the ω_μ-sequence of closed sets is nested. If the ω_μ-diameters go to 0 then, since $(X, \mathfrak{U})$ is ω_μ-complete, by theorem 4.2 the intersection contains a point, x. If the ω_μ-diameters do not go to 0 then there exists a set A' of diameter $> 1_\gamma$ for some $\gamma < \omega_\mu$ contained in the intersection. In either case $A \neq \emptyset$.

We complete the proof by showing that $\{A_a\}$ converges to A. This is done by showing that for any $\beta < \omega_\mu$, if $\gamma > a_{\beta+1}$ then $d_\mu(A_\gamma, A) < 1_\beta$.

First: $A \subset N_{1_\beta}(A_\gamma)$ is true by the following argument: if $x \in A$ then $x \in \overline{N_{1_{\beta+1}}(A_{a_{\beta+1}})}$. So there exists a $y \in A_{a_{\beta+1}}$ such that $\varrho_\mu(x, y) \leqslant 1_{\beta+1}$. Since $d_\mu(A_\gamma, A_{a_{\beta+1}}) < 1_{\beta+1}$, we have that $y \in N_{1_{\beta+1}}(A_\gamma)$ and so there exists a $z \in A_\gamma$ such that $\varrho_\mu(y, z) < 1_{\beta+1}$. Since $\varrho_\mu(x, y) \leqslant 1_{\beta+1} < 1_\beta$, by the lemma, $\varrho_\mu(x, z) < 1_\beta$. If follows that $x \in N_{1_\beta}(A_\gamma)$.

Now $A_\gamma \subset N_{1_\beta}(A)$ is true by the proof below. Let $x \in A_\gamma$; then for all $\delta > \beta+1$, since $d_\mu(A_\gamma, A_{a_\delta}) < 1_{\beta+1}$, it follows that $x \in \overline{N_{1_{\beta+1}}(A_{a_\delta})}$. Hence $x \in N_{1_{\beta+1}}(\overline{N_{1_\delta}(A_{a_\beta})}) \subset N_{1_\beta}(\overline{N_{1_\delta}(A_{a_\delta})}) \subset \overline{N_{1_\beta}(\overline{N_{1_\delta}(A_{a_\delta})})}$. Therefore $x \in \overline{N_{1_\beta}(\overline{N_{1_\zeta}(A_{a_\zeta})})}$ for all $\zeta < \omega_\mu$ because $\overline{N_{1_\zeta}(A_{a_\zeta})} \supset \overline{N_{1_\delta}(A_{1_\delta})}$ if $\zeta < \delta$. So $x \in \bigcap_{\zeta<\omega_\mu} \overline{N_{1_\beta}(\overline{N_{1_\zeta}(A_{a_\zeta})})}$ $= N_{1_\beta}\big(\bigcap_{\zeta<\omega_\mu} \overline{N_{1_\zeta}(A_{a_\zeta})}\big) = N_{1_\beta}(A)$.

Since every Cauchy ω_μ-sequence converges in $(\mathcal{C}, d_\mu)$ it follows that $(\mathcal{C}, d_\mu)$ is ω_μ-complete and hence, by theorem 1.4, $(\mathcal{C}, \mathfrak{U})$ is complete.

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Realization of mappings

by

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1. Realization of a mapping. Let $\mathfrak{C}$ be a category of pairs (X, X_0), where $X_0 \subset X$ are metric spaces and morphisms are continuous. Though many of the results of this section are valid for an arbitrary $\mathfrak{C}$ we shall pay our attention mainly to the three following categories:

a) The category $\mathfrak{S}$ of all metric pairs and all continuous mappings.

b) The category $\mathfrak{P}$ of polyhedral pairs and simplicial mappings. By a polyhedral pair (X, X_0) we understand a finite polyhedron X with a triangulation and a subpolyhedron X_0 of X in this triangulation. Simplicial mappings are considered with respect to the given triangulations. However, the same polyhedral pair may have various triangulations.

c) The category $\mathfrak{M}$ of pairs of differentiable manifolds and differentiable mappings. By a pair of manifolds (X, X_0) we understand a separable manifold X (with boundary or not) of class C^∞ and its submanifold X_0; a differentiable mapping is also of class C^∞.

As it is a frequent practice to do, we identify the pair $(X, \emptyset)$ with the space X alone. If (X, X_0) is an object of $\mathfrak{C}$, then we call X_0 a subobject of X. An isomorphism h of an object A onto a subobject B of an object X is called an *imbedding* of A into X. If such an imbedding exists, the object A is called *imbeddable* in X.

If A, B are subsets of a metric space X and $f\colon A \to B$ is a mapping, then we define $D(f) = \sup_{x\in A} \varrho(x, f(x))$.

Let A, B and X be objects and let $f\colon A \to B$ be a morphism. Let $h\colon B \to X$ be an imbedding of B into X. We say that the morphism f is *realizable* in $X \operatorname{rel} h$ if there exists a sequence $\{h_n\}$ (called a *realization* of $f \operatorname{rel} h$), where $h_n\colon A \to X$ is an imbedding of A into X for $n = 1, 2, \ldots$, such that $\lim_n D(f_n) = 0$ for $f_n = hfh_n^{-1}$.

If an object B is imbeddable in X and if a morphism $f\colon A \to B$ is realizable in $X \operatorname{rel} h$ for any imbedding h of B into X, then we simply say that the morphism f is *realizable* in X. [1]

The definition depends on the category $\mathfrak{C}$ under consideration and we will always make it clear if a statement concerns a particular $\mathfrak{C}$. Usually,

[1] In [8] such a morphism has been called *imbeddable* in X.

we shall withdraw the name of the category and apply one of the adverbs: topologically, simplicially, or differentiably in the case of the category $\mathfrak{S}$, $\mathfrak{P}$, or $\mathfrak{M}$, respectively. If, however, an arbitrary $\mathfrak{C}$ is being considered, then we use the words object and morphism instead of space and mapping, respectively.

Topological realizability of a continuous mapping $f\colon A\to B\,\mathrm{rel}\,h$ is a necessary but not sufficient condition for the existence of a topological imbedding of the mapping cylinder of f which is an extension of h. Another notion which is in a relation to the given above was introduced by M. McCord [4] under the name of approximability by homeomorphisms. However, the definition of McCord requires that A and B are subsets of X and that the homeomorphisms h_n can be extended over neighborhoods of A in X. Thus his definition depends on the situation of A and B in X and is not equivalent to ours even for polyhedra in Euclidean spaces.

We shall now prove some general facts about realizability of morphisms. The definition implies directly the following

1.1. LEMMA. *If a morphism $f\colon A\to B$ is realizable in $X\,\mathrm{rel}\,h$ and $A'\subset A$, $B'\subset B$ are subobjects such that $f(A')\subset B'$, then $f'=f|A'\colon A'\to B'$ is realizable in $X\,\mathrm{rel}\,h|B'$.*

Proof. Indeed, if $\{h_n\}$ is a realization of $f\,\mathrm{rel}\,h$, then $\{h_n|A'\}$ is a realization of $f'\,\mathrm{rel}\,h|B'$.

1.2. LEMMA. *If $f\colon A\to B$ is an isomorphism and the object B is imbeddable in X, then f is realizable in X.*

Proof. Indeed, it is sufficient to verify that f is realizable in $X\,\mathrm{rel}\,h$ for any imbedding h of B into X. But we can define $h_n=hf$ for $n=1,2,\dots$ and this evidently completes the proof.

1.3. LEMMA. *If the morphisms $f\colon A\to B$ and $g\colon B\to C$ are realizable in X, then the composition $gf\colon A\to C$ is also realizable in X.*

Proof. By the assumption, C is imbeddable in X. Let h be an imbedding of C into X. Let ε be a positive number and let us choose an index n such that $D(g_n)<\varepsilon/2$, where $\{h_n\}$ is a realization of the morphism g in $X\,\mathrm{rel}\,h$ and $g_n=hgh_n^{-1}$. Let $\{h'_m\}$ be a realization of the morphism f in $X\,\mathrm{rel}\,h_n$. Then there exists an m such that $D(f_m)<\varepsilon/2$, where $f_m=h_nfh_m'^{-1}$. Thus for any positive ε there exists an imbedding $\chi_\varepsilon=h'_m$ of A into X and a morphism $\varphi_\varepsilon=g_nf_m$ such that $h(gf)=g_nh_nf=g_nf_m\chi_\varepsilon=\varphi_\varepsilon\chi_\varepsilon$ and $D(\varphi_\varepsilon)\leqslant D(g_n)+D(f_m)\leqslant\varepsilon/2+\varepsilon/2=\varepsilon$. This proves the lemma.

1.4. LEMMA. *If the set of imbeddings is dense in the mapping space X^A and there exists an imbedding of B into X, then any morphism $f\colon A\to B$ is realizable in X.*

Proof. Let h be an imbedding of B into X. By the assumption, for any natural number n there exists an imbedding h_n of A into X such that $\varrho\big(hf(a),h_n(a)\big)<1/n$ for $a\in A$ and $n=1,2,\dots$ Evidently $\{h_n\}$ is a realization of the mapping f in $X\,\mathrm{rel}\,h$.

1.5. COROLLARY. *Let $f\colon A\to B$ be a continuous mapping. If A, B are separable, then f is topologically realizable in the Hilbert cube I^ω. If A, B are separable, $\dim A\leqslant m$ and B is topologically imbeddable in the $(2m+1)$-dimensional cube I^{2m+1} (in particular, if $\dim B\leqslant m$), then f is topologically realizable in I^{2m+1}. If A, B are polyhedra, f is simplicial, $\dim A\leqslant m$ and B is simplicially imbeddable in the cube I^{2m+1} (in particular, if $\dim B\leqslant m$), then f is simplicially realizable in I^{2m+1}. If A, B are manifolds, f is differentiable, $\dim A\leqslant m$ and B is differentiably imbeddable in the Euclidean space E^{2m+1} (in particular, if $\dim B\leqslant m$), then f is differentiably realizable in E^{2m+1}.*

Proof. The corollary is an immediate consequence of Lemma 1.4 and of the following theorems, respectively: the Urysohn Theorem ([5], p. 120), the Menger-Nöbeling Theorem ([6], p. 69), the polyhedral lemma to the Menger-Nöbeling Theorem ([7], I.2), and the Whitney Theorem [9].

1.6. THEOREM. *The set of morphisms $f\colon A\to B$, where A is compact, which are realizable in X is closed in the mapping space B^A.*

Proof. Let $f^i\colon A\to B$ be a sequence of mappings realizable in X and let $f=\lim\limits_i f^i$. Let $h\colon B\to X$ be an imbedding and let ε be a positive number. Then there exists an i such that $\mathrm{dist}(hf,hf^i)<\varepsilon/2$ and, since f^i is realizable in $X\,\mathrm{rel}\,h$, there exists an imbedding h^i_j of A into X such that $\mathrm{dist}(f^i_j,\mathrm{id})<\varepsilon/2$ for $f^i_j=hf_i(h^i_j)^{-1}$. Let us define $h_\varepsilon=h^i_j$, $f_\varepsilon=hf(h^i_j)^{-1}$. Then $f_\varepsilon=hfh_\varepsilon^{-1}$ and $\mathrm{dist}(f_\varepsilon,\mathrm{id})\leqslant\mathrm{dist}(f_\varepsilon,f^i_j)+\mathrm{dist}(f^i_j,\mathrm{id})<\varepsilon/2+\varepsilon/2=\varepsilon$. Thus the proof has been completed.

An object A is called *quasi-contractible* to a point $a_0\in A$ if the mapping $\pi\colon A\to\{a_0\}$ is a morphism realizable in $A\,\mathrm{rel}\,\nu$, where $\nu\colon\{a_0\}\to A$ is the inclusion.

1.7. THEOREM. *If A is an object quasi-contractible to a_0 and B is an object such that $A\times B$ is also an object, then any morphism $f\colon A\to B$ is realizable in $A\times B\,\mathrm{rel}\,i_0$, where i_0 is the isomorphism of B onto $(a_0)\times B$ in $A\times B$.*

Proof. Since A is quasi-contractible to a_0, there exists a sequence of imbeddings $\{\bar h_n\}$ of A into A such that $\lim\limits_n\mathrm{diam}\big(\bar h_n(A)\cup\{a_0\}\big)=0$. Let us define the morphism $h_n\colon A\to A\times B$ by the formula: $h_n(a)=\big(\bar h_n(a),f(a)\big)$ for $a\in A$, $n=1,2,\dots$ Since $\bar h_n$ is an imbedding, so is h_n. Next, let us define the morphism $f_n\colon h_n(A)\to(a_0)\times B\subset A\times B$ by the formula: $f_n\big(\bar h_n(a),f(a)\big)=\big(a_0,f(a)\big)$ for $\big(\bar h_n(a),f(a)\big)\in h_n(A)$, $n=1,2,\dots$ Then $\varrho\Big(f_n\big(\bar h_n(a),f(a)\big),\big(\bar h_n(a),f(a)\big)\Big)=\varrho\big(\bar h_n(a),a_0\big)\leqslant\mathrm{diam}\big(h_n(A)\cup\{a_0\}\big)$; consequently, $\lim\limits_n D(f_n)=0$. Finally, we have $f_nh_n(a)=\big(a_0,f(a)\big)=i_0f(a)$, and this completes the proof.

An object A is called *uniformly quasi-contractible* if for every point $a \in A$ the mapping $\pi^a\colon A \to \{a\}$ is a morphism with a realization $\{h_n^a\}$ in $A \operatorname{rel} \nu^a$, where $\nu^a\colon \{a\} \to A$ is the inclusion, and for each n the mapping $h_n\colon A \times A \to A$ defined by $h_n(a', a'') = h_n^{a'}(a'')$ is a morphism.

1.8. THEOREM. *Let A be an object in a multiplicative category and let A be uniformly quasi-contractible. Let $f_j\colon A \to A$ be a morphism for $j = 1, 2, \ldots$ $\ldots, m$. Let $a_0 \in A$ be a point and let i_0 be the isomorphism of A^m onto $A^m \times (a_0)$ in A^{m+1}. Then the product of mappings $f = f_1 \times f_2 \times \ldots \times f_m\colon A^m \to A^m$ is realizable in $A^{m+1} \operatorname{rel} i_0$.*

Proof. We have $i_0 f(a_1, a_2, \ldots, a_m) = \big(f_1(a_1), f_2(a_2), \ldots, f_m(a_m), a_0\big)$ for every $(a_1, a_2, \ldots, a_m) \in A^m$. Since A is uniformly quasi-contractible, there exists a sequence of imbeddings $\{\bar{h}_n^a\}$ of A into A such that $\lim\limits_n \operatorname{diam}\big(\bar{h}_n^a(A) \cup \{a\}\big) = 0$ and $\bar{h}_n\colon A \times A \to A$, where $\bar{h}_n(a', a'') = \bar{h}_n^{a'}(a'')$, is a morphism for each n. Let us define the morphism $h_n\colon A^m \to A^{m+1}$ by the formula: $h_n(a_1, a_2, \ldots, a_m) = \big(\bar{h}_n^{f_1(a_1)}(a_2), \ldots, \bar{h}_n^{f_{m-1}(a_{m-1})}(a_m), \bar{h}_n^{f_m(a_m)}(a_1), \bar{h}_n^{a_0}(a_1)\big)$, where $(a_1, a_2, \ldots, a_m) \in A^m$ and $n = 1, 2, \ldots$ We shall prove that h_n is an imbedding. Since each coordinate is a morphism, it is sufficient to verify that the mapping is one-to-one. Let us suppose that $h_n(a_1', a_2', \ldots, a_m') = h_n(a_1'', a_2'', \ldots, a_m'')$. Then, according to the last coordinates, we have $\bar{h}_n^{a_0}(a_1') = \bar{h}_n^{a_0}(a_1'')$, whence $a_1' = a_1''$. Consequently, according to the first coordinates, we have $a_2' = a_2''$ and then, by induction, we get $a_j' = a_j''$ for $j = 1, 2, \ldots, m$.

On the other hand, we have $\lim\limits_n \bar{h}_n^{f_j(a_j)}(a_{j+1}) = f_j(a_j)$ for $j = 1, 2, \ldots$ $\ldots, m-1$, $\lim\limits_n \bar{h}_n^{f_m(a_m)}(a_1) = f_m(a_m)$, $\lim\limits_n \bar{h}_n^{a_0}(a_1) = a_0$, and this completes the proof.

2. Realization in the plane. In this section we consider the category $\mathfrak{S}$ of all metric spaces and continuous mappings and the following problem. Let A and B be two planar (i.e. topologically imbeddable in the plane) continua. Give a necessary and sufficient condition that any continuous mapping $f\colon A \to B$ can be topologically realized in the plane. A partial answer to this question is given by the following theorem.

Let I denote a closed segment, let T denote a triod i.e. the union of three closed segments disjoint beyond one common end-point, let O denote a circle, and let Q denote the union $O \cup I$, where $O \cap I$ consists of an end-point of I. We write $A \underset{\text{top}}{\subset} A'$ to denote that A is topologically imbeddable in A'. If $A \underset{\text{top}}{\subset} A'$ and $B \underset{\text{top}}{\subset} B'$, then we simply write $(A, B) \underset{\text{top}}{\subset} (A', B')$.

2.1. THEOREM. *Let A and B be locally connected continua and let $\dim A \leqslant 1$. A necessary and sufficient condition that every continuous*

mapping $f\colon A \to B$ be topologically realizable in the plane is that either $(A, B) \underset{\text{top}}{\subset} (T, O)$ or $(A, B) \underset{\text{top}}{\subset} (Q, I)$.

In order to prove the necessity of the condition, we first prove the following lemmas.

Let H denote the union of three closed segments I_0, I_1, I_2 such that $I_1 \cap I_2 = 0$ and $I_0 \cap I_1$, $I_0 \cap I_2$ are the end-points of I_0. Let K denote the union of four closed segments disjoint beyond one common end-point.

2.2. LEMMA. *Let C be a locally connected continuum. Then*

(i) *$C \underset{\text{top}}{\not\subset} T$ if and only if either $K \underset{\text{top}}{\subset} C$ or $H \underset{\text{top}}{\subset} C$ or $O \underset{\text{top}}{\subset} C$;*

(ii) *$C \underset{\text{top}}{\not\subset} O$ if and only if $T \underset{\text{top}}{\subset} C$;*

(iii) *$C \underset{\text{top}}{\not\subset} Q$ if and only if either $K \underset{\text{top}}{\subset} C$ or $H \underset{\text{top}}{\subset} C$;*

(iv) *$C \underset{\text{top}}{\not\subset} I$ if and only if either $T \underset{\text{top}}{\subset} C$ or $O \underset{\text{top}}{\subset} C$.*

Proof. The sufficiency of the condition is obvious in any case. In the proof of necessity we make use of the ramification theory for locally connected continua ([6], section 46).

(i) Let $K, H, O \underset{\text{top}}{\not\subset} C$; then $\operatorname{ord} C \leqslant 3$, there exists at most one point of ramification order 3, and C is acyclic. This implies that C is either a point or I or T.

(ii) Let $T \underset{\text{top}}{\not\subset} C$; then $\operatorname{ord} C \leqslant 2$ and C is either a point or I or O.

(iii) Let $K, H \underset{\text{top}}{\not\subset} C$; then $\operatorname{ord} C \leqslant 3$ and there exists at most one point of ramification order 3. If C is acyclic, then by (i), $C \underset{\text{top}}{\subset} T \underset{\text{top}}{\subset} Q$. Otherwise, there exists exactly one simple closed curve in C and consequently $C = Q$.

(iv) Let $T, O \underset{\text{top}}{\not\subset} C$; then $\operatorname{ord} C \leqslant 2$ and C is acyclic. Hence $C \underset{\text{top}}{\subset} I$.

2.3. LEMMA. *Let A and B be locally connected continua such that neither $(A, B) \underset{\text{top}}{\subset} (T, O)$ nor $(A, B) \underset{\text{top}}{\subset} (Q, I)$. Then the pair (A, B) contains topologically one of the following pairs: (I, T), (H, I), (K, I), or (O, O).*

Proof. We consider the four cases possible: $A \underset{\text{top}}{\not\subset} Q$, $A \underset{\text{top}}{\not\subset} T$ and $B \underset{\text{top}}{\not\subset} Q$, $A \underset{\text{top}}{\not\subset} O$ and $B \underset{\text{top}}{\not\subset} I$, $B \underset{\text{top}}{\not\subset} O$. In any case an easy application of Lemma 2.2 gives the conclusion.

Proof of Theorem 2.1. Necessity. Let us note that for any continuous mapping $f'\colon A' \to B'$, where $(A', B') \underset{\text{top}}{\subset} (A, B)$ and $(A', B') = (I, T), (H, I), (K, I)$, there exists a continuous extension over A, for the graphs are Absolute Retracts. The same is valid if $(A', B') = (O, O)$ and $\dim A \leqslant 1$ ([6], 48. VI). If, however, $(O, O) = (A', B') \underset{\text{top}}{\subset} (A, B)$ and

$\dim A = 2$, then also $(K, I) \underset{\text{top}}{\subset} (A, B)$, for A is locally connected. Thus, by Lemmas 2.3 and 1.1, it is sufficient to give for each pair $(A, B) = (I, T), (H, I), (K, I)$, or (O, O) an example of a continuous mapping $f\colon A \to B$ which is not realizable in the plane rel to any imbedding of B in the plane. Since the plane is topologically homogeneous with respect to the graphs under consideration, it is sufficient to built such examples for a particular imbedding h.

a). Example. Let $I = [-5, 5]$, $T = \overline{(-2, 0), (2, 0)} \cup \overline{(0, 0), (0, 1)}$ and let i be the inclusion of T into the plane. We define the piece-wise linear mapping $f\colon I \to T$ by the conditions: $f(-5) = (-2, 0)$, $f(-4) = f(2) = (-1, 0)$, $f(-3) = f(-1) = f(1) = f(3) = (0, 0)$, $f(-2) = f(4) = (1, 0)$, $f(5) = (2, 0)$, $f(0) = (0, 1)$. Suppose that $\{h_n\}$ is a realization of f in the plane rel$i$. Let $a_n^\nu = h_n(\nu)$ for $\nu \in I$ and $n = 1, 2, \ldots$ Then for n sufficiently large there exist in the plane two arcs $L' = \widehat{a_n^{-5}, a_n^2}$, $L'' = \widehat{a_n^{-2}, a_n^5}$ and a triod Y with the end-points a_n^{-5}, a_n^0, a_n^5 such that L', L'', Y are mutually disjoint and disjoint with $h_n(I)$ beyond the end-points. Thus $h_n(I) \cup \cup L' \cup L'' \cup Y$ would be one of the Kuratowski's curves imbedded in the plane.

b). Example. Let $H = \overline{(0, -2), (0, 3)} \cup \overline{(1, -5), (1, 6)} \cup \overline{(0, 0), (1, 0)}$ $I = [0, 5]$ and let i be the natural inclusion of I into the x-axis of the plane. We define the piece-wise linear mapping $f\colon H \to I$ by the conditions: $f(0, -2) = f(1, -5) = 0$, $f(0, -1) = f(1, -4) = f(1, 2) = 1$, $f(0, 0) = f(1, -3) = f(1, 1) = f(1, 3) = 2$, $f(0, 1) = f(1, -2) = f(1, 0) = f(1, 4) = 3$, $f(0, 2) = f(1, -1) = f(1, 5) = 4$, $f(0, 3) = f(1, 6) = 5$. Suppose that $\{h_n\}$ is a realization of f in the plane rel$i$. Let $a_n^{\nu,\mu} = h_n(\nu, \mu)$ for $(\nu, \mu) \in H$ and $n = 1, 2, \ldots$ Then for n sufficiently large there exist in the plane two disjoint triods: Y' with the end-points $a_n^{0,-2}, a_n^{1,-5}, a_n^{1,2}$ and Y'' with the end-points $a_n^{0,3}, a_n^{0,6}, a_n^{1,-1}$ which are disjoint with $h_n(H)$ beyond the end-points. Thus $h_n(H) \cup Y' \cup Y''$ would be one of the Kuratowski's curves imbedded in the plane.

c). Example. Let $K = \overline{(-3, 0), (7, 0)} \cup \overline{(0, -9), (0, 5)}$, $I = [0, 6]$ and let i be the natural inclusion of I into the x-axis of the plane. We define the piece-wise linear mapping $f\colon K \to I$ by the conditions: $f(-3, 0) = f(7, 0) = 0$, $f(-2, 0) = f(6, 0) = f(0, -4) = 1$, $f(-1, 0) = f(5, 0) = f(0, -3) = f(0, -5) = f(0, 1) = 2$, $f(0, 0) = f(4, 0) = f(0, -2) = f(0, -6) = f(0, 2) = 3$, $f(3, 0) = f(1, 0) = f(0, -1) = f(0, -7) = f(0, 3) = 4$, $f(2, 0) = f(0, -8) = f(0, 4) = 5$, $f(0, -9) = f(0, 5) = 6$. Suppose that $\{h_n\}$ is a realization of f in the plane rel$i$. Let $a_n^{\nu,\mu} = h_n(\nu, \mu)$ for $(\nu, \mu) \in K$ and $n = 1, 2, \ldots$ Then for n sufficiently large there exist in the plane two arbitrarily small arcs $L' = \widehat{a_n^{-1,0}, a_n^{0,1}}$, $L'' = \widehat{a_n^{0,-1}, a_n^{1,0}}$ which are mutually disjoint and disjoint with $h_n(K)$ beyond the end-points.

Replacing the arc $\widehat{a_n^{-1,0}, a_n^{0,0}} \subset h_n(K)$ by L' and the arc $\widehat{a_n^{0,-1}, a_n^{0,0}} \subset h_n(K)$ by L'' we get the same situation as in Example b and the reasoning of that example can be continued.

d). Example. Let O be the set of complex numbers z for which $|z| = 1$, let i be the inclusion of O in the plane, and let $f\colon O \to O$ be given by the formula $f(z) = z^2$ for $z \in O$. Evidently $\deg f = 2$. Suppose that $\{h_n\}$ is a realization of f in the plane rel$i$. Then for n sufficiently large we have $(0, 0) \notin h_n(O)$ and, by the Borsuk Theorem on Disconnecting 0 from ∞ [3], we get $|\deg f| \leqslant 1$.

Before we pass to the proof of sufficiency of the condition given by Theorem 2.1 let us prove the following

2.4. Lemma. *Any continuous mapping $f\colon Q \to I$ is topologically realizable in the plane.*

Proof. Since the plane E^2 is topologically homogeneous with respect to closed arcs, it is evidently sufficient to prove that any continuous mapping f of Q onto I is realizable in E^2rel$i$, where i is the natural inclusion of the interval $I = [0, 1]$ into the x-axis. Moreover, by virtue of Theorem 1.6 and by the simplicial approximation theorem, we can assume that the mapping f is simplicial in some triangulations of Q and of I. By the same reason, we can assume that f maps any 1-simplex of the triangulation of Q onto a 1-simplex of the triangulation of I.

Let $a^0, a^1, \ldots, a^{p-1}, a^p = a^0$ be those consecutive vertices in $Q = O \cup I$ which lie in O and let $a^0 = t^0, t^1, \ldots, t^q$ be those vertices in Q which lie in I. Let $0 = b^0 < b^1 < \ldots < b^r = 1$ be the vertices of the triangulation of I, where $r \geqslant 2$.

By a fold of the mapping f we understand any sequence $a^\nu, \ldots, a^{\nu+\mu}, \ldots \ldots, a^{\nu+2\mu}, \ldots, a^{\nu+3\mu}$, where $\mu > 0$ and indices are reduced mod$p$, such that $f(a^{\nu+\alpha}) = f(a^{\nu+2\mu-\alpha}) = f(a^{\nu+2\mu+\alpha})$ for $\alpha = 0, 1, \ldots, \mu$.

First of all let us note that if f has no folds, then it can be realized in E^2rel$i$. Indeed, in this case $p = 2r$ and there exists $0 \leqslant \nu \leqslant p$ such that $f(a^{\nu+\alpha}) = f(a^{\nu-\alpha}) = b^\alpha$ for $\alpha = 0, 1, \ldots, r$, where indices are reduced mod$p$. We can assume without a real loss of generality that $0 \leqslant \nu \leqslant r$. Let n be a natural number. We define a function h_n' of the set of vertices of Q into E^2 as follows: $h_n'(a^\nu) = \left(b^0, \dfrac{2}{3n}\right)$, $h_n'(a^{\nu+\alpha}) = \left(b^\alpha, \dfrac{1}{n}\right)$ for $\alpha = 1, 2, \ldots \ldots, r-1$. $h_n'(a^{\nu-\alpha}) = \left(b^\alpha, \dfrac{1}{3n}\right)$ for $\alpha = 1, 2, \ldots, r-1$, $h_n'(a^{\nu+r}) = h_n'(a^{\nu-r}) = \left(b^r, \dfrac{2}{3n}\right)$; $h_n'(t^\beta) = \left(f(t^\beta), \dfrac{1}{3n} - \dfrac{\beta}{3qn}\right)$ for $\beta = 0, 1, \ldots, q$. It is easy to see that the piece-wise linear extension h_n of h_n' over Q is a homeomorphism of Q into E^2 and that $\{h_n\}$ is a realization of f in E^2rel$i$.

We now procede by induction with respect to p. Let $p = 2r$. Since f is onto, it has no folds, and can therefore be realized in $E^2 \operatorname{rel} i$.

Let us suppose that the conclusion is valid for all numbers s which satisfy the inequalities $2r \leqslant s \leqslant p-1$ and we shall prove it for the number p. Let $a^{\nu}, \ldots, a^{\nu+\mu}, \ldots, a^{\nu+2\mu}, \ldots, a^{\nu+3\mu}$ be a fold of f. Let us choose two of the three arcs $L^1 = \widehat{a^{\nu}, a^{\nu+\mu}}$, $L^2 = \widehat{a^{\nu+\mu}, a^{\nu+2\mu}}$, $L^3 = \widehat{a^{\nu+2\mu}, a^{\nu+3\mu}}$ in O which do not contain the point a^0 in the interiors, then let us cut off their interiors and identify the two end-points of each of them. Suppose that the arc L^{k_0}, where $1 \leqslant k_0 \leqslant 3$, has not been canceled. The new triangulation of Q has $p - 2^{\mu} < p$ vertices in O. Moreover, by the definition of a fold, the mapping f induces in an obvious manner a mapping f' of Q onto I which is simplicial in the new triangulation of Q. By the inductive assumption, there exists a realization $\{h'_n\}$ of f' in $E^2 \operatorname{rel} i$. Let ε be a positive number and let $D(if'h'^{-1}_n) < \varepsilon/2$ for some n.

It is clear that we can replace the arc $h'_n(L^{k_0})$ by three arcs $M^k = \widehat{x^k, y^k}$ $(k = 1, 2, 3)$ which are distant from $h'_n(L^{k_0})$ less than $\varepsilon/2$, are mutually disjoint and disjoint with $h'_n(O)$, and are such that at most one of them meets $h'_n(I)$. Then, by a suitable modification of h'_n on I we can achieve that if $h'_n(I)$ meets one of the arcs M^k $(k = 1, 2, 3)$, then the intersection consists of the point $h'_n(a^0)$. Finally, we match pair-wise the end-points of M^k $(k = 1, 2, 3)$ and of $h'_n(L^{k_0})$ in such a manner that $h'_n(O) - h'_n(L^{k_0}) \cup \cup M^1 \cup M^2 \cup M^3$ is again a simple closed curve which meets $h'_n(I)$ only at the point $h'_n(a^0)$. The construction naturally induces a homeomorphism h_n of Q into E^2 which agrees with h'_n on $Q - (L^1 \cup L^2 \cup L^3)$ and maps the arcs L^1, L^2, L^3 onto M^1, M^2, M^3 (possibly not in the same order). Since evidently $D(ifh_n^{-1}) < \varepsilon$, we infer that $\{h_n\}$ is a realization of f in $E^2 \operatorname{rel} i$.

Thus the proof is completed by induction.

Since I is an Absolute Retract, any continuous function f': $A \to I$, where $A \subset Q$, can be extended over Q. Thus Lemmas 1.1 and 2.4 imply

2.5. Corollary. *Any continuous mapping f: $A \to B$, where A, B are continua satisfying $(A, B) \underset{\text{top}}{\subset} (Q, I)$ is topologically realizable in the plane.*

2.6. Lemma. *Any continuous mapping f: $I \to O$ is topologically realizable in the plane.*

Proof. By topological homogeneity of the plane E^2 with respect to simple closed curves, it is sufficient to prove that any continuous mapping f: $I \to O$ is realizable in $E^2 \operatorname{rel} i$, where i is the natural inclusion of the unit circle in E^2. In order to do this we define h_n: $I \to E^2$ by means of the formula $h_n(x) = \left(1 - \frac{x}{n}\right) \cdot f(x)$. Evidently h_n is a homeomorphism for each $n = 1, 2, \ldots$ and $\{h_n\}$ is a realization of f in $E^2 \operatorname{rel} i$.

Proof of Theorem 2.1. Sufficiency. If $(A, B) \underset{\text{top}}{\subset} (Q, I)$, then by Corollary 2.5, any continuous mapping f: $A \to B$ is realizable in the plane. Let us now suppose that f: $A \to B$, where $(A, B) \underset{\text{top}}{\subset} (T, O)$. Since A is contractible to a point, there exist continuous mappings φ: $A \to I$ and ψ: $I \to B$ such that $f = \psi\varphi$. It is now sufficient to make use of Lemma 1.3, Corollary 2.5, Lemmas 2.6 and 1.1.

Since any mapping f of the segment I into the "condensed sinusoid" S is in fact a mapping of I into a closed arc, Theorem 2.1 implies that f is topologically realizable in the plane. Thus the assumption of local connectedness is essential for that theorem.

In connection with Theorem 2.1 the following problem can be raised.

2.7. Problem. *Which plane continua are images of the segment under a continuous mapping realizable in the plane?*

We shall now consider more exactly the case of mappings of a circle into itself. Namely, we shall prove the following

2.8. Theorem. *A continuous mapping f: $O \to O$ is topologically realizable in the plane if and only if $|\deg f| \leqslant 1$.*

Proof. Since the plane is topologically homogeneous with respect to simple closed curves, it is sufficient to consider realizability $\operatorname{rel} i$, where i: $O \to E^2$ is the natural inclusion. Necessity of the condition is proved by the same reasoning as in the proof of Theorem 2.1 (Example d). In order to prove that the condition is sufficient we consider the two cases:

Case 1. $\deg f = 0$. Then there exist continuous mappings φ: $O \to I$ and ψ: $I \to O$ such that $f = \psi\varphi$. Now the conclusion follows from Theorem 2.1 and Lemma 1.3.

Case 2. $\deg f = \pm 1$. It is clear that we can assume that $\deg f = 1$. By virtue of the simplicial approximation theorem and by Theorem 1.6, it is sufficient to consider the case of an f which is simplicial in some triangulations O_1 and O_2 of O. Let v be a vertex of O_2 and let $\{w_1, w_2, \ldots, w_k\} = f^{-1}(v)$. We can evidently assume that $k > 1$ and that $v = (1, 0)$. Let L be a lemniscate consisting of two circles L' and L'' with a node u. For each $i = 1, 2, \ldots, k+1 \pmod{k}$ let $M'_i = \widehat{w_i, w_{i+1}} \subset O_1$ and $M''_i = O_1 - \operatorname{int} M'_i$. Then for each i there exist two simplicial mappings (with respect to a triangulation of L): φ_i: $O_1 \to L$, ψ_i: $L \to O_2$ such that $\varphi_i(M'_i) = L'$, $\varphi_i(M''_i) = L''$, $\varphi_i(w_i) = \varphi_i(w_{i+1}) = u$ and $f = \psi_i\varphi_i$. The mapping ψ_i induces two continuous mappings ψ'_i: $L' \to O_2$, ψ''_i: $L'' \to O_2$. It is clear that $|\deg\psi'_i| \leqslant 1$ for $i = 1, 2, \ldots, k$ and $\sum_{i=1}^{k} \deg\psi'_i = \deg f = 1$. Hence there exists an i_0 such that $\deg\psi'_{i_0} = 1$. On the other hand, $\deg\psi'_i + \deg\psi''_i = \deg f = 1$ for $i = 1, 2, \ldots, k$. Hence $\deg\psi''_{i_0} = 0$.

Let ε be a positive number. Let $\alpha(\vartheta)$ be that branch of $\operatorname{arc}\tan\vartheta$ which satisfies $0 < \alpha(\vartheta) < \pi$. Let (r, ϑ) denote polar coordinates of a point in the plane and let us define $I = \left\{(r, \vartheta)\colon\ 0 \leqslant \vartheta \leqslant 2\pi,\ r = \frac{2\varepsilon}{\pi}\cdot\alpha(\vartheta)+1-\frac{\varepsilon}{2}\right\}$, $J = \left\{(r, \vartheta)\colon\ -\infty < \vartheta < \infty,\ r = \frac{2\varepsilon}{\pi}\cdot\alpha(\vartheta)+1-\varepsilon\right\}$. By the covering homotopy theorem, there exist continuous mappings $\tau'\colon M'_{i_0}\to I$ and $\tau''\colon M''_{i_0}\to J$ such that $f|M'_{i_0} = \xi\tau'$, $f|M''_{i_0} = \xi\tau''$, where $\xi\colon E^2-(0)\to O$ is the radial projection.

We define a mapping h_ε of O_1 into E^2 as follows. Let δ be a real-valued function defined on O_1 such that $\delta(w_{i_0}) = -1$, $\delta(w_{i_0+1}) = 1$, and which is monotone on M'_{i_0} and on M''_{i_0}. If $p \in M'_{i_0}$ and $\tau'(p) = (r, \vartheta)$, then we define $h_\varepsilon(p) = \left(r+\frac{\varepsilon}{4}\cdot\delta(p), \vartheta\right)$; if $p \in M''_{i_0}$ and $\tau''(p) = (r, \vartheta)$, then we define $h_\varepsilon(p) = \left(r-\frac{\varepsilon}{4}\cdot\delta(p), \vartheta\right)$. It is easy to see that h_ε is a homeomorphism and that $\xi h_\varepsilon(p) = f(p)$. Since $\varrho\big(h_\varepsilon(p), f(p)\big) < \varepsilon$, the proof has been completed.

3. Mappings of cubes and of spheres. Let $I = [-1, 1]$ and $I^m = I_1\times I_2\times\ldots\times I_m$, where $I_j = I$ for $j = 1, 2, \ldots, m$. The natural inclusion of the cube I^m into a cube I^k, where $k \geqslant m$, will be denoted by i. Let $Q^m = \{(x_1, x_2, \ldots, x_m) \in E^m\colon\ x_1^2+x_2^2+\ldots+x_m^2 \leqslant 1\}$. The natural inclusion of the ball Q^m into a ball Q^k, where $k \geqslant m$, will also be denoted by i. Finally, let S^m be the boundary of Q^{m+1}; we also denote by i the natural inclusion of S^m into E^k, where $k > m$.

3.1. Theorem. *Any continuous (simplicial) mapping $f\colon I^m\to I^m$ is topologically (simplicially) realizable in the cube $I^{2m}\operatorname{rel} i$. Any differentiable mapping $f\colon Q^m\to Q^m$ is differentiably realizable in the ball $Q^{2m}\operatorname{rel} i$.*

Proof. Let us note that the cube I^m is topologically (and even simplicially) quasi-contractible to the point 0. Since the categories $\mathfrak{S}$ and $\mathfrak{P}$ are multiplicative, we can apply Theorem 1.7 and obtain the conclusion. The category $\mathfrak{M}$ is not multiplicative, however the open ball $\operatorname{int}Q^m$ is a manifold differentiably quasi-contractible to the point 0 and the product $Q^m\times\operatorname{int}Q^m$ is again a manifold with boundary. Thus, by Theorem 1.7, the differentiable mapping $f\colon Q^m\to Q^m$ is differentiably realizable in $Q^m\times\operatorname{int}Q^m\operatorname{rel} i_0$, where i_0 is the natural inclusion of Q^m into $Q^m\times\operatorname{int}Q^m$. There exists, however, a diffeomorphism of $Q^m\times\operatorname{int}Q^m$ onto a submanifold of Q^{2m} which maps $i_0(Q^m)$ onto $i(Q^m)$ and this implies the conclusion of the theorem.

Let us define the number $c(m)$ to be the least natural number k such that any continuous mapping $f\colon I^m\to I^m$ is topologically realizable in the cube $I^k\operatorname{rel} i$. Theorem 3.1 assures that $c(m) \leqslant 2m$ for $m = 1, 2, \ldots$ On the other hand, we have the following

3.2. Lemma. *Let X be a locally connected continuum and $f\colon X\to X$ a continuous mapping. Let us suppose that $f(x') = f(x'')$ for some $x', x'' \in X$ and that there exist open neighborhoods U' and U'' of the points x' and x'', respectively, such that $\overline{U}'\cap\overline{U}'' = \emptyset$ and $f|U'$, $f|U''$ are homeomorphisms. Then the mapping f cannot be topologically realized in $X\operatorname{rel}$ identity on X.*

Proof. (²) Suppose that $\{h_n\}$ is a topological realization of f in $X\operatorname{rel} id_X$ and let $y = f(x') = f(x'')$. We shall show that $y \in h_n(\overline{U}')$ for sufficiently large n. Indeed, let assume that $y \notin h_n(\overline{U}')$ for $n = 1, 2, \ldots$ Since $\lim_n\varrho\big(h_n(x'), y\big) = \lim_n\varrho\big(h_n(x'), f(x')\big) = 0$, by local arcwise connectivity of X, there exist arbitrarily small arcs joining $h_n(x')$ with y for sufficiently large n. Thus there exists a sequence $p_n \in \operatorname{Fr}h_n(\overline{U}') = h_n(\operatorname{Fr}\overline{U}') = h_n(\operatorname{Fr}U')$ such that $\lim_n p_n = y$. Let $p_n = h_n(u_n)$, where $u_n \in U'$ for $n = 1, 2, \ldots$

Let us consider a positive ε such that if $x \in X$ and $\varrho(x, x') < \varepsilon$, then $x \in U'$. Since $f|\overline{U}'$ is a homeomorphism, there exists a $\delta > 0$ such that if $x \in \overline{U}'$ and $\varrho\big(f(x), y\big) < \delta$, then $\varrho(x, x') < \varepsilon$. Let us consider an integer number n such that $\varrho\big(f(u_n), h_n(u_n)\big) < \delta/2$ and $\varrho(p_n, y) < \delta/2$. Then $\varrho\big(f(u_n), y\big) \leqslant \varrho\big(f(u_n), h_n(u_n)\big)+\varrho\big(h_n(u_n), y\big) \leqslant \delta/2+\delta/2 = \delta$. Hence $\varrho(u_n, x') < \varepsilon$ and consequently $u_n \in U'$, contrary to $u_n \in \operatorname{Fr}U'$. The contradiction proves that $y \in h_n(\overline{U}')$ for sufficiently large n.

By the same reason, $y \in h_n(\overline{U}'')$ for sufficiently large n. Thus $\emptyset \neq h_n(\overline{U}')\cap h_n(\overline{U}'') = h_n(\overline{U}'\cap\overline{U}'')$, contrary to $\overline{U}'\cap\overline{U}'' = \emptyset$.

The lemma we have just proved together with Theorem 3.1 imply the following

3.3. Corollary. $m+1 \leqslant c(m) \leqslant 2m$ *for* $m = 1, 2, \ldots$

Proof. Indeed, the mapping $f\colon I^m\to I^m$ defined by the formula $f(x_1, x_2, \ldots, x_m) = (x_1^2, x_2, \ldots, x_m)$ for $(x_1, x_2, \ldots, x_m) \in I^m$ satisfies the assumptions of that lemma.

3.4. Corollary. $c(1) = 2$.

We shall give an example allowing to improve one inequality of 3.3.

3.5. Example. The mapping $f\colon Q^2\times I^m\to Q^2\times I^m$ defined by the formula $f(z, x_1, x_2, \ldots, x_m) = (z^2, x_1, x_2, \ldots, x_m)$, where $z \in Q^2$ is a complex number and $(x_1, x_2, \ldots, x_m) \in I^m$, cannot be topologically realized in $Q^2\times I^{m+1}\operatorname{rel} i$, where i is the natural inclusion of $Q^2\times I^m$ into $Q^2\times I^{m+1}$.

Proof. Let us suppose that $\{h_n\}$ is a topological realization of f in $Q^2\times I^{m+1}\operatorname{rel} i$ and let h_n satisfies the inequality $\varrho\big(h_n(p), if(p)\big) < 1/8$ for $p \in Q^2\times I^m$. For every $z \in Q^2$ such that $|z| = 2^{-1/2}$ let $\varphi(z) = \inf\{t\colon (z^2, 0, \ldots, 0, t) \in h_n(Q^2)\}$. Then exactly one of the points $h_n(z, 0, \ldots, 0)$, $h_n(-z, 0, \ldots, 0)$ can be joined in $h_n(Q^2\times I^m)$ with the point

(²) A stronger formulation of the lemma announced in [8] is false.

$(z^2, 0, \ldots, 0, \varphi(z))$ by an arc of diameter less than $\frac{1}{4}$. It can be proved that this property defines a continuous selection of one point from each pair $(z, -z)$, where $|z| = 2^{-1/2}$, which is evidently impossible and this contradiction proves our property.

3.6. Corollary. *$m+2 \leqslant c(m) \leqslant 2m$ for $m = 2, 3, \ldots$*

3.7. Corollary. $c(2) = 4$.

In connection with 3.4 and 3.7 the following problem can be raised.

3.7 Problem. *Does the equality $c(m) = 2m$ hold for each natural number m?*

From Theorem 1.8 we deduce the following

3.8. Theorem. *If $f_j\colon I^k \to I^k$ is a continuous mapping for $j = 1, 2, \ldots, m$, then the product $f = f_1 \times f_2 \times \ldots \times f_m\colon I^{m\cdot k} \to I^{m\cdot k}$ is topologically realizable in $I^{m\cdot k+k}\operatorname{rel} i$, where i is the natural inclusion.*

Proof. We have to verify that the cube I^k is topologically uniformly quasi-contractible. It is, however, easy to see that for any point $a_1 \in I^k$ the formula $h_n^{a_1}(a_2) = \dfrac{1}{n}(a_2 - a_1) + a_1$, where $a_2 \in I^k$ and $n = 1, 2, \ldots$, defines a topological realization $\{h_n^{a_1}\}$ of the mapping $\pi^{a_1}\colon I^k \to \{a_1\}$ in $I^k \operatorname{rel}$ the inclusion $\nu^{a_1}\colon \{a_1\} \to I^k$. Moreover, $h_n^{a_1}(a_2)$ is a continuous function of the pair (a_1, a_2).

3.9. Corollary. *If $f_j\colon I \to I$ is a continuous mapping for $j = 1, 2, \ldots, m$ then the product $f = f_1 \times f_2 \times \ldots \times f_m\colon I^m \to I^m$ is topologically realizable in $I^{m+1}\operatorname{rel} i$, where i is the natural inclusion.*

The property exhibited in this corollary plays a fundamental role in the proof of the R. Bennet's theorem [1] on imbedding of products of chainable continua.

If mappings of spheres into spheres are being considered, the degree of a mapping has an influence on its realizability in an Euclidean space. The first part of Theorem 2.8 can be, by using of the same methods, generalized to the following

3.10. Theorem. *If a continuous mapping $f\colon S^m \to S^m$ can be realized in $E^{m+1}\operatorname{rel} i$, then $|\deg f| \leqslant 1$.*

However, the converse theorem is not necessarily true as it is shown by the following example.

3.11. Example. Let $\varphi\colon I^m \to I^m$ be a continuous mapping which cannot be realized in E^{m+1} (comp. Example 3.5). By making a suitable homeomorphism we can assume that φ maps the hemisphere S_+^m into itself, where $S_+^m = \{(x_1, x_2, \ldots, x_{m+1}) \in E^{m+1}\colon x_{m+1} \geqslant 0\}$. Let $\pi\colon S^m \to S_+^m$ be the projection given by $\pi(x_1, x_2, \ldots, x_m, x_{m+1}) = (x_1, x_2, \ldots, x_m, |x_{m+1}|)$ and let $\iota\colon S_+^m \to S^m$ be the inclusion. Then it is easy to see that the mapping

$f = \iota\varphi\pi\colon S^m \to S^m$ cannot be realized in $E^{m+1}\operatorname{rel} i$, though $\deg f = 0$. Thus, in particular, there exists a continuous mapping $f\colon S^2 \to S^2$ of degree 0 which cannot be realized in $E^3\operatorname{rel} i$. A slight modification of this example, found by A. Trybulec, gives a mapping of degree 1 with the same property concerning realizability.

4. Embedding of inverse limits. Let $f\colon A \to B$ be a mapping of metric spaces. We define the Lipschitz number of the mapping f as

$$L(f) = \sup_{\substack{a', a'' \in A \\ a' \neq a''}} \left\{ \frac{\varrho\big(f(a'), f(a'')\big)}{\varrho(a', a'')} \right\}.$$

Let $f\colon A \to B$ be a morphism and let $\{h_n\}$ be a realization of f in a space $X \operatorname{rel}$ an imbedding h of B into X. We say that $\{h_n\}$ is a Lipschitz realization if $\limsup L(f_n) \leqslant 1$, where $f_n = hfh_n^{-1}$ for $n = 1, 2, \ldots$

4.1 Theorem. *Let $\{A_j, a_j\}$, where $a_j\colon A_{j+1} \to A_j$ for $j = 1, 2, \ldots$, be an inverse system of objects and morphisms such that the inverse limit $A = \underleftarrow{\operatorname{Lim}}\{A_j, a_j\}$ (not necessarily in the category) is compact. If the morphisms a_j can be Lipschitz realized in a complete object X for $j = 1, 2, \ldots$, then A can be topologically imbedded in X.*

Proof. By the assumption of realizability of a_1 in X, there exists an isomorphism h_1 of A_1 onto a subobject B_1 of X. Then, by means of induction, we use the assumptions to extend the inverse sequence $\{A_j, a_j\}$ to the following diagram:

$$\begin{array}{ccccccccccc}
A_1 & \xleftarrow{\alpha_1} & A_2 & \xleftarrow{\alpha_2} & \ldots & \xleftarrow{\alpha_{j-1}} & A_j & \xleftarrow{\alpha_j} & A_{j+1} & \xleftarrow{\alpha_{j+1}} & \ldots \\
\downarrow{\scriptstyle h_1} & & \downarrow{\scriptstyle h_2} & & & & \downarrow{\scriptstyle h_j} & & \downarrow{\scriptstyle h_{j+1}} & & \\
B_1 & \xleftarrow{\beta_1} & B_2 & \xleftarrow{\beta_2} & \ldots & \xleftarrow{\beta_{j-1}} & B_j & \xleftarrow{\beta_j} & B_{j+1} & \xleftarrow{\beta_{j+1}} & \ldots
\end{array}$$

where h_j is an isomorphism of A_j onto a subobject B_j of X, β_j is a morphism, $h_j a_j = \beta_j h_{j+1}$, $D(\beta_j) < 2^{-j}$, and $L(\beta_j) < 1 + 2^{-j}$ for $j = 1, 2, \ldots$ This implies, in particular, that the inverse limit $B = \underleftarrow{\operatorname{Lim}}\{B_j, \beta_j\}$ is homeomorphic to A. It is therefore sufficient to imbed B in X.

Let us define a mapping $h\colon B \to X$ as follows. If $b = \{b_j\} \in B$, then $\varrho(b_j, b_{j+k}) = \varrho\big(\beta_j \ldots \beta_{j+k-1}(b_{j+k}), b_{j+k}\big) \leqslant 2^{-(j+k)}$; thus $\{b_j\}$ is a Cauchy sequence in X and let $h(b) = \lim\limits_j b_j$.

The mapping h is continuous. Indeed, if $b^n = \{b_j^n\} \in B$ for $n = 0, 1, \ldots$ and $\lim\limits_n b^n = b^0$ in B, then $\lim\limits_n b_j^n = b_j^0$ for $j = 1, 2, \ldots$ Consequently, $h(b^0) = \lim\limits_j b_j^0 = \lim\limits_j \lim\limits_n b_j^n = \lim\limits_n \lim\limits_j b_j^n = \lim\limits_n h(b^n)$.

The mapping h is one-to-one. Indeed, let $b' = \{b'_j\}$ and $b'' = \{b''_j\}$ belong to B and let $b' \neq b''$. There exists a j_0 such that for $j \geqslant j_0$ we have $b'_j \neq b''_j$. Consequently,

$$\frac{\varrho\big(\beta_j(b'_j),\, \beta_j(b''_j)\big)}{\varrho(b'_j,\, b''_j)} \leqslant L(\beta_j) \leqslant 1+2^{-j},$$

whence $\varrho\big(\beta_j(b'_j),\, \beta_j(b''_j)\big) \leqslant \varrho(b'_j,\, b''_j)(1+2^{-j})$ for $j \geqslant j_0$.

Thus

$$0 < \varrho(b'_{j_0},\, b''_{j_0}) \leqslant (1+2^{-j_0})(1+2^{-(j_0+1)})\ldots(1+2^{-j})\cdot\varrho(b'_j,\, b''_j) \quad \text{for} \quad j \geqslant j_0 .$$

Writing $p = \prod\limits_{j=1}^{\infty}(1+2^{-j}) < \infty$ we have $0 < \varrho(b'_{j_0},\, b''_{j_0}) \leqslant p\cdot\varrho(b'_j,\, b''_j)$, whence $\varrho(b',\, b'') \geqslant \dfrac{\varrho(b'_{j_0},\, b''_{j_0})}{p} > 0$ for $j \geqslant j_0$. But this implies that $h(b') = \lim\limits_j b'_j \neq \lim\limits_j b''_j = h(b'')$. Since B is compact, we infer that h is a topological imbedding of B into X and this completes the proof.

We shall now prove that the assumption of Lipschitz realizability in Theorem 4.1 can be replaced by usual realizability in the case of the polyhedral category. Let us begin with two lemmas.

4.2. Lemma. *Let P be a subpolyhedron of a polyhedron X. For any $\varepsilon > 0$ there exists a $\delta > 0$ such that for each subpolyhedron Q of X and a simplicial mapping $f\colon P \to Q$ satisfying $D(f) < \delta$ we have*

$$\frac{\varrho\big(f(p'),f(p'')\big)}{\varrho(p',\, p'')} < 1+\varepsilon$$

for any two distinct points $p',\, p''$ which either belong to a common simplex of P not degenerated by f or belong to disjoint simplexes of P.

Proof. Let us suppose that $p',\, p''$ belong to a simplex $s(p_0, p_1, \ldots, p_k)$ and $p' = \sum\limits_{i=0}^{k}\lambda'_i p_i$, $p'' = \sum\limits_{i=0}^{k}\lambda''_i p_i$, where $\sum\limits_{i=0}^{k}\lambda'_i = \sum\limits_{i=0}^{k}\lambda''_i = 1$. Since the points $q_i = f(p_i)$, where $i = 0, 1, \ldots, k$, are linearly independent, the function $\varphi(\lambda_0, \lambda_1, \ldots, \lambda_k) = \big|\sum\limits_{i=0}^{k}\lambda_i\cdot q_i\big|$ has a positive minimum μ on the unit sphere $\sum\limits_{i=0}^{k}\lambda_i^2 = 1$. Thus for $D(f)$ sufficiently small we have $\big|\sum\limits_{i=0}^{k}\lambda_i p_i\big| > \mu/2$. On the other hand,

$$\frac{\varrho\big(f(p'),f(p'')\big)}{\varrho(p',\, p'')} = \frac{\big|\sum\limits_{i=0}^{k}(\lambda'_i-\lambda''_i)q_i\big|}{\big|\sum\limits_{i=0}^{k}(\lambda'_i-\lambda''_i)p_i\big|} = \frac{\big|\sum\limits_{i=0}^{k}\lambda_i q_i\big|}{\big|\sum\limits_{i=0}^{k}\lambda_i p_i\big|}, \quad \text{where} \quad \lambda_i = \frac{\lambda'_i-\lambda''_i}{\sqrt{\sum\limits_{i=0}^{k}(\lambda'_i-\lambda''_i)^2}}$$

and consequently $\sum\limits_{i=0}^{k}\lambda_i^2 = 1$. Thus

$$\frac{\varrho\big(f(p'),f(p'')\big)}{\varrho(p',\, p'')} \leqslant 1+\frac{\big|\sum\limits_{i=0}^{k}\lambda_i(q_i-p_i)\big|}{\big|\sum\limits_{i=0}^{k}\lambda_i p_i\big|} \leqslant 1+\frac{k\cdot\delta}{\mu/2} \quad \text{if} \quad D(f) < \delta .$$

The conclusion follows, since P consists of finitely many simplexes.

Suppose now that $p',\, p''$ belong to disjoint simplexes of P. Then there exists a positive μ such that $\varrho(p',\, p'') > \mu$. Thus

$$\frac{\varrho\big(f(p'),f(p'')\big)}{\varrho(p',\, p'')} = 1+\frac{\varrho\big(f(p'),f(p'')\big)-\varrho(p',\, p'')}{\varrho(p',\, p'')} \leqslant 1+\frac{2\delta}{\mu} \quad \text{if} \quad D(f) < \delta.$$

The conclusion follows.

4.3. Lemma. *Let P be a subpolyhedron of a polyhedron X. There exists positive numbers $\delta,\, \eta$ such that if Q is a subpolyhedron of X, $f\colon P \to Q$ is a simplicial mapping satisfying $D(f) < \delta$, $s',\, s''$ are simplexes of P not degenerated under f and having a non-empty proper face s^* as their intersection, and $p' \in s'-s^*$, $p'' \in s''-s^*$, then $\inf\limits_{p \in s^*} \measuredangle\, (\overline{p',p},\, \overline{p'',p}) > \eta$.*

Proof. By compactness of Q there exists a positive number η such that if $t',\, t''$ are simplexes of Q which have a non-empty proper face t^* as their intersection, and if $q' \in t'-t^*$, $q'' \in t''-t^*$, then $\inf\limits_{q \in t^*} \measuredangle\, (\overline{q',q},\, \overline{q'',q}) > 2\eta$. It is now sufficient to use the assumption that f does not degenerate s' nor s'' (so it does not degenerate s^*) and find a suitable δ replacing 2η by η.

4.4. Theorem. *Let $\{A_j,\, \alpha_j\}$, where $\alpha_j\colon A_{j+1} \to A_j$ for $j = 1, 2, \ldots$, be an inverse system of polyhedra and simplicial mappings. If the mappings α_j can be simplicially realized in a polyhedron X for $j = 1, 2, \ldots$, then $A = \underleftarrow{\operatorname{Lim}}\{A_j,\, \alpha_j\}$ can be topologically imbedded in X.*

Proof. Making use of Lemmas 4.2 and 4.3, we extend, exactly in the same manner like in the proof of Theorem 4.1, the sequence $\{A_j,\, \alpha_j\}$ to the following diagram:

$$\begin{array}{ccccccccccc}
A_1 & \xleftarrow{\alpha_1} & A_2 & \xleftarrow{\alpha_2} & \ldots & \xleftarrow{\alpha_{j-1}} & A_j & \xleftarrow{\alpha_j} & A_{j+1} & \xleftarrow{\alpha_{j+1}} & \ldots \\
\downarrow{\scriptstyle h_1} & & \downarrow{\scriptstyle h_2} & & & & \downarrow{\scriptstyle h_j} & & \downarrow{\scriptstyle h_{j+1}} & & \\
B_1 & \xleftarrow{\beta_1} & B_2 & \xleftarrow{\beta_2} & \ldots & \xleftarrow{\beta_{j-1}} & B_j & \xleftarrow{\beta_j} & B_{j+1} & \xleftarrow{\beta_{j+1}} & \ldots
\end{array}$$

where for any $j = 1, 2, \ldots$ h_j is a simplicial homeomorphism of A_j onto a subpolyhedron B_j of X, B_j is a simplicial mapping, $h_j\alpha_j = \beta_j h_{j+1}$,

$D(\beta_j) < 2^{-j}$, $\dfrac{\varrho(\beta_j(b'_j), \beta_j(b''_j))}{\varrho(b'_j, b''_j)} < 1+2^{-j}$ for any two distinct points b'_j, b''_j which either belong to a common simplex of B_j not degenerated under β_{j-1} or belong to disjoint simplexes of B_j, and $\inf_{b \in s_j^*} \sphericalangle(\overline{b'_j, b}, \overline{b''_j, b}) > \eta > 0$ for any points $b'_j \in s'_j - s_j^*$, $b''_j \in s''_j - s_j^*$, where s'_j, s''_j are simplexes of B_j which are not degenerated under β_{j-1} and have a non-empty proper face s_j^* as their intersection.

Then, as in that proof, we define the mapping h of $B = \underleftarrow{\mathrm{Lim}}\{B_j, \beta_j\}$ into X by the formula $h(b) = \lim_j b_j$, where $b = \{b_j\} \in B$, and verify that it is continuous.

It remains to verify that h is one-to-one. Suppose that $b' = \{b'_j\}$ and $b'' = \{b''_j\}$ belong to B and that $b' \neq b''$. Then there exists a j_0 such that for $j \geqslant j_0$ we have $b'_j \neq b''_j$. Let s'_j denote the carrier of the point b'_j in B and s''_j — that of b''_j. Evidently $\beta_{j-1}(s'_j) = s'_{j-1}$, $\beta(s''_j) = s''_{j-1}$. Since X is a polyhedron and any B_j is a subset of X, the dimensions of B_j are bounded by $\dim X$. Thus we can assume without any loss of generality that for $j \geqslant j_0$ neither s'_j nor s''_j is degenerated under β_{j-1}.

For any j one of the following possibilities holds:

(i) at least one of the simplexes s'_j, s''_j is a face of the other;

(ii) the simplexes s'_j, s''_j are disjoint;

(iii) the simplexes s'_j, s''_j have a non-empty proper face as their intersection.

Moreover, since $\beta_{j-1}(s'_j) = s'_{j-1}$, $\beta_{j-1}(s''_j) = s''_{j-1}$ and β_{j-1} is simplicial we can assume without any loss of generality that the same case (i), (ii) or (iii) holds for each $j \geqslant j_0$.

If one of the cases (i), (ii) holds, then by the construction of β_j we have

$$\frac{\varrho(\beta_j(b'_j), \beta_j(b''_j))}{\varrho(b'_j, b''_j)} < 1+2^{-j}.$$

By the same calculation as in the proof of Theorem 4.1 this yields $h(b') \neq h(b'')$.

Let us now suppose that case (iii) holds for $j \geqslant j_0$. Let s_j^* be the intersection of s'_j with s''_j. We can evidently assume that $b'_j \in s'_j - s_j^*$, $b''_j \in s''_j - s_j^*$ for $j \geqslant j_0$. Since s'_j, s''_j are not degenerated under β_{j-1}, neither is s_j^*. Moreover, we have $\beta_{j-1}(s_j^*) = s_{j-1}^*$. There exists, therefore, an element $b^* = \{b_j^*\} \in B$ such that $b_j^* \in s_j^*$ for $j \geqslant j_0$. For each of the pairs b'_j, b_j^* and b''_j, b_j^* case (i) holds, thus by the proved part of the theorem, we have $h(b') \neq h(b^*) \neq h(b'')$. In order to prove that $h(b') \neq h(b'')$ let us suppose the contrary. Then $\lim_j \varrho(b'_j, b''_j) = 0$ but there exist positive numbers η', η'' such that $\varrho(b'_j, b_j^*) > \eta'$ and $\varrho(b''_j, b_j^*) > \eta''$ for $j \geqslant j_0$. This however implies that $\lim_j \sphericalangle(\overline{b'_j, b_j^*}, \overline{b''_j, b_j^*}) = 0$ contrary to the construction. This completes the proof of the theorem.

We shall now pass to applications of Theorem 4.1 to the category of manifolds. Let M^m be a compact submanifold (possibly with boundary) of the Euclidean space E^k. Let δ be a positive number and for each point $p \in M^m$ let $Q_\delta^{k-m}(p)$ be the ball about p of radius δ in the $(k-m)$-dimensional hyperplane normal to M^m. It is a well-known fact of the theory of differentiable manifolds that there exists $\delta > 0$ such that $Q_\delta^{k-m}(p') \cap Q_\delta^{k-m}(p'') = \emptyset$ for $p' \neq p''$ and the natural retraction r_δ of $T_\delta^k = \bigcup_{p \in M^m} Q_\delta^{k-m}(p)$ to M^m is a differentiable fiber mapping. We can consider r_δ as a differentiable mapping of a closed domain in E^k into itself and let $r'_\delta(p)$ be the derived linear operator at the point $p \in M^m$.

4.5. Lemma. *The norm of $r'_\delta(p)$ at any point $p \in M^m$ is equal to* 1.

Proof. By the ortogonality of the projection onto M^m, we can choose an ortonormal basis at the point p with respect to which $r'_\delta(p)$ is represented by a diagonal matrix. The conslusion follows.

By the continuity of the norm and by compactness of M^m, we infer that for any positive ε there exists a $\delta > 0$ such that $\|r'_\delta(p)\| < 1+\varepsilon$ for $p \in T_\delta^k$. Thus, a well-known theorem of functional analysis implies directly

4.6. Lemma. *For any $\varepsilon > 0$ there exists a $\delta > 0$ such that $\varrho(r_\delta(p'), r_\delta(p'')) \leqslant (1+\varepsilon)\varrho(p', p'')$ for any $p', p'' \in T_\delta^k$.*

We shall use the lemma for the proof of the following

4.7. Lemma. *If a differentiable mapping $f\colon P^p \to M^m$ of compact differentiable manifolds is differentiably realizable in the space E^k* rel *an imbedding $h\colon M^m \to E^k$, then f can be approximated by differentiable mappings of P^p into M^m which are Lipschitz differentiably realizable in E^k* rel n.

Proof. Let η be a positive number and let $\{h_n\}$ be a differentiable realization of f in $E^k \mathrm{rel}\, h$. By a small modification of h_n near the boundary of $h(M^m)$ we can assume that for sufficiently large n_0 we have $h_{n_0}(P^p) \subset T_\eta^k$, where T_η^k denotes the tubular neighborhood of $h(M^m)$ in E^k. Let $d_{\eta,\delta}$ denotes the natural deformation of T_η^k to T_δ^k for $0 < \delta < \eta$. It is easy to see that the sequence $\{d_{\eta,1/n} h_{n_0}\}$ is a differentiable realization of the mapping $h^{-1} r_\eta h_{n_0} \mathrm{rel}\, h$. Moreover, by Lemma 4.6, it is a Lipschitz realization. On the other hand, $h^{-1} r_\eta h_{n_0}$ converges to f as η converges to 0.

Using a differentiable manifold K^k instead of the space E^k and replacing normal lines by geodetics in a Riemannian metric we can easily generalize Lemma 4.7 to the following

4.8. LEMMA. *If a differentiable mapping $f\colon P^p \to M^m$ of compact differentiable manifolds is differentiably realizable in a differentiable manifold K^k* rel *an imbedding $h\colon M^m \to K^k$, then f can be appoximated by differentiable mappings of P^p into M^m which are Lipschitz differentiably realizable in K^k* rel *h.*

Theorem 4.1 and Lemma 4.8 together with the approximation theorem of M. Brown ([2], Theorem 3) imply the following

4.9. THEOREM. *Let $\{A_j, a_j\}$, where $a_j\colon A_{j+1} \to A_j$ for $j = 1, 2, \ldots$, be an inverse system of compact differentiable manifolds and differentiable mappings. If the mappings a_j can be differentiably realized in a differentiable manifold M for $j = 1, 2, \ldots$, then the inverse limit $A = \varprojlim\{A_j, a_j\}$ can be topologically imbedded in M.*

In the case of the differential category Theorem 3.1 can be easily generalized to the following

4.10. THEOREM. *Any differentiable mapping $f\colon Q^m \to Q^m$ is differentiably realizable in Q^{2m}.*

Proof. Indeed, let $h\colon Q^m \to Q^{2m}$ be a differentiable imbedding. We can evidently assume that $h(Q^m) \subset \operatorname{int} Q^{2m}$. Since Q^m is contractible to a point, the tubular neighborhood of $h(Q^m)$ is a trivial fibre bundle and is, therefore, diffeomorphic to $Q^m \times Q^m$. By a suitable adaptation of the radii in the tubular neighborhood, we can assume that it is diffeomorphic to Q^{2m} and that $h(Q^m)$ is mapped onto the canonical ball Q^m in Q^{2m} under this diffeomorphism. Now we apply the corresponding part of Theorem 3.1.

Theorems 4.9 and 4.10 give the following

4.11. COROLLARY. *Let $\{Q_j^m, a_j\}$, where $a_j\colon Q_{j+1}^m \to Q_j^m$ for $j = 1, 2, \ldots$, be an inverse system of m-dimensional balls and differentiable mappings. Then $\varprojlim\{Q_j, a_j\}$ can be topologically imbedded in Q^{2m}.*

Making use again of the theorem of M. Brown [2] and of the Weierstrass Approximation Theorem, we deduce the following

4.14. COROLLARY. *Any m-cell-like continuum can be topologically imbedded in Q^{2m}.*

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