Fixed points of multiple-valued transformations

by

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1. Introduction. A multiple-valued transformation $F: X \to Y$ is a law assigning to each point $x$ of a topological space $X$, a closed non-empty subset $F(x)$ of a topological space $Y$. A multiple-valued transformation $F: X \to Y$ is called upper semi-continuous if $F^{-1}(B)$ is closed in $X$ for each closed set $B$ in $Y$. By a multiple-valued map, it is meant an upper semi-continuous multiple-valued transformation.

A fixed point of a multiple-valued map $F: X \to X$ is a point satisfying $x_0 \in F(x_0)$. The purpose of this paper is to associate a Lefschetz number [3] to a class of multiple-valued self-maps of a compact polyhedron, the non-vanishing of which implies the existence of a fixed point under the maps. Eilenberg and Montgomery [1] defined the Lefschetz number $L(F)$ associated with multiple-valued self-maps $F: X \to X$ of a compact polyhedron $X$ (or a metric absolute neighbourhood retract) when the image sets $F(x)$ consist of acyclic sets for all $x \in X$ and proved that $L(F) \neq 0$ implies the existence of a point $x_0$ satisfying $x_0 \in F(x_0)$. Maxwell defined the Lefschetz number $L(f)$ associated with single-valued continuous maps $f: X \to X/\Sigma_k$ (here $X/\Sigma_k$ is the orbit space under the natural action of the permutation group of letters $\Sigma_k$ on the $k$th Cartesian product of the compact polyhedron $X$) and proved that $L(f) \neq 0$ implies the existence of a point $x_0$ such that $x_0$ is one of the coordinates of $f(x_0)$. We shall extend the method used by Maxwell in his study of fixed points of symmetric product mappings and relate it to the study of fixed points of multiple-valued maps under convenient assumptions. Our result may be regarded as an extension of the work of Maxwell [4] and that of Eilenberg and Montgomery [1] on the fixed points of multiple-valued maps.

It is likely that our results can be proved or generalized by a possible generalized form of the Vietoris–Begle theorem and the methods of Eilenberg and Montgomery. Some results have been obtained in these directions and will be presented in a subsequent paper.

2. Let $X$ be a compact metric space with metric $\rho$. For two subsets $A$, $B$ of $X$, we denote by:

$$d(x, A) = \inf \{\rho(x, y) \mid y \in A \} ,$$
\[ d(A, B) = \sup \{ d(a, B) \mid a \in A \} \]
\[ \delta(A, B) = \max \{ d(A, B), d(B, A) \} \]

Here \( d(A, B) \) is the well-known Hausdorff distance between the sets \( A \) and \( B \). Let \( K^n(X) \) denote the space of all sets consisting of \( n \) distinct elements of \( X \). A metric is introduced in the set \( K^n(X) \) and it becomes a topological space with the topology induced by this metric. We make an assumption that it is possible to introduce arbitrarily fine decompositions in the spaces \( X \) and \( K^n(X) \). The identification map \( \tau: X^n \rightarrow K^n(X) \) is defined by \( \tau(x_1, x_2, \ldots, x_n) = \{ x_1, x_2, \ldots, x_n \} \) where \( \{ x_1, x_2, \ldots, x_n \} \) is an element of the product space \( X^n \) and \( \{ x_1, x_2, \ldots, x_n \} \) is the set consisting of \( n \) distinct elements of \( X \) and an element of \( K^n(X) \). Let \( X, Y \) be compact metric spaces. A continuous (single-valued) map \( f: X \rightarrow Y \) induces in a natural manner \( f: K^n(X) \rightarrow K^n(Y) \) given by
\[ f(x_1, x_2, \ldots, x_n) = \{ f(x_1), f(x_2), \ldots, f(x_n) \}. \]

If \( f_1, f_2: X \rightarrow Y \) are homotopic, then so also \( \tilde{f}_1, \tilde{f}_2: K^n(X) \rightarrow K^n(Y) \).

**Proposition 2.1.** Let \( \tau: X^n \rightarrow K^n(X) \) be the identification map. Then the chain homomorphism \( \tau_*: C_n(K^n(X), L) \rightarrow C_n(X^n, L) \) (where \( L \) is a field of coefficients) induces a homomorphism \( \lambda_\tau: C_n(K^n(X), L) \rightarrow C_n(X^n, L) \). \( \lambda_\tau \) given here is similar to the transfer homomorphism given by Floyd [2].

**Proof.** Let \( e_i \) be a generator of the chain group \( C_n(K^n(X)) \), then there exists an element \( e_i \in C_n(X^n) \) with \( \tau_0 e_i = e_i \). We define \( \lambda_\tau: C_n(K^n(X)) \rightarrow C_n(X^n) \) by setting \( \lambda_\tau e_i = e_i \) for some \( e_i \in L \). For another generator \( e_i' \in C_n(K^n(X)) \), we have the equation \( \lambda_\tau e_i' = e_i' \) for some \( e_i' \in L \) and \( e_i' \in C_n(X^n) \). Now we define \( \lambda_\tau (e_i + e_i') = \lambda_\tau e_i + \lambda_\tau e_i' \). It can be easily verified that \( \lambda_\tau \) is a chain homomorphism. We need to verify that \( \lambda_\tau d_i = d_\lambda \) \( (\lambda \) is the usual boundary homomorphism). Consider \( \partial_\lambda e_i = \partial e_i = e_i \) \( (\lambda \) is the usual boundary homomorphism). We have shown above that \( \partial_\lambda = \partial_\lambda \). We have shown in above that the generators can be easily extended by linearity to all elements of \( C_n(K^n(X)) \).

3. Let \( F \) be a multiple-valued map \( F: X \rightarrow X \) where \( X \) is a compact metric space, such that for each \( x \in X \), \( F(x) \) consists of \( n \) distinct elements of \( X \). The single-valued (continuous) map \( f: X \rightarrow K^n(X) \) is defined as before by \( f(x) = F(x) \) for each \( x \in X \). We have the sequence of homomorphisms:
\[ C_n(X) \rightarrow C_n(K^n(X)) \rightarrow C_n(X^n) \rightarrow C_n(X^n) \rightarrow C_n(X) \]
where \( f_* \) is the usual chain homomorphism induced by the map \( f, \lambda_\tau \) is the homomorphism given by the proposition 2.1 and \( \lambda_\tau \) is the homomorphism induced by the projection map \( \pi(x_1, x_2, \ldots, x_n) = x_i \).

On the level of homology groups (simplicial theory with a field of coefficients is used throughout), we have the sequence of homomorphisms
\[ H_n(X) \rightarrow H_n(K^n(X)) \rightarrow H_n(X^n) \rightarrow H_n(X^n) \rightarrow H_n(X) \]

The composite homomorphism \( \lambda_{\tau_0} \lambda_{\tau_1} \lambda_{\tau_2} \) is a linear transformation of the (finite dimensional) vector space \( H_n(X) \). The Lefschetz number associated with the multiple-valued map \( F \) (of the type defined earlier) is given by
\[ L(F) = \sum (-1)^i \text{trace}(\lambda_{\tau_0} \lambda_{\tau_1} \lambda_{\tau_2}) \]

The number \( L(F) \) depends only on the homotopy class of the single-valued map \( f \) associated with the multiple-valued map \( F \). In case \( n = 1 \), this reduces to the well known Lefschetz number associated with a single-valued map. This generalizes the Lefschetz number associated with a map \( f: X \rightarrow X^n \) where \( X^n / S_n \) is the orbit space under the action of the permutation group of \( n \) letters on \( X^n \) as defined by Maxwell [4]. Now we come to the main theorem of this paper.

**Theorem.** Let \( F \) be a multiple-valued map from a compact metric space \( X \) into itself such that \( F(x) \) consists of \( n \) distinct points for each \( x \in X \). Let the Lefschetz number \( L(F) \) be defined as in the preceding. Then the equation \( L(F) = 0 \) implies the existence of a fixed point under \( F \).

**Proof.** The single-valued map \( f: X \rightarrow K^n(X) \) is defined by \( f(x) = F(x) \) for each \( x \in X \). The distance \( d(x, f(x)) \) from \( x \) to the nonempty closed set \( f(x) \) is defined, as before, by \( d(x, f(x)) = \inf \{ d(x, y) \mid y \in f(x) \} \).

We observe here that \( d(x, f(x)) = 0 \) if and only if \( x \in \text{fix} f \), i.e. \( x \) is a fixed point under \( f \).

Consider \( X \times X \rightarrow X \times K^n(X) \rightarrow X \) where maps \( 1 \times f \) and \( \psi \) are defined by
\[ (1 \times f)(x) = (x, f(x)) \]
\[ \psi(x, y) = d(x, y) = \inf \{ d(x, y) \mid y \in f(x) \} \]

(here \( \psi \) denotes the set \( f(x) \)).

Suppose on the contrary there exists no fixed point under \( F \), then \( (1 \times f)(x) > 0 \) for each \( x \in X \). By compactness of \( X \), there exists a \( \epsilon > 0 \) such that
\[ (1 \times f)(x) > \epsilon \quad \text{for each} \; x \in X \]

We have assumed earlier that the spaces \( X \) and \( K^n(X) \) admit simplicial decompositions and we now choose sufficiently fine triangulations such
that mesh $X < \varepsilon/2$ and mesh $K'(\varepsilon/2) < \varepsilon/2$. By the simplicial approximation theorem, there exists a barycentric subdivision $X'$ of $X$ and a simplicial map $\pi: X' \rightarrow K'(\varepsilon/2)$ where $\pi$ is homotopic to $f$ and $\phi(\pi(\sigma), f(\sigma)) < \varepsilon/2$.

We consider the composite of the sequence of maps

$$X' \xrightarrow{\pi} K'(\varepsilon/2) \xrightarrow{\phi} (X'/\varepsilon) \rightarrow \rho(x) \rightarrow X$$

where $\phi$ is a simplicial map homotopic to $f$ which exists by the simplicial approximation theorem, $\pi^{-1}$ is the inverse of the identification map $\pi: X' \rightarrow K'(\varepsilon/2)$ and is set-valued, $\phi$ is induced by the subdivision map of the barycentric subdivision of $(X'/\varepsilon)$ of $X$ and $\rho$ is the projection map on the first coordinate defined by $\pi(x_1, x_2, \ldots, x_n) = x_1$. We define the set-valued map $\tau: X \rightarrow 2^{\varepsilon/2}$ by $\tau(\sigma) = \pi \rho^{-1}(\rho(\pi(\sigma)))$ for each simplex $\sigma$ of the triangulation of $X$. It can be easily verified that $|\sigma| \cap \tau(\sigma) = \emptyset$ for each simplex $\sigma$ of the triangulation of $X$ by considering the definition of the set-valued map $\tau$ and the choice of the triangulations of the spaces $X$ and $K'(\varepsilon/2)$.

We now consider the sequence of chain homomorphisms

$$G_p(X) \xrightarrow{(S\delta)_p} G_p(X') \xrightarrow{S \delta} G_p(K'(\varepsilon/2)) \xrightarrow{\phi} G_p(X)$$

where $(S\delta)_p$ is the chain homomorphism induced by the map of a space $X$ into its barycentric subdivision $X'$, $\pi_1$ is induced by the projection map $\pi(x_1, x_2, \ldots, x_n) = x_1$ and $\lambda_1$ is the transfer homomorphism as defined in proposition 2.1. Define the composite chain map $\delta_1 = \pi_1 \lambda_1 \phi \delta_1$. This chain map $\delta_1$ has no fixed elements because of $|\sigma| \cap \tau(\sigma) = \emptyset$ and by a theorem due to Lefschetz [3], $L(\delta_1) = 0$. Hence,

$$L(\delta_1) = \sum \left( -1 \right)^i \text{trace} \delta_1$$

$$= \sum \left( -1 \right)^i \text{trace} \delta_1 \quad \text{(by a theorem due to H. Hopf)}$$

$$= \sum \left( -1 \right)^i \text{trace} [\pi_1 \lambda_1 \phi \delta_1 \delta_1]$$

$$= \sum \left( -1 \right)^i \text{trace} [\pi_1 \lambda_1 \phi f]$$

$$= \sum \left( -1 \right)^i \text{trace} F$$

$$= 0$$

Hence no fixed points under $F$ implies $L(F) = 0$ and our proof is complete.

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**References**


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