Fixed points of multiple-valued transformations

by

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1. Introduction. A multiple-valued transformation $F\colon X\to Y$ is a law assigning to each point x of a topological space X, a closed non-empty subset $F(x)$ of a topological space Y. A multiple-valued transformation $F\colon X\to Y$ is called upper semi-continuous if $F^{-1}(B)$ is closed in X for each closed set B in Y. By a multiple-valued map, it is meant an upper semi-continuous multiple-valued transformation.

A *fixed point* of a multiple-valued map $F\colon X\to X$ is a point satisfying $x_0\in F(x_0)$. The purpose of this paper is to associate a Lefschetz number [3] to a class of multiple-valued self-maps of a compact polyhedron, the non-vanishing of which implies the existence of a fixed point under the maps. Eilenberg and Montgomery [1] defined the Lefschtez number $L(F)$ associated with multiple-valued self-maps $F\colon X\to X$ of a compact polyhedron X (or a metric absolute neighbourhood retract) when the image sets $F(x)$ consist of acyclic sets for all $x\in X$ and proved that $L(F)\neq 0$ implies the existence of a point x_0 satisfying $x_0\in F(x_0)$. Maxwell defined the Lefschetz number $L(f)$ associated with single-valued continuous maps $f\colon X\to X^n/S_n$ (here X^n/S_n is the orbit space under the natural action of the permutation group of n letters S_n on the nth Cartesian product of a compact polyhedron X) and proved that $L(f)\neq 0$ implies the existence of a point x_0 such that x_0 is one of the coordinates of $f(x_0)$. We shall extend the method used by Maxwell in his study of fixed points of symmetric product mappings and relate it to the study of fixed points of multiple-valued maps under convenient assumptions. Our result may be regarded as an extension of the work of Maxwell [4] and that of Eilenberg and Montgomery [1] on the fixed points of multiple-valued maps.

It is likely that our results can be proved or generalized by a possible generalized form of the Vietoris–Begle theorem and the methods of Eilenberg and Montgomery. Some results have been obtained in these directions and will be presented in a subsequent paper.

2. Let X be a compact metric space with metric ϱ. For two subsets A, B of X, we denote by:

$$d(x, A) = \inf\{\varrho(x, y)\mid y\in A\},$$

$$d(A, B) = \sup\{d(x, B) \mid x \in A\},$$
$$\delta(A, B) = \max\{d(A, B), d(B, A)\}.$$

Here $\delta(A, B)$ is the well-known Hausdorff distance between the sets A and B. Let $K^n(X)$ denote the space of all sets consisting of n distinct elements of X. A metric is introduced in the set $K^n(X)$ and it becomes a topological space with the topology induced by this metric. We make an assumption that it is possible to introduce arbitrarily fine simplicial decompositions in the spaces X and $K^n(X)$. The identification map $\tau\colon X^n \to K^n(X)$ is defined by $\tau(x_1, x_2, \ldots, x_n) = [x_1, x_2, \ldots, x_n]$ where $(x_1, x_2, \ldots, x_n)$ is an element of the product space X^n and $[x_1, x_2, \ldots, x_n]$ is the set consisting of n distinct elements of X and is an element of $K^n(X)$. Let X, Y be compact metric spaces. A continuous (single-valued) map $f\colon X \to Y$ induces in a natural manner $\tilde{f}\colon K^n(X) \to K^n(Y)$ given by

$$\tilde{f}[x_1, x_2, \ldots, x_n] = [f(x_1), f(x_2), \ldots, f(x_n)].$$

If $f_1, f_2,\colon X \to Y$ are homotopic, then so also $\tilde{f}_1, \tilde{f}_2,\colon K^n(X) \to K^n(Y)$.

PROPOSITION 2.1. *Let $\tau\colon X^n \to K^n(X)$ be the identification map. Then the chain homomorphism $\tau_\#\colon C_i(X^n, L) \to C_i(K^n(X), L)$ (where L is a field of coefficients) induces a homomorphism $\lambda_\#\colon C_i(K^n(X), L) \to C_i(X^n, L)$. ($\lambda_\#$ given here is similar to the transfer homomorphism given by Floyd* [2]).

Proof. Let e_i be a generator of the chain group $C_i(K^n(X))$, then there exists an element $c_i \in C_i(X^n)$ with $\tau_\# c_i = e_i$. We define $\lambda_\#\colon C_i(K^n(X)) \to C_i(X^n)$ by setting $\lambda_\# c_i = l_i c_i$ for some $l_i \in L$. For another generator $e'_i \in C_i(K^n(X))$, we have the equation $\lambda_\# c'_i = l'_i e'_i$ for some $l' \in L$ and $c'_i \in C_i(X^n)$. Now we define $\lambda_\#(e_i + e'_i) = l_i c_i + l'_i c'_i$. It can easily be verified that $\lambda_\#$ thus defined on the generators of the group $C_i(K^n(X))$ is linear and can thus be linearly extended to all elements of $C_i(K^n(X))$. To prove that $\lambda_\#$ is a chain homomorphism, we need to verify that $\lambda_\# \partial = \partial \lambda_\#$ (where ∂ is the usual boundary homomorphism). Consider $\partial \lambda_\# e_i = \partial l_i c_i = l_i \partial c_i = \lambda_\# \partial e_i = \lambda_\# \partial \tau_\# c_i = \lambda_\# \tau_\# \partial c_i = (\lambda_\# \tau_\#) \partial c_i = l \partial c_i$ (we observe that $\partial \tau_\# = \tau_\# \partial$), what we have shown in above for the generators can be easily extended by linearity to all elements of $C_i(K^n(X))$.

3. Let F be a multiple-valued map $F\colon X \to X$ where X is a compact metric space; such that for each $x \in X$, $F(x)$ consists of n distinct elements of X. The single-valued (continuous) map $f\colon X \to K^n(X)$ is defined as before by $f(x) = F(x)$ for each $x \in X$. We have the sequence of homomorphisms:

$$(3.1) \qquad C_i(X) \xrightarrow{f\# i} C_i(K^n(X)) \xrightarrow{\lambda\# i} C_i(X^n) \xrightarrow{\pi\# i} C_i(X)$$

where $f_\#$ is the usual chain homomorphism induced by the map f, $\lambda_\#$ is the homomorphism given by the proposition 2.1 and $\pi_\#$ is the homomorphism induced by the projection map $\pi(x_1, x_2, \ldots, x_n) = x_1$.

On the level of homology groups (simplicial theory with a field of coefficients is used throughout), we have the sequence of homomorphisms

$$(3.2) \qquad H_i(X) \xrightarrow{f^* i} H_i(K^n(X)) \xrightarrow{\lambda^* i} H_i(X^n) \xrightarrow{\pi^* i} H_i(X).$$

The composite homomorphism $\pi_{*_i} \lambda_{*_i} f_{*_i}$ is a linear transformation of the (finite dimensional) vector space $H_i(X)$. The *Lefschetz number* associated with the multiple-valued map F (of the type defined earlier) is given by

$$L(F) = \sum_i (-1)^i \operatorname{trace}(\pi_{*_i} \lambda_{*_i} f_{*_i}).$$

The number $L(F)$ depends only on the homotopy class of the single-valued map f associated with the multiple-valued map F. In case $n = 1$, this reduces to the well known Lefschetz number associated with a single-valued map. This generalizes the Lefschetz number associated with a map $f\colon X \to X^n/S_n$ where X^n/S_n is the orbit space under the action of the permutation group of n letters on X^n as defined by Maxwell [4]. Now we come to the main theorem of this paper.

THEOREM. *Let F be a multiple-valued map from a compact metric space X into itself such that $F(x)$ consists of n distinct points for each $x \in X$. Let the Lefschetz number $L(F)$ be defined as in the preceding. Then the equation $L(F) \neq 0$ implies the existence of a fixed point under F.*

Proof. The single-valued map $f\colon X \to K^n(X)$ is defined by $f(x) = F(x)$ for each $x \in X$. The distance $\varrho(x, f(x))$ from x to the nonempty closed set $f(x)$ is defined, as before, by $\varrho(x, f(x)) = \inf\{\varrho(x, y) \mid y \in f(x) = F(x)\}$. We observe here that $\varrho(x, f(x)) = 0$ if and only if $x \in f(x)$, i.e. x is fixed point under F.

Consider $X \xrightarrow{1\times f} X \times K^n(X) \xrightarrow{\psi} R$ where maps $1 \times f$ and ψ are defined by

$$(1 \times f)(x) = (x, f(x)),$$

$$\psi(x, \bar{y}) = \varrho(x, \bar{y}) = \inf\{\varrho(x, y_i) \mid y_i \in f(x)\}$$

(here $\bar{y}$ denotes the set $f(x)$).

Suppose on the contrary there exists no fixed point under F, then $(1 \times f)(x) > 0$ for each $x \in X$. By compactness of X, there exists a $\varepsilon > 0$ such that

$$\psi(1 \times f)(x) > \varepsilon \quad \text{for each} \quad x \in X.$$

We have assumed earlier that the spaces X and $K^n(X)$ admit simplicial decompositions and we now choose sufficiently fine triangulations such that mesh $X < \varepsilon/2$ and mesh $K^n(X) < \varepsilon/2$. By the simplicial approximation theorem, there exists a barycentric subdivision X' of X and a simplicial map $s\colon X' \to K^n(X)$ where s is homotopic to f and $\varrho\big(s(x), f(x)\big) < \varepsilon/2$.

We consider the composite of the sequence of maps

$$X' \xrightarrow{S\simeq f} K^n(X) \xrightarrow{\tau^{-1}} (X^n)' \xrightarrow{\varphi} X^n \xrightarrow{\pi} X$$

where S is a simplicial map homotopic to f which exists by the simplicial approximation theorem, τ^{-1} is the inverse of the identification map $\tau\colon X^n \to K^n(X)$ and is set-valued, φ is induced by the subdivision map of the barycentric subdivision of $(X^n)'$ of X and π is the projection map on the first coordinate defined by $\pi(x_1, x_2, \ldots, x_n) = x_1$. We define the set-valued map $t\colon X \to 2^{|X|}$ by $t(\sigma) = \pi\varphi\tau^{-1}S(|\sigma|)$ for each simplex σ of the triangulation of X. It can be easily verified that $|\sigma| \cap t(|\sigma|) = \emptyset$ for each simplex σ of the triangulation of X by considering the definition of the set-valued map t and the choice of the triangulations of the spaces X and $K^n(X)$.

We now consider the sequence of chain homomorphisms

$$C_i(X) \xrightarrow{(Sd)_{\#_i}} C_i(X') \xrightarrow{S_{\#_i}} C_i\big(K^n(X)\big) \xrightarrow{\pi_{\#_i}\lambda_{\#_i}} C_i(X)$$

where $(Sd)_\#$ is the chain homomorphism induced by the map of a space X into its barycentric subdivision X', $\pi_\#$ is induced by the projection map $\pi(x_1, x_2, \ldots, x_n) = x_1$ and $\lambda_\#$ is the transfer homomorphism as defined in proposition 2.1. Define the composite chain map $\theta_{\#_i} = \pi_{\#_i}\lambda_{\#_i}s_{\#_i}(Sd)_{\#_i}$. This chain map $\theta_\#$ has no fixed elements because of $|\sigma| \cap t(|\sigma|) = \emptyset$ and by a theorem due to Lefschetz [3], $L(\theta_\#) = 0$. Hence,

$$\begin{aligned} L(\theta_\#) &= \sum_i (-1)^i \operatorname{trace}\theta_{\#_i} \\ &= \sum_i (-1)^i \operatorname{trace}\theta_{*_i} \quad \text{(by a theorem due to H. Hopf)} \\ &= \sum_i (-1)^i \operatorname{trace}\big(\pi_{*_i}\lambda_{*_i}s_{*_i}(Sd)_{*_i}\big) \\ &= \sum_i (-1)^i \operatorname{trace}(\pi_{*_i}\lambda_{*_i}s_{*_i}) \\ &= \sum_i (-1)^i \operatorname{trace}(\pi_{*_i}\lambda_{*_i}f_{*_i}) \\ &= \sum_i (-1)^i \operatorname{trace}F_{*_i} \\ &= 0. \end{aligned}$$

Hence no fixed points under F implies $L(F) = 0$ and our proof is complete.

COROLLARY. *If X is a acyclic polyhedron, then $L(F) \neq 0$ and every multiple-valued map $F\colon X \to X$ of the type discussed in the above theorem admits a fixed point.*

Remark. O'Neill [5] by using a different method has obtained a similar theorem, our procedure is an extension of the work of Maxwell [4]. In the statement of the theorem the image sets $F(x)$ under $F\colon X \to X$ may consist of n distinct acyclic components by an application of the Vietoris mapping theorem.

References

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