point p of X is an *endpoint* of X provided $X \setminus p$ is connected. It follows quickly from I.2 that p is an endpoint of X iff p is an endpoint of each subarc of X which contains it. Denote the set of endpoints of X by E. We shall investigate the relationships between E and B, the branchpoints of X.

II.1. Proposition. *If $e \in E$ is a limit point of E, then e is a limit point of B.*

Proof. Let U be a connected open set containing e. Then U contains two endpoints e_1 and e_2 distinct from e. Since X is locally connected, $[e_1, e] \cup [e_2, e] \subset U$. Now $[e_1, e] \cap [e_2, e]$ is a proper subarc of $[e_1, e]$ and $[e_2, e]$, and the endpoint of this arc different from e is a branchpoint of X.

II.2. Proposition. *If $x \in X$ is a limit point of B, then x is a limit point of E.*

Proof. Let U be an open set containing x and choose V connected and open so that $x \in V \subset V^* \subset U$. Since the components of $X \setminus V^*$ form a cover of $X \setminus U$ by pairwise disjoint open sets, there must be only a finite number of them which meet $X \setminus U$. Label those which do meet $X \setminus U$, $\{C_i\}_{i=1}^{n}$. Now the boundary of each C_i is a single point e_i. (To see this, let y and z be in $C_i^* \setminus C_i$ and $w \in C_i$. Then by I.2, $[y, w]$ and $[z, w]$ lie, except for y and z respectively, entirely in C_i. Since $([y, w] \cup [z, w]) \cap V^* = \{y, z\}$ is connected, we have $z = y$.) For each e_i, e_j with $i \neq j$ consider $[e_i, x] \cap [e_j, x]$. This intersection is either x or an arc $[x, b_{ij}]$ where b_{ij} is a branchpoint of X. Since there are at most a finite number of b_{ij}, there is a branchpoint b in V different from each of them. Let K_1, K_2, and K_3 be distinct components of $X \setminus b$. At most one of these contains x and at most one of the others contains an e_i. Consequently one of them contains neither x nor an e_i. Assume that K_1 has this property. Then $K_1 \subset U$ and any one of its noncut points is an endpoint of X. This completes the proof.

II.3. Theorem. *If E is closed, then B is countable.*

Proof. Fix a branchpoint p. For each branchpoint $b \neq p$, we assert that $[b, p] \cap B$ is finite. For if not, then some x in $[b, p]$ is a limit point of B. Hence by II.2, x is a limit point of E, and since E is closed by assumption, $x \in E$. This is an impossibility since b and p are branchpoints.

Now let $o(b)$ denote the number of points in $[b, p] \cap B$, and let $B_n = \{b \in B|\ o(b) = n\}$ for $n = 2, 3, \ldots$ and let $B_1 = \{p\}$. If B_n is infinite for some n, then B_n has a limit point x, which by II.2 must also be a limit point of E, hence an element of E. Now on the arc $[x, p]$ there are an infinite number of branchpoints; in fact, for each z in (x, p) there is a branchpoint in (z, x). To see this, pick an endpoint e other than x in the component C of $X \backslash z$ which contains x. Then, by I.2, $[e, z]$ and $[z, x]$ lie except for z entirely in C. Now $[e, z] \cap [x, z]$ is a proper subarc of each of $[e, z]$ and $[x, z]$ since e and x are endpoints. Thus the endpoint of this arc different from z is a branchpoint lying in (z, x).

Thus $[x, p] \cap B$ is infinite. Choose b in $[x, p] \cap B$ so that $o(b) > n$. Let C be the component of $X \backslash b$ containing x. Then $b \in [y, p]$ for each $y \in C$, and hence $B_n \cap C = \emptyset$, a contradiction. Thus each B_n is finite. We have already shown that $B = \bigcup_{n=1}^{\infty} B_n$, and therefore B is countable.

II. 4. Proposition. *If A and B are connected subsets of X, then $A \cap B$ is connected.*

Proof. Let x and y be in $A \cap B$. We show that $[x, y] \subset A \cap B$. Suppose that some point z of $[x, y]$ is not in $A \cap B$. Assume $z \notin A$. Let C be the component of $X \backslash z$ containing x. Note $y \notin C$. Thus $A = (C \cap A) \cup \cup ((X \backslash C^*) \cap A)$ is not connected, a contradiction.

II.5. Corollary. *Each subcontinuum of X is a tree.*

Proof. It follows from II.4 that each subcontinuum of X is locally connected.

II.6. Theorem. *If E is an F_σ set in X then B is countable.*

Proof. Write $E = \bigcup_{n=1}^{\infty} E_n$, where E_n is a closed subset of X and $E_n \subset E_{n+1}$. Let X_n be the intersection of all subcontinua of X which contain E_n. It follows from the hereditary unicoherence of X that X_n is a continuum. Thus X_n is a tree by II.5. We assert that the endpoints of X_n consist precisely of the set E_n. That each element of E_n is an endpoint of X_n is clear. Suppose that x is an endpoint of X_n, and let y be some other point in X_n. For each z in $[y, x]$ let C_z be the component of $X_n \backslash z$ which contains x. These form a base for the topology at x. Further, each C_z must contain a point of E_n (otherwise $X_n \backslash C_z$ would be a smaller continuum containing E_n). Therefore x is a limit point of E_n or x is in E_n. We conclude that $x \in E_n$.

Let B_n be the set of branchpoints of X_n. Clearly $B = \bigcup_{n=1}^{\infty} B_n$. By II.3, B_n is countable for each n; hence B is countable. This completes the proof.

II.7. Theorem. *Let A be a closed set of endpoints of X. Let A_1 be the set of isolated points of A and $A_2 = A \backslash A_1$. Then*

(1) *A_1 is countable,*

(2) *A_2 is a G_δ set in X, and*

(3) *if X is first countable at each point of A_1, then A is a G_δ set in X.*

Proof. Let X_1 be the intersection of all subcontinua of X containing A. Then by previous arguments X_1 is a tree whose set of endpoints is precisely A. Let B_1 denote the branchpoints of X_1. Choose $p \in B_1$ and define for each $x \in X$, the *order of* x, $o(x)$, as the number of points in $[x, p] \cap B_1$. We claim that $o(x)$ is infinite iff $x \in A_2$. To prove this, suppose that $[x, p] \cap B_1$ is infinite. Then some point z of $[x, p]$ is a limit point of B_1. By II.2, z is a limit point of A. Since A is closed, z must be in A, and therefore $z = x$. Since x is not isolated in A, $x \in A_2$. Conversely, suppose that $[x, p] \cap B_1$ is finite. Then there is a z in (x, p) such that $(x, z) \cap B_1$ is void. Let C denote the component of $X \backslash z$ which contains x. If $x \in A_2$ then C contains a point y of A distinct from x. Since (x, z) contains no points of B_1, we have $x \in [y, z]$, a contradiction.

Proof of (1): Let $A_{1n} = \{a \in A|\ o(a) = n\}$. If A_{1n} is infinite for some n, then some a in A is a limit point of A_{1n}. Note that $a \in A_2$. Hence $[a, p] \cap B_1$ is infinite. Choose b in $B_1 \cap [a, p]$ so that $o(b) > n$. Then the component of $X \backslash b$ which contains a must contain a point a' of A_{1n}. But since $b \in [a, p]$, and $o(b) > n$, $a' \notin A_{1n}$. Hence A_{1n} is finite for each n Since $A_1 = \bigcup_{n=1}^{\infty} A_{1n}$, A_1 is countable.

Proof of (2): Let $B_{1n} = \{b \in B_1|\ o(b) = n\}$. Let $U_n = \{x \in X|\ (x, p) \cap \cap B_{1n} \neq \emptyset\}$. We claim that U_n is open and $A_2 = \bigcap_{n=1}^{\infty} U_n$. To prove this, suppose $x \in U_n$. Then $(x, p) \cap B_{1n} \neq \emptyset$. Let b be the branchpoint of X in (x, p) such that $o(b) = n$. Then clearly the component of $X \backslash B$ which contains x lies in U_n. Thus U_n is open. Now since $x \in A_2$ iff $[x, p] \cap B$ is infinite, we see that $x \in A_2$ iff $[x, p] \cap B_{1n} \neq \emptyset$ for each n. From this we conclude that $\bigcap_{n=1}^{\infty} U_n = A_2$. Thus A_2 is a G_δ set in X.

Proof of (3): Label the points of A_{1n} $\{a_{ni}\}_{i=1}^{m(n)}$ for each n. Since X is the first countable at a_{ni}, we can find a monotonic sequence $\{a_{nij}\}_{j=1}^{\infty}$ on $[p, a_{ni}]$ which converges to a_{ni}. Further we can choose a_{ni1} so that the component of $X \backslash a_{ni1}$ which contains a_{ni} contains no other point of A. Now for each a_{nij}, let V_{nij} be the component of $X\backslash a_{nij}$ containing a_{ni}. Define

$$V_n = \bigcup \{V_{kin}\colon\ k = 1, \ldots, n\} \quad \text{and} \quad W_n = U_n \cup V_n$$

where U_n is the same as in the proof of (2). We observe that $A_{11} \cup A_{12} \cup \ldots \cup A_{1n} \subset V_n$ and that $A_{1i} \subset U_n$ for $i > n$. Hence W_n is an open set containing A. Now suppose $y \in X\backslash A$. Then as in the proof of (2) there is an n such that $y \notin U_n$. For each a_{ki} where $k = 1, \ldots, n$, choose a j so that $y \notin V_{kij}$, and then choose m to be larger than n and all of the j's choosen above. Then we have $y \notin \bigcup \{V_{kim}\colon\ k = 1, \ldots, n\} \subset V_m$. Consequently $y \notin W_m$.

III. Metrizability of Souslin trees. Let X be a tree. We call X a *Souslin tree* provided (1) each arc in X is separable, and (2) there does not exist in X an uncountable family of pairwise disjoint open sets.

In this section we obtain some metrizability conditions for Souslin trees. The question of the existence of a non-metrizable Souslin tree is not answered by these results; however, they do indicate that it would be difficult to construct such an object. It seems to me that this question is equivalent to Souslin's question about linearly ordered spaces [5].

Throughout this section X will denote a Souslin tree with branchpoints B and endpoints E.

III.1. THEOREM. *X is separable iff B is countable.*

Proof. If X is separable, then B is countable by I.4. Suppose that B is countable. Consider $X\backslash B^*$. The components of this set form a collection of pairwise disjoint open sets. Hence there are only countably many of them. Label them $C_1, C_2, \ldots$ It is seen that C_i^* is an arc for each i and hence has a countable dense subset D_i. Clearly $D = B \cup \bigcup_{i=1}^{\infty} D_i$ is a countable dense subset of X.

III.2. THEOREM. *If E is an F_σ set in X, then X is separable.*

Proof. This follows immediately from II.6 and III.1.

III.3. PROPOSITION. *X is first countable.*

Proof. Let $x \in X$. Note that there are only countably many components of $X\backslash x$ since these form a collection of pairwise disjoint open sets. Choose a point y_i from each of these components and a sequence $\{y_{ij}\}_{j=1}^{\infty}$ on $[y_i, x]$ which is monotonically converging to x. Let C_{ij} denote the component of $X\backslash y_{ij}$ which contains x and let $U_n = \bigcap_{i=1}^{n} C_{in}$. We need only show that $\bigcap_{n=1}^{\infty} U_n = x$. Suppose $y \in X/x$. Then y lies in the same component of $X\backslash x$ as some y_i. Furthermore, $[y, y_i]$ lies in this component and $[y, y_i] \cap [y_i, x)$ is either y_i or an arc $[y_i, z]$. Choose n so that $y_{in} \in (z, x)$. Then clearly $y \notin C_{in} \supset U_n$, and thus $x = \bigcap_{n=1}^{\infty} U_n$.

III.4. THEOREM. *Each closed subset of X is G_δ in X.*

Proof. Let A be a closed subset of X. Then $X\backslash A$ has only countably many components; label them C_i. It can be seen that C_i^* is a Souslin tree and that $A_i = A \cap C_i^*$ is a closed subset of the endpoints of C_i^*. Thus if follows from II.4 and III.3 that A_i is G_δ in C_i^*. Let $\{U_{ij}\}_{j=1}^{\infty}$ be a collection of open sets in C_i^* such that $A_i = \bigcap_{j=1}^{\infty} U_{ij}$. Now define $U_n = X\backslash \bigcup_{i=1}^{n} (C_i^*\backslash U_{in})$. It is easily verified that U_n is open and that $\bigcap_{i=1}^{\infty} U_n = A$. This completes the proof.

We now examine some implications of III.4 which we interpret to mean that a non-metrizable Souslin tree would be difficult to construct.

Define the *core* of X, $K(X)$, to be the set of all x in X such that $U \cap B$ is uncountable for each open set U containing x. We note that $K(X)$ is closed.

III.5. COROLLARY. *X is separable iff $K(X)$ has a void interior.*

Proof. If X is separable, then $K(X)$ itself is void. Suppose that $K(X)$ has a void interior. By III.4, $K(X) = \bigcap_{n=1}^{\infty} U_n$, where U_n is open in X. Furthermore by the normality of X we can assume $U_{n+1}^* \subset U_n$. Note that $X\backslash U_n \cap B$ is countable for each n. Hence for each n, the closures of the components of $X\backslash U_n^*$ are Souslin trees with only a countable number of branchpoints. Consequently they are separable. Since there are only a countable number of these components, we conclude that $X\backslash U_n$ is separable. Let D_n be a countable dense subset of $X\backslash U_n$. Then $D = \bigcup_{n=1}^{\infty} D_n$ is a countable dense subset of $X\backslash \operatorname{int} K(X) = X$, since we have assumed $\operatorname{int} K(X) = \emptyset$. This completes the proof.

III.6. COROLLARY. *If X is not separable, then there is a Souslin tree $X_1 \subset X$ such that $K(X_1) = X_1$.*

Proof. If X is not separable, then $\operatorname{int} K(X)$ is nonvoid. Let C be a component of $\operatorname{int} K(X)$ and let $X_1 = C^*$.

The author has recently become aware of a very nice metrizability theorem for trees which can be obtained quickly from a result of Isbell ([2], p. 629) and a result in [5], p. 426; namely, *a tree is metrizable if and only if it is an absolute retract*. As a consequence of this fact, the following question has a yes answer: *Is every one-dimensional factor space of a Tychonoff cube metrizable?* This question was the starting point of the work presented here.

References

[1] J. G. Hocking and G. S. Young, *Topology*, Reading, Mass. 1961.
[2] J. R. Isbell, *Uniform neighborhood retracts*, Pac. Jour. of Math. 11 (1961), pp. 609-648.
[3] K. Kuratowski, Topologie II, Warszawa 1961.
[4] M. A. Maurice, *Compact ordered spaces*, Mathematical Centre Tracts, Mathematisch Centrum, Amsterdam, 1964.
[5] M. Souslin, *Problème 3*, Fund. Math. 1 (1920), p. 223.
[6] L. E. Ward, Jr., *Mobs, trees, and fixed points*, Proc. AMS 8 (1957), pp. 798-804.
[7] G. T. Whyburn, *Analytic Topology*, AMS Colloq. Pub. XXVIII, 1942.

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A dense set of sewings of two crumpled cubes yields S^3

by

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I. Introduction. In 1963 Hosay [8] and Lininger [9] proved that the space obtained by sewing a crumpled cube to a 3-cell with a homeomorphism between their boundaries is actually S^3. At the Wisconsin Topology Seminar in 1965 Lininger asked several questions about sewings of one crumpled cube with another. The primary result of this paper is Theorem 1 which answers Question 7 of [10]; this result shows that, given a sewing of two crumpled cubes, there is another sewing near the first (in the metric sense) which yields S^3.

Results by Harrold and Moise [7], Ball [1], and Martin [12] indicate that not every sewing of two crumpled cubes yields S^3. Neither Theorem 1 nor the techniques of its proof show which homeomorphisms do produce S^3. Section 3 contains some information about this problem in certain cases. The strongest result is Theorem 2, which shows that any sewing matching the wild points of one crumpled cube with points of a tame Sierpiński curve in the other yields S^3. Theorem 3 proves a necessary and sufficient condition that a sewing gives S^3 for special crumpled cubes.

A *crumpled cube* C is defined as a space homeomorphic to the closure of the interior of a topological 2-sphere in E^3. The *boundary* of C, denoted $\mathrm{Bd}\,C$, consists of the points where C fails to be a 3-manifold.

When two crumpled cubes K_1 and K_2 are sewn together by a homeomorphism h of $\mathrm{Bd}\,K_1$ to $\mathrm{Bd}\,K_2$, the resulting space S is obtained from the union (disjoint) of K_1 and K_2 by identifying each x in $\mathrm{Bd}\,K_1$ with $h(x)$ in $\mathrm{Bd}\,K_2$. The homeomorphism h is referred to as a *sewing* of K_1 and K_2, and S is called the *sum* of K_1 and K_2.

Suppose that C is a crumpled cube and p is a point in $\mathrm{Bd}\,C$. The statement that p is a *piercing point* of C means that there exists an embedding f of C in S^3 so that $f(\mathrm{Bd}\,C)$ can be pierced by a tame arc at $f(p)$. Similarly, a Sierpiński curve X on $\mathrm{Bd}\,C$ is *tame* if $f(X)$ is tame under an embedding f of C into S^3 so $\mathrm{Cl}\big(S^3-f(C)\big)$ is a 3-cell. It follows from Theorem 11 of [11] that a Sierpiński curve X on $\mathrm{Bd}\,C$ is tame if and only if it is tame under some embedding of C in S^3.

The reader is referred to [2] for definition of other terms used in this paper.