Addendum (August 14, 1969)

Theorem. Let $G$ be a torsion-free LCA group satisfying any one of the following conditions:

(a) $G$ is separable.
(b) $G$ satisfies countable chain condition, i.e. any family of disjoint open sets in $G$ is countable.
(c) $G$ is $\omega_1$-compact.
(d) $G$ is Lindelöf.
(e) Any uncountable family of open sets has an uncountable subfamily with non empty intersection.

Then $G$ is self-dual if and only if $G$ is of the form mentioned in the preceding theorem.

Proof. If $G$ is of the form mentioned in the preceding theorem then it is self-dual by Lemma 6. Suppose now $G$ is self-dual. Then $G = R^+ \oplus A$ where $A$ has a compact open subgroup $H$. $G = R^+ \oplus A$ where $A$ has a compact open subgroup $H^+$ which is the dual of the discrete $A/H$. Let $G$ satisfy any one of the conditions (a)-(c). Observe that (a) $\Rightarrow$ (b) and (c) $\Rightarrow$ (d). We assert that $A/H$ is countable. If not, $A = \bigcup H_n$, a set union of disjoint cosets and $A$ is an uncountable set. Each of these cosets is open in $A$. If we now consider $(R^n \times H_n)_{\infty}$ we easily arrive at a contradiction. So $A/H$ is countable. Now $G = R^+ \oplus A$ where $A$ has a compact open subgroup $H^+$ which is now the dual of the countable discrete group $A/H$. Hence $H^+$ is metrizable. Since $G$ is isomorphic to $\hat{G}$, by a similar reasoning we get $\hat{A}/\hat{H}^+$ is countable. Hence $A$, the dual of $A/H^+$ is metrizable. Since $H^+$ is metrizable and $A/H$ is countable discrete and hence metrizable we get $A$ is metrizable. Already $R^+$ is metrizable. Hence $G$ is metrizable. Then the preceding theorem completes the proof.

References


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Results on $\omega_n$-metric spaces

by

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§ 1. Introduction and preliminary results. A linearly ordered abelian group is a set $A$, together with a binary operation $\cdot$, and an order relation $>$, such that $(A, \cdot)$ is an abelian group $(A, >)$ is a linearly ordered set and the following condition is satisfied: if $a > b$ then $ac > bc$.

The group $A$ has character $\omega_n$ if there exists a decreasing $\omega_n$-sequence converging to 0 in the order topology on $A$. Here $\omega_n$ denotes the $n$th infinite cardinal number. Cardinal numbers are considered as initial ordinal numbers and each cardinal coincides with the set of all smaller ordinals.

The power of $\omega_n$, is denoted by $\kappa_n$. We will be concerned with only that $\omega_n$ which represents the least character of $A$ and it is easily shown that such an $\omega_n$ must be a regular cardinal number.

Let $X$ be a set and $g$ a function from $X \times X$ to $(A, >)$ such that

(i) $g(x, x) = 0$ if $x = y$,
(ii) $g(x, y) = g(y, x) > 0$ if $x \neq y$,
(iii) $g(x, y) < g(x, z) + g(z, y),$
then $g$ is called an $\omega_n$-metric and $(X, g)$ is an $\omega_n$-metric space. Sikorski [10] has done the most extensive study of $\omega_n$-spaces; other references include Hausdorff [3], Cohen and Goffman [1] and [2], Parrowzenko [6], and most recently, Sun-Tang [7].

The $\omega_n$-metric $g$ on $X$ induces a topology $\tau_g$ on $X$; a base for the topology consisting of sets of the form $F_s(x)$ where $N_s(x) = \{y \in X: g(x, y) < s\}, a < s$, and $a > 0$. Also $g$ induces a uniformity $U_s$ on $X$: a base for the uniformity consisting of sets $U_s$ where $U_s = \{(x, y): g(x, y) < a\}, a \in A, a > 0$. It is easily shown that the $\omega_n$-metric topology and the $\omega_n$-uniform topology are identical.

An $\omega$-additive space is a topological space $(X, \tau)$ which satisfies the condition that for any family of open sets $F_s$ of power $< \kappa_n$, it follows that $\bigcap F_s$ is an open set. Clearly every topological space is an $\omega_0$-additive space. It is easily shown that if $(X, \tau)$ is an $\omega_n$-metric space then $(X, \tau)$ is an $\omega_n$-additive space. Sikorski defines the following concepts on an $\omega_n$-additive space $(X, \tau)$. The space $(X, \tau)$ has a basis iff it has a base
of power \( \kappa_2 \) \((X, \mathcal{V})\) is separable iff there exists an everywhere dense subset \( Y \) of \( X \) of power \( \leq \kappa_2 \); \((X, \mathcal{V})\) is compact iff every \( \omega_0 \)-sequence in \( X \) has a convergent \( \omega_0 \)-sequence; \( (X, \mathcal{V}) \) is \( \omega_0 \)-bicompact iff every open cover of \( X \) has a subcover of power \( \leq \kappa_2 \). For an \( \omega_0 \)-metric space \((X, \rho)\) Sikorski gives these definitions: \((X, \mathcal{V})\) is totally bounded iff for any \( a \in A, a > 0 \), there exists a subset \( Y \) of \( X \) of power \( \leq \kappa_2 \) such that \( \bigcup \{ \mathcal{V}(y) : y \in Y \} = X \); \( (X, \mathcal{V}) \) is complete iff every Cauchy \( \omega_0 \)-sequence converges.

A Cauchy \( \omega_0 \)-sequence is an \( \omega_0 \)-sequence \( \{a_n\} \) satisfying the condition that for all \( 0 > a \in A \) there exists \( n \in \omega \) such that if \( \beta, \gamma > a \) then \( \rho(a_\beta, a_\gamma) < a \).

Sikorski states some theorems relating these topological and uniform concepts. We give a more complete list below, omitting the proofs since each is a straightforward generalization of the proof of the standard topological theorem. For Sikorski’s terms: basis, separable, compact, \( \omega_0 \)-bicompact, totally bounded and complete, we will use; \( \omega_0 \)-countable, \( \omega_0 \)-separable, \( \omega_0 \)-compact (for both compact and \( \omega_0 \)-bicompact since they are equivalent by theorem 1.3), \( \omega_0 \)-totally bounded, and \( \omega_0 \)-complete, respectively.

If \( \rho \) is an \( \omega_0 \)-metric on \( X \) then:

**Theorem 1.1.** \((X, \mathcal{V})\) is \( \omega_0 \)-countable iff it is \( \omega_0 \)-separable.

**Theorem 1.2.** If \((X, \rho)\) is \( \omega_0 \)-totally bounded then \((X, \mathcal{V})\) is \( \omega_0 \)-separable.

**Theorem 1.3.** The following three statements are equivalent on \((X, \mathcal{V})\):

(i) Every \( \omega_0 \)-sequence in \( X \) has a convergent \( \omega_0 \)-subsequence in \( X \).

(ii) Every open cover of \( X \) of power \( \kappa_2 \) has a subcover of power \( \leq \kappa_2 \).

(iii) Every open cover of \( X \) of power \( \kappa_2 \) has a subcover of power \( \leq \kappa_2 \).

**Theorem 1.4.** \((X, \rho)\) is \( \omega_0 \)-complete iff \((X, \mathcal{V})\) is complete in the uniform sense.

**Theorem 1.5.** If \((X, \mathcal{V})\) is \( \omega_0 \)-compact then \((X, \rho)\) is \( \omega_0 \)-complete and \( \omega_0 \)-totally bounded.

The converse of Theorem 1.5 is true for \( \mu = 0 \). Sikorski has shown that for accessible regular cardinals the converse is not true in general; specifically: if \( \mu = \nu + 1 \), if \( \omega_0 \) is regular, and if \( 2^\kappa = \kappa_2 \) for \( \kappa > \nu \), then there exists an \( \omega_0 \)-metric space which is \( \omega_0 \)-complete, \( \omega_0 \)-totally bounded, but not \( \omega_0 \)-compact. (See [10] pp. 132, 133.) For inaccessible cardinals the converse of Theorem 1.5 remained an open question.

**§ 2. A counterexample for an \( \omega_0 \)-compactness theorem for inaccessible cardinals.** In 1964 Monk and Scott [6] showed that if \( \omega_0 \) is the first uncountable inaccessible cardinal then \( 2^\kappa \) is not \( \omega_0 \)-compact” in the \( \omega_0 \)-product topology, \( \mathcal{V}_\kappa \). The set \( 2^\kappa \) is essentially the set of all \( \omega_0 \)-sequences of 0’s and 1’s; the term “\( \omega_0 \)-compact” is identical to Sikorski’s term “\( \omega_0 \)-bicompact” which is equivalent to Sikorski’s term “compact” [8]; the \( \omega_0 \)-product topology on \( 2^\kappa \) has a base made up of all \( \omega_0 \)-sequences of \( 0 \)'s and \( 1 \)'s for all \( \omega_0 \)-sequences \( x, y \) agree on all coordinates in \( T \).

Sikorski had studied the space of \( \omega_0 \)-sequences of 0’s and 1’s, which he denoted by \( \mathcal{D}_\kappa \), along with the metric \( \varrho \) such that \( \varrho(x, y) = 0 \) iff \( x = y \) and \( \varrho(x, y) = 1/\varrho \) if \( x \neq y \), where \( \varrho \) is the first ordinal where the sequences \( x \) and \( y \) differ. Here \( \varrho \) is the least algebraic field containing the set of all ordinals \( \omega_0 \); see [9]. Sikorski showed that \((\mathcal{D}_\kappa, \varrho)\) is \( \omega_0 \)-complete for all \( \mu \) and \((\mathcal{D}_\kappa, \varrho)\) is \( \omega_0 \)-totally bounded for all \( \omega_0 \) which are inaccessible cardinals \( \omega_0 \); in particular \((\mathcal{D}_\kappa, \varrho)\) is \( \omega_0 \)-complete and \( \omega_0 \)-totally bounded for the first inaccessible cardinal \( \omega_0 \). Now it is easily shown that \( \mathcal{V}_\kappa \) is identical to \( \mathcal{V}_\kappa \) and hence the result of Monk and Scott that \((\mathcal{D}_\kappa, \mathcal{V}_\kappa)\) is not \( \omega_0 \)-compact resolves the open question referred to in § 1.

Perhaps a conceptually simpler \( \omega_0 \)-metric than the metric \( \varrho \) is the one we define below; it too induces the topology \( \mathcal{V}_\kappa \) on \( \mathcal{D}_\kappa \). Let \( \varrho_2 : \mathcal{D}_\kappa \times \mathcal{D}_\kappa \to \mathcal{D}_\kappa \) be such that \( \varrho_2(x, y) = (0, 0, ... \) iff \( x = y \) and \( \varrho_2(x, y) = 1 \) if \( x \neq y \), where \( \alpha \) is the first ordinal at which the sequences \( x \) and \( y \) differ, and \( 1 \) is the \( \omega_0 \)-sequence which is 1 in the \( \alpha \)th coordinate and 0 elsewhere. Given that \( \mathcal{D}_\kappa \) is a subset of the ordered abelian group \((J_\kappa, +, \cdot) \), where \( J_\kappa \) is the family of all \( \omega_0 \)-sequences of integers, \( + \) is coordinate-wise addition, and \( \cdot \) is lexicographic order, it is easy to show that \( \varrho_2 \) is an \( \omega_0 \)-metric on \( \mathcal{D}_\kappa \). It is also easily shown that \((\mathcal{D}_\kappa, \varrho_2)\) is \( \omega_0 \)-complete for all \( \mu \) and \((\mathcal{D}_\kappa, \varrho_2)\) is \( \omega_0 \)-totally bounded for all \( \omega_0 \) which are inaccessible. Now \( \mathcal{V}_\kappa \) is identical to \( \mathcal{V}_\kappa \), so if \( \omega_0 \) is the first uncountable inaccessible cardinal \( \mathcal{D}_\kappa, \varrho_2) \) provides us with an \( \omega_0 \)-metric space which is \( \omega_0 \)-complete and \( \omega_0 \)-totally bounded but not \( \omega_0 \)-compact. Thus:

**Theorem 2.** It is not necessarily true for inaccessible cardinals that an \( \omega_0 \)-complete, \( \omega_0 \)-totally bounded \( \omega_0 \)-metric space is \( \omega_0 \)-compact.

The same space \((\mathcal{D}_\kappa, \varrho_2)\) also provides us with an example of a \( \omega_0 \)-metric space which is \( \omega_0 \)-compact, and perfect, and is of power \( 2^\kappa \). It is true that every compact, perfect metric space has power \( 2^\kappa \) but Sikorski noted that he knew of no example of a compact, perfect \( \omega_0 \)-metric space of power \( > \kappa_2 \).

**§ 3. An \( \omega_0 \)-metrization theorem.** Sikorski remarks that every \( \omega_0 \)-additive, \( \omega_0 \)-countable topological space is \( \omega_0 \)-metrizable. This is a generalization of Urysohn’s metrization theorem. Shu-Tang [7] generalized the Nagata–Smirnov metrization theorem with this result: If \( (X, \mathcal{V}) \) is a regular (topologically speaking) \( \omega_0 \)-additive space then \((X, \mathcal{V})\) is \( \omega_0 \)-metrizable iff there exists a \( \kappa_2 \) basis for \( \mathcal{V} \). The range space of Sikorski’s
We define $q : X \times X \rightarrow J^*$ as follows: $[q(x, y)](a) = 0$ if $q(x, y) = 0$, otherwise $[q(x, y)](a) = 1$. Actually, the exact range space of $q$ is $\mathcal{D}_q$.

First we show that $q$ is an $\omega_q$-metric.

(i) Clearly $q(x, y) \geq 0$ for all $x, y \in X$ (0 here is the $\omega_q$-0-sequence).

(ii) Clearly $q(x, x) = 0$.

(iii) Suppose $x \neq y$. Since $\mathcal{U}$ is separated there exists an $a < \omega_q$ such that $\mathcal{U}(x, y) \notin V_a$; hence $q\mathcal{U}(x, y) \geq a > 0$.

(iv) Clearly $q(x, y) = q(y, x)$.

(v) Since $[q(x, y)] + q(y, x)](a) = 0$ implies $[q(x, y)](a) = 0$ and $[q(y, x)](a) = 0$ which in turn implies that $[q(x, z)](a) = 0$ we have that $q(x, y) = q(y, x)$.

Now we show that $\mathcal{U}_q = \mathcal{U}_q$. Let us note that $(\mathcal{U}_q : a < \omega_q)$ is a base for $\mathcal{U}_q$.

Suppose that $\mathcal{V}_a \subseteq \mathcal{U}_q$ is given. Then $\mathcal{U}_q(\mathcal{V}_a) \subseteq \mathcal{U}_q$. We also have $\mathcal{U}_q(\mathcal{V}_a) \subseteq \mathcal{U}_q$ because if $(x, y) \in \mathcal{U}_q$ then $\mathcal{U}_q(x, y) = 0 < \epsilon_q$, and so $(x, y) \in \mathcal{U}_q(\mathcal{V}_a)$. Hence $\mathcal{U}_q \subseteq \mathcal{V}_a$.

Suppose that $\mathcal{U}_q$ is given. For each $\mathcal{U}_q(1/n)$, $\beta < a$ there exists a $\mathcal{V}_a(\mathcal{U}_q(1/n))$ such that $\mathcal{V}_a(\mathcal{U}_q(1/n))$ is contained in $\mathcal{U}_q(1/(n+1))$ where $\mathcal{V}_a(\mathcal{U}_q(1/(n+1))$.

Now there exists a $\gamma < \omega_q$ such that $\mathcal{V}_a \subseteq \mathcal{U}_q(1/(n+1))$, because there are only $\mathcal{U}_q$-entourages of the form $\mathcal{V}_a(\mathcal{U}_q(1/(n+1)))$ where $\mathcal{U}_q(\mathcal{V}_a(\mathcal{U}_q(1/(n+1))))$.

We conclude the proof by showing that $\mathcal{V}_a \subseteq \mathcal{U}_q$. Let $(x, y) \in \mathcal{V}_a$. Then $q(x, y) < 1/n$ for all $a < \omega_q$ and $\beta < a$; hence $q(x, y) < 0$ for $\beta < a$; that is, $q(x, y) < 0$ for $\beta < a$. But this just means that $\mathcal{V}_a < 1/n$ or, equivalently, $(x, y) < 1/n$.

In a sense this is a generalization of the theorem that a uniform space is metrizable iff it has a denumerable base; because if $(X, \mathcal{U})$ has a denumerable base $(\mathcal{U}_a : a = 1, 2, \ldots)$ then it has an equivalent linearly ordered base $(\mathcal{V}_a : a = 1, 2, \ldots)$ where $\mathcal{V}_a = \bigcup a < \omega_q$.

The fact that $\mathcal{D}_q$ is the exact range space of $q$ is important. Sierpinski [5] proved that $\mathcal{D}_q$ is order complete and therefore lub and glb of sets in $\mathcal{D}_q$ exist. Furthermore $\mathcal{D}_q$ is order complete as a subset of $J^*$.

Hence we may generalize metric concepts such as diameter of sets, distance between sets and the Hausdorff metric on closed sets, which depend upon the completeness of the real number system.

§ 4. Generalized metric concepts, and Hausdorff $\omega_q$-metric spaces. A uniform space with linearly ordered base of least power $\omega_q$ will be called an $L_q$ space. The $\mu$-distance, $\sigma$, between sets $A$ and $B$ in an $L_q$ space is defined by $\sigma(A, B) = \text{gld}\{q(x, y) : x \in A, y \in B\}$ where $\omega_q$ is the $\omega_q$-metric defined by theorem 3. The $\mu$-diameter, $\delta$, of set $A$ in an $L_q$ space is defined by $d(A) = \text{lub}\{q(x, y) : x, y \in A\}$. 

\begin{itemize}
  \item [\textbf{Results on $\omega_q$-metric spaces}]
  \item We define $q : X \times X \rightarrow J^*$ as follows: $[q(x, y)](a) = 0$ if $q(x, y) = 0$, otherwise $[q(x, y)](a) = 1$. Actually, the exact range space of $q$ is $\mathcal{D}_q$.
  \item First we show that $q$ is an $\omega_q$-metric.
  \item (i) Clearly $q(x, y) \geq 0$ for all $x, y \in X$. (0 here is the $\omega_q$-0-sequence).
  \item (ii) Clearly $q(x, x) = 0$.
  \item (iii) Suppose $x \neq y$. Since $\mathcal{U}$ is separated there exists an $a < \omega_q$ such that $\mathcal{U}(x, y) \notin V_a$; hence $q\mathcal{U}(x, y) \geq a > 0$.
  \item (iv) Clearly $q(x, y) = q(y, x)$.
  \item (v) Since $[q(x, y)] + q(y, x)](a) = 0$ implies $[q(x, y)](a) = 0$ and $[q(y, x)](a) = 0$ which in turn implies that $[q(x, z)](a) = 0$ we have that $q(x, y) = q(y, x)$.
  \item Now we show that $\mathcal{U}_q = \mathcal{U}_q$. Let us note that $(\mathcal{U}_q : a < \omega_q)$ is a base for $\mathcal{U}_q$.
  \item Suppose that $\mathcal{V}_a \subseteq \mathcal{U}_q$ is given. Then $\mathcal{U}_q(\mathcal{V}_a) \subseteq \mathcal{U}_q$. We also have $\mathcal{U}_q(\mathcal{V}_a) \subseteq \mathcal{U}_q$ because if $(x, y) \in \mathcal{U}_q$ then $\mathcal{U}_q(x, y) = 0 < \epsilon_q$, and so $(x, y) \in \mathcal{U}_q(\mathcal{V}_a)$. Hence $\mathcal{U}_q \subseteq \mathcal{V}_a$.
  \item Suppose that $\mathcal{U}_q$ is given. For each $\mathcal{U}_q(1/n)$, $\beta < a$ there exists a $\mathcal{V}_a(\mathcal{U}_q(1/n))$ such that $\mathcal{V}_a(\mathcal{U}_q(1/n))$ is contained in $\mathcal{U}_q(1/(n+1))$ where $\mathcal{V}_a(\mathcal{U}_q(1/(n+1)))$.
  \item Now there exists a $\gamma < \omega_q$ such that $\mathcal{V}_a \subseteq \mathcal{U}_q(1/(n+1))$, because there are only $\mathcal{U}_q$-entourages of the form $\mathcal{V}_a(\mathcal{U}_q(1/(n+1)))$.
  \item We conclude the proof by showing that $\mathcal{V}_a \subseteq \mathcal{U}_q$. Let $(x, y) \in \mathcal{V}_a$. Then $q(x, y) < 1/n$ for all $a < \omega_q$ and $\beta < a$; hence $q(x, y) < 0$ for $\beta < a$; that is, $q(x, y) < 0$ for $\beta < a$. But this just means that $\mathcal{V}_a < 1/n$ or, equivalently, $(x, y) < 1/n$.
\end{itemize}
We should note here that since an \( \omega_2 \)-metric space need not have a range space which is order complete, the concept of \( \mu \)-distance and \( \mu \)-diameter as well as the Hausdorff \( \mu \)-metric (which will be defined later) are not definable unless you consider the \( L_\mu \)-space induced by the given \( \omega_2 \)-metric space, with the metric \( \varphi_\mu \) of theorem 3. In this sense, \( \mu \)-distance, \( \mu \)-diameter, and the Hausdorff \( \mu \)-metric are "\( L_\mu \)-concepts" and are therefore defined on \( L_\mu \)-spaces.

The following three theorems are generalizations of familiar theorems for metric spaces. The proofs are analogous to the metric case and are therefore not included.

**Theorem 4.1.** Let \( A \) denote the \( \mu \)-distance between sets on an \( L_\mu \)-space \((X, \varphi_\mu)\).

(i) If \( A \subset B = 0 \) then \( \sigma(A, B) = 0 \).

(ii) If \( A \subset B = \emptyset \) and \( A \) is closed and \( B \) is \( \omega_2 \)-compact then \( \sigma(A, B) > 0 \).

**Theorem 4.2.** An \( L_\mu \)-space is complete iff every nested \( \omega_2 \)-sequence of closed sets whose \( \mu \)-diameters go to 0 contains exactly one point.

**Theorem 4.3.** An \( L_\mu \)-space is complete iff it is closed in every \( L_\mu \)-space in which it can be uniformly isomorphically embedded.

Let \( C \) be the set of all non-empty closed sets in the metric space \((X, \varphi)\). Define \( d: C \times C \rightarrow \mathbb{R}_{\geq 0} \) non-negative reals as follows: \( d(A, B) = \text{glb}(\varphi: A \subset N_\mu(B) \text{ and } B \subset N_\mu(A)) \). Here \( X_\mu(A) = \bigcup N_\mu(A) \). Then \( d \) is a metric and \((C, d)\) is called the Hausdorff metric space associated with \((X, \varphi)\). If \( C \) is the set of all non-empty closed sets on uniform space \((X, \varphi_\mu)\), the Hausdorff uniform space \((C, \varphi_\mu)\) associated with \((X, \varphi_\mu)\) is defined as follows: \( \mathcal{U} = \{V_U: U \in \mathcal{U}\} \) where \( V_U = \{\{A, B\): B \subset U(A) \text{ and } A \subset U(B)\} \). The following relationships hold between a Hausdorff space and its associated space. See [4] for proofs of these theorems. \((X, \varphi)\) is totally bounded, (complete), (compact) iff \((C, d)\) is totally bounded, (complete), (compact); \((X, \varphi_\mu)\) is totally bounded, (compact) iff \((C, \varphi_\mu)\) is complete then \((X, \varphi_\mu)\) is complete, the converse is not true in general.

We now define a Hausdorff \( L_\mu \)-space as follows: given an \( L_\mu \)-space \((X, \varphi_\mu)\), let \( \delta_\mu \) be the induced \( \omega_2 \)-metric (by theorem 3), let \( d_\mu(A, B) = \text{glb}(\varphi \in \delta_\mu: A \subset N_\mu(B) \text{ and } B \subset N_\mu(A)) \) where \( A, B \in C \), then \( d_\mu \) is an \( \omega_2 \)-metric; letting \( \mathcal{U}_\mu \) denote the linearly ordered uniformity induced by \( d_\mu \), we define \((C, \mathcal{U}_\mu)\) as the Hausdorff \( L_\mu \)-space associated with \((X, \varphi_\mu)\). It is straightforward to show that the Hausdorff \( L_\mu \)-uniformity is identical to the ordinary Hausdorff uniformity associated with \( \mathcal{U}_\mu \). The relationships holding between \( L_\mu \)-spaces and their associated Hausdorff \( L_\mu \)-spaces are given below.

**Theorem 4.4.** The \( L_\mu \)-space \((X, \varphi_\mu)\) is complete iff \((C, \mathcal{U}_\mu)\) is complete.

**Theorem 4.5.** If \((C, \mathcal{U}_\mu)\) is \( \omega_2 \)-totally bounded then \((X, \varphi_\mu)\) is \( \omega_2 \)-totally bounded. The converse is true if \( \omega_2 \) is an inaccessible cardinal, but the converse is not true when \( \omega_2 \) is an accessible regular cardinal.

**Theorem 4.6.** If \((C, \mathcal{U}_\mu)\) is \( \omega_2 \)-compact then \((X, \varphi_\mu)\) is \( \omega_2 \)-compact. The converse is not true if \( \omega_2 \) is an accessible cardinal.

Whether the converse of 4.6 is true or not for inaccessible cardinals has not yet been determined.

A counterexample to the conjectures of theorems 4.5 and 4.6 is the space \((D_\mu, \mathcal{U}_{\omega_2}(D_\mu)) \text{ where } D_\mu \subset 2^\omega \), and \( x \in D_\mu \Leftrightarrow x = 1_\omega \text{ for some } a < \omega_2 \). Here we assume \( \omega_2 \) is a regular inaccessible cardinal and \( \mu \) is such that \( \nu < \mu, 2^\nu > \omega_2 \). Clearly \((D_\mu, \mathcal{U}_{\omega_2}(D_\mu))\) is \( \omega_2 \)-compact since it has fewer than \( \omega_2 \) points. But the associated Hausdorff \( L_\mu \)-space \((C, \varphi_\mu)\) is not \( \omega_2 \)-totally bounded. This follows because every subset of \( D_\mu \) is closed, hence there are \( 2^\nu > \omega_2 \text{ members of } \mathcal{C} \), the Hausdorff distance \( d_\mu \) between any two sets \( A, B \in C \) is \( > \omega_2 \).

The affirmative parts of the theorems 4.5 and 4.6 are straightforward and will not be included here. Theorem 4.4 is perhaps the most interesting of the results and the proof is provided below.

**Lemma.** Suppose that \( \varphi: X \times X \rightarrow D_\mu \) is an \( \omega_2 \)-metric defined on \( X \).

Then:

(i) If \( \varphi(x, y) < 1_\mu \), \( \varphi(y, z) < 1_\mu \), and \( y \neq 1_\mu \) then \( \varphi(x, z) < 1_\mu \).

(ii) \( N_{\omega_2}(\varphi_\mu(A)) = N_{\omega_2}(\varphi_\mu(A)) \text{ for } y \neq 1_\mu, A \subset C \).

Proof. (i) Suppose that \( \varphi(x, y) < 1_\mu \) and \( \varphi(y, z) < 1_\mu \). Then \( \varphi(x, z) = 0 \text{ if } a < b \text{ and } \varphi(y, z) = 0 \text{ for all } a < b \text{ in } \omega_2 \).\( \text{Now } y \neq 1_\mu \text{ so we have } \varphi(x, y) + \varphi(y, z) = 0 \text{ for all } a < b \text{. Therefore } \varphi(x, z) < 1_\mu \).

(ii) Clearly \( N_{\omega_2}(\varphi_\mu(A)) \subset N_{\omega_2}(\varphi_\mu(A)) \) since \( \varphi(x, y) < 1_\mu \) such that \( \varphi(x, y) < 1_\mu \). Since \( y \neq 1_\mu \), there exists \( x \in A \) such that \( \varphi(x, y) < 1_\mu \). By part (i) if follows that \( \varphi(x, z) < 1_\mu \); hence, \( \varphi(x, z) < 1_\mu \).

Proof of theorem 4.6. If \((C, \mathcal{U}_\mu)\) is complete then \((X, \varphi_\mu)\) is complete by the known result for uniform spaces.

Suppose that \((X, \varphi_\mu)\) is complete. Then it follows from theorem 4.4 that \((X, \varphi_\mu)\) is \( \omega_2 \)-complete (where \( \varphi_\mu \) is the \( \omega_2 \)-metric induced \( \mathcal{U}_\mu \)). Let \( \{A_n\} \) be a Cauchy \( \omega_2 \)-sequence in \((C, \delta_\mu)\). For each \( \beta < \omega_2 \), there exists an \( \alpha < \omega_2 \) such that \( \alpha > \beta \) and for \( y, \alpha > \alpha \), we have \( d_\mu(A_\alpha, A_\beta) < 1_\mu \). Let \( A = \bigcup \mathcal{U}_\mu(A_n) \).

We first show that \( A \neq \emptyset \). Consider the \( \omega_2 \)-sequence of closed sets \( N_\mu(A_n) \). By construction, if \( y > 1_\mu \) then \( N_\mu(A_n) \supset N_\mu(A_{n+1}) \). Now, by the lemma, \( N_\mu(A_n) = N_\mu(N_\mu(A_n)) \supset N_\mu(A_{n+1}) \) and hence \( N_\mu(A) \supset N_\mu(A_{n+1}) \). Therefore
Realization of mappings

by

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1. Realization of a mapping. Let \( \mathcal{C} \) be a category of pairs \((X, X_0)\), where \( X_0 \subset X \) are metric spaces and morphisms are continuous. Though many of the results of this section are valid for an arbitrary \( \mathcal{C} \) we shall pay our attention mainly to the three following categories:

a) The category \( \mathcal{C} \) of all metric pairs and all continuous mappings.

b) The category \( \mathcal{B} \) of polyhedral pairs and simplicial mappings. By a polyhedral pair \((X, X_0)\) we understand a finite polyhedron \( X \) with a triangulation and a subpolyhedron \( X_0 \) of \( X \) in this triangulation. Simplicial mappings are considered with respect to the given triangulations. However, the same polyhedral pair may have various triangulations.

c) The category \( \mathcal{M} \) of pairs of differentiable manifolds and differentiable mappings. By a pair of manifolds \((X, X_0)\) we understand a separable manifold \( X \) (with boundary or not) of class \( C^m \) and its submanifold \( X_0 \); a differentiable mapping is also of class \( C^m \).

As it is a frequent practice to do, we identify the pair \((X, X_0)\) with the space \( X_0 \) alone. If \((X, X_0)\) is an object of \( \mathcal{C} \), then we call \( X_0 \) a subobject of \( X \). An isomorphism \( h \) of an object \( A \) onto a subobject \( B \) of an object \( X \) is called an imbedding of \( A \) into \( X \). If such an imbedding exists, the object \( A \) is called imbeddable in \( X \).

If \( A, B \) are subsets of a metric space \( X \) and \( f : A \to B \) is a mapping, then we define \( D(f) = \sup_{x \in A} d(x, f(x)) \).

Let \( A, B \) and \( X \) be objects and let \( f : A \to B \) be a morphism. Let \( h : B \to X \) be an imbedding of \( B \) into \( X \). We say that the morphism \( f \) is realizable in \( X \) if there exists a sequence \( \{h_n\} \) (called a realization of \( f \)) in \( X \), where \( h_n : A \to X \) is an imbedding of \( A \) into \( X \) for \( n = 1, 2, \ldots \), such that \( \lim D(f_n) = 0 \) for \( f_n = h_n \).

If an object \( B \) is imbeddable in \( X \) and if a morphism \( f : A \to B \) is realizable in \( X \) for any imbedding \( h \) of \( B \) into \( X \), then we simply say that the morphism \( f \) is realizable in \( X \).

The definition depends on the category \( \mathcal{C} \) under consideration and we will always make it clear if a statement concerns a particular \( \mathcal{C} \). Usually,

(1) In [8] such a morphism has been called imbeddable in \( X \).