Addendum (August 14, 1969)

THEOREM. *Let G be a torsion-free* LCA *group satisfying any one of the following conditions:*

(a) *G is separable.*

(b) *G satisfies countable chain condition, i.e. any family of disjoint open sets in G is countable.*

(c) *G is σ-compact.*

(d) *G is Lindelöf.*

(e) *Any uncountable family of open sets has an uncountable subfamily with non empty intersection.*

Then G is self-dual if and only if G is of the form mentioned in the preceding theorem.

Proof. If G is of the form mentioned in the preceding theorem then it is self-dual by Lemma 6. Suppose now G is self-dual. Then $G = R^n \oplus A$ where A has a compact open subgroup H. $\hat{G} \simeq \hat{R}^n \oplus \hat{A}$ where $\hat{A}$ has a compact open subgroup $H^\perp$ which is the dual of the discrete A/H. Let G satisfy anyone of the conditions (a)–(e). Observe that (a) $\Rightarrow$ (b) and (c) $\Rightarrow$ (d). We assert that A/H is countable. If not, $A = \bigcup_{a \in I} Hx_a$, a set union of disjoint cosets and I is an uncountable set. Each of these cosets is open in A. If we now consider $\{R^n + Hx_a\}_{a \in I}$ we easily arrive at a contradiction. So A/H is countable. Now $\hat{G} \simeq \hat{R}^n \oplus \hat{A}$ where $\hat{A}$ has a compact open subgroup $H^\perp$ which is now the dual of the countable discrete group A/H. Hence $H^\perp$ is metrizable. Since G is isomorphic to $\hat{G}$, by a similar reasoning we get $\hat{A}/H^\perp$ is countable. Hence H, the dual of $\hat{A}/H^\perp$ is metrizable. Since H is metrizable and A/H is countable discrete and hence metrizable we get A is metrizable. Already R^n is metric-Hence G is metrizable. Then the preceding theorem completes the proof.

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Results on ω_μ-metric spaces

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§ 1. Introduction and preliminary results. A linearly ordered abelian group is a set A, together with a binary operation $\cdot$, and an order relation $>$, such that $(A, \cdot)$ is an abelian group $(A, >)$ is a linearly ordered set and the following condition is satisfied: if $a > b$ then $ac > bc$. The group A has character ω_μ iff there exists a decreasing ω_μ-sequence converging to 0 in the order topology on A. Here ω_μ denotes the μth infinite cardinal number. Cardinal numbers are considered as initial ordinal numbers and each ordinal coincides with the set of all smaller ordinals. The power of ω_μ is denoted by $\aleph_\mu$. We will be concerned with only that ω_μ which represents the least character of A and it is easily shown that such an ω_μ must be a regular cardinal number.

Let X be a set and ϱ a function from $X \times X$ to $(A, \cdot, >)$ such that

(i) $\varrho(x, y) = 0$ iff $x = y$,

(ii) $\varrho(x, y) = \varrho(y, x) > 0$ if $x \neq y$,

(iii) $\varrho(x, y) \leqslant \varrho(x, z) + \varrho(z, y)$,

then ϱ is called an ω_μ-metric and (X, ϱ) is an ω_μ-metric space. Sikorski [10] has done the most extensive study of ω_μ-spaces; other references include Hausdorff [3], Cohen and Goffman [1] and [2], Parovicenko [6], and most recently, Shu-Tang [7].

The ω_μ-metric ϱ on X induces a topology $\mathcal{T}_\varrho$ on X; a base for the topology consisting of sets of the form $N_a(x)$ where $N_a(x) = \{y \in X : \varrho(x, y) < a\}$, $a \in A$, and $a > 0$. Also ϱ induces a uniformity $\mathcal{U}_\varrho$ on X: a base for the uniformity consisting of sets U_a where $U_a = \{(x, y) : \varrho(x, y) < a\}$, $a \in A$, $a > 0$. It is easily shown that the ω_μ-metric topology and the ω_μ-uniform topology are identical.

An ω_μ-additive space is a topological space $(X, \mathcal{T})$ which satisfies the condition that for any family of open sets $\mathcal{F}$, of power $< \aleph_\mu$ it follows that $\bigcap \mathcal{F}$ is an open set. Clearly every topological space is an ω_0-additive space. It is easily shown that if (X, ϱ) is an ω_μ-metric space then $(X, \mathcal{T}_\varrho)$ is an ω_μ-additive space. Sikorski defines the following concepts on an ω_μ-additive space $(X, \mathcal{T})$. The space $(X, \mathcal{T})$ has a basis iff it has a base of power $\aleph_\mu$; $(X, \mathcal{T})$ is separable iff there exists an everywhere dense subset Y of X of power $\leqslant \aleph_\mu$; $(X, \mathcal{T})$ is compact iff every ω_μ-sequence in X has a convergent ω_μ-subsequence; $(X, \mathcal{T})$ is ω_μ-bicompact iff every open cover of X has a subcover of power $< \aleph_\mu$. For an ω_μ-metric space (X, ϱ) Sikorski gives these definitions: (X, ϱ) is totally bounded iff for any $a \in A$, $a > 0$, there exists a subset Y of X of power $< \aleph_\mu$ such that $\bigcup_{y \in Y} N_a(y) = X$; (X, ϱ) is complete iff every Cauchy ω_μ-sequence converges. A Cauchy ω_μ-sequence is an ω_μ-sequence $\{x_a\}$ satisfying the condition that for all $a > 0$, $a \in A$ there exists $\alpha < \omega_\mu$ such that if $\beta, \gamma > \alpha$ then $\varrho(x_\beta, x_\gamma) < a$.

Sikorski states some theorems relating these topological and uniform concepts. We give a more complete list below, omitting the proofs since each is a straightforward generalization of the proof of the standard topological theorem. For Sikorski's terms: basis, separable, compact, ω_μ-bicompact, totally bounded and complete, we will use; *ω_μ-countable*, *ω_μ-separable*, *ω_μ-compact* (for both compact and ω_μ-bicompact since they are equivalent by theorem 1.3), *ω_μ-totally bounded*, and *ω_μ-complete*, respectively.

If ϱ is an ω_μ-metric on X then:

THEOREM 1.1. *$(X, \mathcal{T}_\varrho)$ is ω_μ-separable iff it is ω_μ-countable.*

THEOREM 1.2. *If (X, ϱ) is ω_μ-totally bounded then $(X, \mathcal{T}_\varrho)$ is ω_μ-separable.*

THEOREM 1.3. *The following three statements are equivalent on $(X, \mathcal{T}_\varrho)$:*

(i) *Every ω_μ-sequence in X has a convergent ω_μ-subsequence in X.*

(ii) *Every open cover of X of power $\aleph_\mu$ has a subcover of power $< \aleph_\mu$.*

(iii) *Every open cover of X has a subcover of power $< \aleph_\mu$.*

THEOREM 1.4. *(X, ϱ) is ω_μ-complete iff $(X, \mathcal{U}_\varrho)$ is complete in the uniform sense.*

THEOREM 1.5. *If $(X, \mathcal{T}_\varrho)$ is ω_μ-compact then (X, ϱ) is ω_μ-complete and ω_μ-totally bounded.*

The converse of Theorem 1.5 is true for $\mu = 0$. Sikorski has shown that for accessible regular cardinals the converse is not true in general; specifically: if $\mu = \nu + 1$, if ω_ν is regular, and if $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ for $\alpha < \nu$ then there exists an ω_μ-metric space which is ω_μ-complete, ω_μ-totally bounded, but not ω_μ-compact. (See [10] pp. 132, 133). For inaccessible cardinals the converse of Theorem 1.5 remained an open question.

§ 2. A counterexample for an ω_μ-compactness theorem for inaccessible cardinals. In 1964 Monk and Scott [6] showed that if ω_μ is the first uncountable inaccessible cardinal then 2^{ω_μ} is not "ω_μ-compact" in the ω_μ-product topology, $\mathcal{T}_\mu$. The set 2^{ω_μ} is essentially

the set of all ω_μ-sequences of 0's and 1's; the term "ω_μ-compact" is identical to Sikorski's term "ω_μ-bicompact" which is equivalent to Sikorski's term "compact"; the ω_μ-product topology on 2^{ω_μ} has a base made up of the sets G_x^T where $x \in 2^{\omega_\mu}$, T is a subset of ω_μ of power $< \aleph_\mu$, and $y \in G_x^T$ iff the sequences x and y agree on all coordinates in T.

Sikorski had studied the space of ω_μ-sequences of 0's and 1's, which he denoted by $\mathfrak{D}_\mu$, along with the metric σ, where $\sigma(x, y) = 0$ iff $x = y$ and $\sigma(x, y) = 1/\xi_0$ if $x \neq y$ where $1/\xi_0 \in W_\mu$ and ξ_0 is the first ordinal where the sequences x and y differ. Here W_μ denotes the least algebraic field containing the set of all ordinals $\alpha < \omega_\mu$; see [9]. Sikorski showed that $(\mathfrak{D}_\mu, \sigma)$ is ω_μ-complete for all μ and $(\mathfrak{D}_\mu, \sigma)$ is ω_μ-totally bounded for all ω_μ which are inaccessible cardinals so, in particular $(\mathfrak{D}_\mu, \sigma)$ is ω_μ-complete and ω_μ-totally bounded for the first inaccessible cardinal ω_μ. Now it is easily shown that $\mathcal{T}_\sigma$ is identical to $\mathcal{T}_\mu$ and hence the result of Monk and Scott that $(\mathfrak{D}_\mu, \mathcal{T}_\mu)$ is not ω_μ-compact resolves the open question referred to in § 1.

Perhaps a conceptually simpler ω_μ-metric than the metric σ is the one we define below; it too induces the topology $\mathcal{T}_\mu$ on $\mathfrak{D}_\mu$. Let ϱ_μ: $\mathfrak{D}_\mu \times \mathfrak{D}_\mu \to \mathfrak{D}_\mu$ be such that $\varrho(x, y) = (0, 0, \ldots)$ iff $x = y$ and $\varrho(x, y) = 1_\alpha$ if $x \neq y$ where α is the least ordinal at which the sequences x and y differ and 1_α is the ω_μ-sequence which is 1 in the αth coordinate and 0 elsewhere. Given that $\mathfrak{D}_\mu$ is a subset of the ordered abelian group $(J_\mu, +, >)$ where J_μ is the family of all ω_μ-sequences of integers, $+$ is coordinatewise addition, and $>$ is lexicographic order, it is easy to show that ϱ_μ is an ω_μ-metric on $\mathfrak{D}_\mu$. It is also easily shown that $(\mathfrak{D}_\mu, \varrho_\mu)$ is ω_μ-complete for all μ and $(\mathfrak{D}_\mu, \varrho_\mu)$ is ω_μ-totally bounded for all ω_μ which are inaccessible. Now $\mathcal{T}_{\varrho_\mu}$ is identical to $\mathcal{T}_\mu$ so if ω_μ is the first uncountable inaccessible cardinal $(\mathfrak{D}_\mu, \varrho_\mu)$ provides us with an ω_μ-metric space which is ω_μ-complete and ω_μ-totally bounded but not ω_μ-compact. Thus:

THEOREM 2. *It is not necessarily true for inaccessible cardinals that an ω_μ-complete, ω_μ-totally bounded ω_μ-metric space is ω_μ-compact.*

The same space $(\mathfrak{D}_\mu, \varrho_\mu)$ also provides us with an example of a ω_μ-metric space which is ω_μ-compact, and perfect, and is of power $2^{\aleph_\mu}$. It is true that every compact, perfect metric space has power $2^{\aleph_0}$ but Sikorski noted that he knew of no example of a compact, perfect ω_μ-metric space of power $> \aleph_\mu$.

§ 3. A ω_μ-metrization theorem. Sikorski remarks that every ω_μ-additive, ω_μ-countable topological space is ω_μ-metrizable. This is a generalization of Urysohn's metrization theorem. Shu-Tang [7] generalized the Nagata–Smirnov metrization theorem with this result: If $(X, \mathcal{T})$ is a regular (topologically speaking) ω_μ-additive space then $(X, \mathcal{T})$ is ω_μ-metrizable iff there exists a $\aleph_\mu$ basis for $\mathcal{T}$. The range space of Sikorski's

ω_μ-metric is W_μ and the range of Shu-Tang's ω_μ-metric is the family of ω_μ-sequences of real numbers.

We give a metrization theorem for uniform spaces.

LEMMA. *Let $(X, \mathfrak{U})$ be a uniform space with a linearly ordered base ($U_1 < U_2$ iff $U_1 \supset U_2$ for $U_1, U_2 \in U$). Let $\aleph_\mu$ be the least power of such a base. Then there exists an equivalent well ordered base of power $\aleph_\mu$.*

Proof. Let $\mathfrak{B}$ be a linearly ordered base of least power. So $\mathfrak{B} = \{B_\alpha: \alpha < \omega_\mu\}$ (here $\alpha < \beta$ does not imply that $B_\alpha \supset B_\beta$). Let $V_{\alpha_0} = \bigcap_{\alpha \leqslant \alpha_0} B_\alpha$ and let $\mathfrak{V} = \{V_\alpha: \alpha < \omega_\mu\}$. Clearly now, $\alpha < \beta$ does imply that $V_\alpha \supset V_\beta$, so we have a well ordered set $\mathfrak{V}$.

We now show that $\mathfrak{V}$ is equivalent to $\mathfrak{B}$.

(i) Clearly, for any B_α, we have $V_\alpha \subset B_\alpha$.

(ii) Let V_{α_0} be given. Suppose there does not exist a $\beta < \omega_\mu$ such that $B_\beta \subset V_{\alpha_0}$. Since $\mathfrak{B}$ is linearly ordered this would mean that for all $\beta > \alpha_0$ there exists an $\alpha < \alpha_0$ such that $B_\alpha \subset B_\beta$. Therefore $\{B_\alpha: \alpha_\alpha < \alpha_0\}$ is a linearly ordered base for $(X, \mathfrak{U})$. But the power of α_0 is less than $\aleph_\mu$ because $\alpha_0 < \omega_\mu$ and this contradicts the assumption that $\aleph_\mu$ is the least power of all linearly ordered bases of $\mathfrak{U}$. This contradiction establishes that for each V_{α_0} there exists a B_α such that $B_\alpha \subset V_{\alpha_0}$. Hence, $\mathfrak{V}$ is a base for $\mathfrak{U}$.

THEOREM 3. *A separated uniform space $(X, \mathfrak{U})$ is ω_μ-metrizable iff $(X, \mathfrak{U})$ has a linearly ordered base and $\aleph_\mu$ is the least power of such a base.*

Proof. If $(X, \mathfrak{U})$ is ω_μ-metrizable then by definition there exists a ω_μ-metric ϱ, such that the uniformity U_ϱ, induced by ϱ is identical to $\mathfrak{U}$. But clearly U_ϱ has a linearly ordered base (in fact, well ordered) given by U_α where $U_\alpha = \{(x, y): \varrho(x, y) < a_\alpha\}$ and $\{a_\alpha\}$ is the ω_μ-sequence converging to 0 in the group $(A, \cdot, >)$ of character ω_μ. Since ω_μ is the least character of A, $\aleph_\mu$ is the least power of a well ordered base. Since every linearly ordered base has an equivalent well ordered base, $\aleph_\mu$ is the least power of any linearly ordered base.

Now suppose that $(X, \mathfrak{U})$ has a linearly ordered base of least power $\aleph_\mu$. If $\mu = 0$ then a standard result gives us that $(X, \mathfrak{U})$ is metrizable and hence ω_0-metrizable. Suppose $\mu > 0$ and let the linearly ordered base $\mathfrak{V} = \{V_\alpha: \alpha < \omega_\mu\}$ be of power $\aleph_\mu$. We may assume, by the lemma that $\mathfrak{V}$ is well ordered. Since $(X, \mathfrak{U})$ is a uniform space there exists an augmented family of pseudo-metrics $\{\varrho_i: i \in I\}$ such that $(X, [\varrho_i])$ is identical to $(X, \mathfrak{U})$. So for all $\alpha < \omega_\mu$ there exists ϱ_{i_α} and ε_α such that $U_{\varrho_{i_\alpha}}(\varepsilon_\alpha) \subset V_\alpha$. Let $(J_\mu, +, >)$ be the ordered group defined in § 2. This group has character ω_μ. This follows easily because $\{1_\alpha: \alpha < \omega_\mu\}$ is an ω_μ-sequence converging to 0 and no ω_ν-sequence converges to 0 for $\nu < \mu$ because ω_μ is a regular cardinal.

We define ϱ: $X \times X \to J_\mu$ as follows: $[\varrho(x, y)](\alpha) = 0$ if $\varrho_{i_\alpha}(x, y) = 0$, otherwise $[\varrho(x, y)](\alpha) = 1$. Actually, the exact range space of ϱ is $\mathfrak{D}_\mu$.

First we show that ϱ is an ω_μ-metric.

(i) Clearly $\varrho(x, y) \geqslant 0$ for all $x, y \in X$ (0 here is the ω_μ-0-sequence).

(ii) Clearly $\varrho(x, x) = 0$.

(iii) Suppose $x \neq y$. Since $\mathfrak{U}$ is separated there exists an $\alpha < \omega_\mu$ such that $(x, y) \notin V_\alpha$; hence $\varrho_{i_\alpha}(x, y) \geqslant \varepsilon_\alpha > 0$.

(iv) Clearly $\varrho(x, y) = \varrho(y, x)$.

(v) Since $[\varrho(x, y) + \varrho(y, z)](\alpha) = 0$ implies $[\varrho(x, y)](\alpha) = 0$ and $[\varrho(y, z)](\alpha) = 0$ which in turn implies that $[\varrho(x, z)](\alpha) = 0$ we have that $\varrho(x, y) + \varrho(y, z) \geqslant \varrho(x, z)$.

Now we show that $\mathfrak{U} = \mathfrak{U}_\varrho$. Let us note that $\{U_{1_\alpha}: \alpha < \omega_\mu\}$ is a base for $\mathfrak{U}_\varrho$.

Suppose that $V_\alpha \in \mathfrak{V}$ is given. Then $U_{\varrho_{i_\alpha}}(\varepsilon_\alpha) \subset V_\alpha$. We also have $U_{\varrho_{i_\alpha}}(\varepsilon_\alpha) \supset U_{1_\alpha}$ because if $(x, y) \in U_{1_\alpha}$ then $\varrho_{i_\alpha}(x, y) = 0 < \varepsilon_\alpha$, and so $(x, y) \in U_{\varrho_{i_\alpha}}(\varepsilon_\alpha)$. Hence $U_{1_\alpha} \subset V_\alpha$.

Suppose that U_{1_α} is given. For each $U_{\varrho_{i_\beta}}(1/n)$, $\beta < \alpha$ there exists a $V_{\gamma(\beta,n)}$ such that $V_{\gamma(\beta,n)}$ is contained in $U_{\varrho_{i_\beta}}(1/n)$ where $\gamma(\beta, n) < \omega_\mu$. Now there exists a $\gamma < \omega_\mu$ such that $V_\gamma \subset \bigcap \{V_{\gamma(\beta,n)}: \beta \leqslant \alpha, n < \omega_0\}$, because there are only $\aleph_\nu$ entourages of the form $V_{\gamma(\beta,n)}$ where $\aleph_\nu = \max(|\alpha|, \aleph_0)$ and $\omega_\nu < \omega_\mu$ because $\alpha < \omega_\mu$ and $\omega_0 < \omega_\mu$. (Here $|\alpha|$ denotes the power of α.) We conclude the proof by showing that $V_\gamma \subset U_{1_\alpha}$. Let $(x, y) \in V_\gamma$. Then $\varrho_{i_\beta}(x, y) < 1/n$ for all $n < \omega_0$ and $\beta \leqslant \alpha$; hence $\varrho_{i_\beta}(x, y) = 0$ for $\beta \leqslant \alpha$; that is, $\varrho(x, y)(\beta) = 0$ for $\beta \leqslant \alpha$. But this just means that $\varrho(x, y) < 1_\alpha$ or, equivalently, $(x, y) \in U_{1_\alpha}$.

In a sense this is a generalization of the theorem that a uniform space is metrizable iff it has a denumerable base; because if $(X, \mathfrak{U})$ has a denumerable base $\{U_k: k = 1, 2, \ldots\}$ then it has an equivalent linearly ordered base $\{V_k: k = 1, 2, \ldots\}$ where $V_k = \bigcap_{i=1}^{k} \mathfrak{U}_i$.

The fact that $\mathfrak{D}_\mu$ is the exact range space of ϱ is important. Sierpiński [8] proved that $\mathfrak{D}_\mu$ is order complete and therefore lubs and glbs of sets in $\mathfrak{D}_\mu$ exist. Furthermore $\mathfrak{D}_\mu$ is order complete as a subset of J_μ. Hence we may generalize metric concepts such as diameter of sets, distance between sets and the Hausdorff metric on closed sets, which depend upon the completeness of the real number system.

§ 4. Generalized metric concepts, and Hausdorff ω_μ-metric spaces. A uniform space with linearly ordered base of least power $\aleph_\mu$ will be called an L_μ space. The μ-distance, σ, between sets A and B in an L_μ space is defined by $\sigma(A, B) = \text{glb}\{\varrho_\mu(x, y): x \in A, y \in B\}$ where ϱ_μ is the ω_μ-metric defineable by theorem 3. The μ-diameter, d, of set A in an L_μ space is defined by $d(A) = \text{lub}\{\varrho_\mu(x, y): x, y \in A\}$.

We should note here that since an ω_μ-metric space need not have a range space which is order complete, the concept of μ-distance and μ-diameter as well as the Hausdorff μ-metric (which will be defined later) are not defineable unless you consider the L_μ-space induced by the given ω_μ-metric space, with the metric ϱ_μ of theorem 3. In this sense, μ-distance, μ-diameter, and the Hausdorff μ-metric are "L_μ-concepts" and are therefore defined on L_μ-spaces.

The following three theorems are generalizations of familiar theorems for metric spaces. The proofs are analogous to the metric case and are therefore not included.

THEOREM 4.1. *Let σ denote the μ-distance between sets on an L_μ space $(X, \mathfrak{U})$.*

(i) *If $A \cap B = \emptyset$ then $\sigma(A, B) = 0$.*

(ii) *If $A \cap B \neq \emptyset$ and A is closed and B is ω_μ-compact then $\sigma(A, B) > 0$.*

THEOREM 4.2. *An L_μ space is complete iff every nested ω_μ-sequence of closed sets whose μ-diameters go to 0 contains exactly one point.*

THEOREM 4.3. *An L_μ space is complete iff it is closed in every L_μ space in which it can be uniformly isomorphically embedded.*

Let $\mathcal{C}$ be the set of all non-emtpy closed sets in the metric space (X, ϱ). Define d: $\mathcal{C} \times \mathcal{C} \to$ non-negative reals as follows: $d(A, B) = \operatorname{glb}\{\varepsilon\colon A \subset N_\varepsilon(B)$ and $B \subset N_\varepsilon(A)\}$. Here $N_\varepsilon(A) = \bigcup_{x \in A} N_\varepsilon(x)$. Then d is a metric and $(\mathcal{C}, d)$ is called the Hausdorff metric space associated with (X, ϱ), If $\mathcal{C}$ is the set of all non-empty closed sets on uniform space $(X, \mathfrak{U})$, the Hausdorff uniform space $(\mathcal{C}, \mathfrak{V})$ associated with $(X, \mathfrak{U})$ is defined as follows: $\mathfrak{V} = \{V_U\colon U \in \mathfrak{U}\}$ where $V_U = \{(A, B)\colon B \subset U(A)$ and $A \subset U(B)\}$. The following relationships hold between a Hausdorff space and its associated space. See [4] for proofs of these theorems. (X, ϱ) is totally bounded, (complete), (compact) iff $(\mathcal{C}, d)$ is totally bounded, (complete), (compact); $(X, \mathfrak{U})$ is totally bounded, (compact) iff $(\mathcal{C}, \mathfrak{V})$ is totally bounded (compact); if $(\mathcal{C}, \mathfrak{V})$ is complete then $(X, \mathfrak{U})$ is complete, the converse is not true in general.

We now define a Hausdorff L_μ-space as follows: given an L_μ-space $(X, \mathfrak{U})$, let ϱ_μ be the induced ω_μ-metric (by theorem 3), let $d_\mu(A, B) = \operatorname{glb}\{\alpha \in \mathfrak{D}_\mu\colon A \subset N_\alpha(B),\ B \subset N_\alpha(A)\}$ where $A, B \in \mathcal{C}$, then d_μ is an ω_μ-metric; letting $\mathfrak{V}_\mu$ denote the linearly ordered uniformity induced by d_μ we define $(\mathcal{C}, \mathfrak{V}_\mu)$ as the Hausdorff L_μ-space associated with $(X, \mathfrak{U})$. It is straightforward to show that the Hausdorff L_μ uniformity is identical to the ordinary Hausdorff uniformity associated with $\mathfrak{U}$. The relationships holding between L_μ-spaces and their associated Hausdorff L_μ-spaces are given below.

THEOREM 4.4. *The L_μ space $(X, \mathfrak{U})$ is complete iff $(\mathcal{C}, \mathfrak{V}_\mu)$ is complete.*

THEOREM 4.5. *If $(\mathcal{C}, \mathfrak{V}_\mu)$ is ω_μ-totally bounded then $(X, \mathfrak{U})$ is ω_μ-totally bounded. The converse is true if ω_μ is an inaccessible cardinal, but the converse is not true when ω_μ is an accessible regular cardinal.*

THEOREM 4.6. *If $(\mathcal{C}, \mathcal{T}_{\mathfrak{V}})$ is ω_μ-compact then $(X, \mathcal{T}_{\mathfrak{U}})$ is ω_μ-compact. The converse is not true if ω_μ is an accessible cardinal.*

Whether the converse of 4.6 is true or not for inaccessible cardinals has not yet been determined.

A counterexample to the converses of theorems 4.5 and 4.6 is the space $(D_\nu, \mathfrak{U}_{\varrho_\mu}/D_\nu)$ where $D_\nu \subset \mathfrak{D}_\nu$ and $x \in D_\nu$ iff $x = 1_\alpha$ for some $\alpha < \omega_\nu$. Here we assume ω_μ is a regular accessible cardinal and ν is such that $\nu < \mu$, $2^{\aleph_\nu} \geqslant \aleph_\mu$. Clearly D_ν is ω_μ-compact since it has fewer than $\aleph_\mu$ points. But the associated Hausdorff L_μ-space $(\mathcal{C}, \mathfrak{V})$ is not ω_μ-totally bounded. This follows because every subset of D_ν is closed, hence there are $2^{\aleph_\nu} \geqslant \aleph_\mu$ members of $\mathcal{C}$, and the Hausdorff distance d_μ between any two sets A, B in $\mathcal{C}$ is $\geqslant 1_{\omega_\nu}$.

The affirmative parts of the theorems 4.5 and 4.6 are straight forward and will not be included here. Theorem 4.4 is perhaps the most interesting of the results and the proof is provided below.

LEMMA. *Suppose that $\varrho\colon X \times X \to D_\mu$ is an ω_μ-metric defined on X. Then:*

(i) *If $\varrho(x, y) < 1_\beta$, $\varrho(y, z) < 1_\gamma$, and $\gamma \geqslant \beta$ then $\varrho(x, z) < 1_\beta$.*

(ii) *$N_{1_\gamma}(N_{1_\beta}(A)) = N_{1_\beta}(A)$, for $\gamma \geqslant \beta$, $A \subset X$.*

Proof. (i) Suppose that $\varrho(x, y) < 1_\beta$ and $\varrho(y, z) < 1_\gamma$. Then $(\varrho(x, y))(\alpha) = 0$ for all $\alpha \leqslant \beta$ and $(\varrho(y, z))(\alpha) = 0$ for all $\alpha \leqslant \gamma$. Now $\gamma \geqslant \beta$ so we have $(\varrho(x, y) + \varrho(y, z))(\alpha) = 0$ for all $\alpha \leqslant \beta$. Therefore $\varrho(x, z) \leqslant \varrho(x, y) + \varrho(y, z) < 1_\beta$.

(ii) Clearly $N_{1_\beta}(A) \subset N_{1_\gamma}(N_{1_\beta}(A))$. Suppose that $x \in N_{1_\gamma}(N_{1_\beta}(A))$; then there exists a $y \in N_{1_\beta}(A)$ such that $\varrho(x, y) < 1_\gamma$. Since $y \in N_{1_\beta}(A)$, there exists a $z \in A$ such that $\varrho(y, z) < 1_\beta$. By part (i) if follows that $\varrho(x, z) < 1_\beta$; hence, $x \in N_{1_\beta}(A)$.

Proof of theorem 4.6. If $(\mathcal{C}, \mathfrak{V})$ is complete then $(X, \mathfrak{U})$ is complete by the known result for uniform spaces.

Suppose that $(X, \mathfrak{U})$ is complete. Then it follows from theorem 1.4 that (X, ϱ_μ) is ω_μ-complete (where ϱ_μ is the ω_μ-metric inducing $\mathfrak{U}$). Let $\{A_\alpha\}$ be a Cauchy ω_μ-sequence in $(\mathcal{C}, d_\mu)$. For each $\beta < \omega_\mu$ there exists an $\alpha_\beta < \omega_\mu$ such that $\alpha_\beta > \beta$ and for $\gamma, \delta > \alpha_\beta$ we have $d_\mu(A_\gamma, A_\delta) < 1_\beta$. Let $A = \bigcap_{\beta < \omega_\mu} \overline{N_{1_\beta}(A_{\alpha_\beta})}$.

We first show that $A \neq \emptyset$. Consider the ω_μ-sequence of closed sets, $\overline{N_{1_\beta}(A_{\alpha_\beta})}$. By construction, if $\gamma > \beta$ then $N_{1_\beta}(A_{\alpha_\beta}) \supset A_{\alpha_\gamma}$. Now, by the lemma, $N_{1_\beta}(A_{\alpha_\beta}) = N_{1_\gamma}(N_{1_\beta}(A_{\alpha_\gamma})) \supset N_{1_\gamma}(A_{\alpha_\gamma})$ and hence $\overline{N_{1_\beta}(A_{\alpha_\beta})} \supset \overline{N_{1_\gamma}(A_{\alpha_\gamma})}$. Therefore the ω_μ-sequence of closed sets is nested. If the ω_μ-diameters go to 0 then, since $(X, \mathfrak{U})$ is ω_μ-complete, by theorem 4.2 the intersection contains a point, x. If the ω_μ-diameters do not go to 0 then there exists a set A' of diameter $> 1_\gamma$ for some $\gamma < \omega_\mu$ contained in the intersection. In either case $A \neq \emptyset$.

We complete the proof by showing that $\{A_\alpha\}$ converges to A. This is done by showing that for any $\beta < \omega_\mu$, if $\gamma > \alpha_{\beta+1}$ then $d_\mu(A_\gamma, A) < 1_\beta$.

First: $A \subset N_{1_\beta}(A_\gamma)$ is true by the following argument: if $x \in A$ then $x \in \overline{N_{1_{\beta+1}}(A_{\alpha_{\beta+1}})}$. So there exists a $y \in A_{\alpha_{\beta+1}}$ such that $\varrho_\mu(x, y) \leqslant 1_{\beta+1}$. Since $d_\mu(A_\gamma, A_{\alpha_{\beta+1}}) < 1_{\beta+1}$, we have that $y \in N_{1_{\beta+1}}(A_\gamma)$ and so there exists a $z \in A_\gamma$ such that $\varrho_\mu(y, z) < 1_{\beta+1}$. Since $\varrho_\mu(x, y) \leqslant 1_{\beta+1} < 1_\beta$, by the lemma, $\varrho_\mu(x, z) < 1_\beta$. If follows that $x \in N_{1_\beta}(A_\gamma)$.

Now $A_\gamma \subset N_{1_\beta}(A)$ is true by the proof below. Let $x \in A_\gamma$; then for all $\delta > \beta+1$, since $d_\mu(A_\gamma, A_{\alpha\delta}) < 1_{\beta+1}$, it follows that $x \in N_{1_{\beta+1}}(A_{\alpha\delta})$. Hence $x \in N_{1_{\beta+1}}(N_{1_\delta}(A_{\alpha\beta})) \subset N_{1_\beta}(N_{1_\delta}(A_{\alpha\delta})) \subset N_{1_\beta}(N_{1_\delta}(A_{\alpha\delta}))$. Therefore $x \in N_{1_\beta}(N_{1_\zeta}(A_{\alpha\zeta}))$ for all $\zeta < \omega_\mu$ because $\overline{N_{1_\zeta}(A_{\alpha\zeta})} \supset \overline{N_{1_\delta}(A_{1_\delta})}$ if $\zeta < \delta$. So $x \in \bigcap_{\zeta<\omega_\mu} N_{1_\beta}(N_{1_\zeta}(A_{\alpha\zeta}))$

$$= N_{1_\beta}\Big(\bigcap_{\zeta<\omega_\mu} \overline{N_{1_\zeta}(A_{\alpha\zeta})}\Big) = N_{1_\beta}(A).$$

Since every Cauchy ω_μ-sequence converges in $(\mathcal{C}, d_\mu)$ it follows that $(\mathcal{C}, d_\mu)$ is ω_μ-complete and hence, by theorem 1.4, $(\mathcal{C}, \mathfrak{V})$ is complete.

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Realization of mappings

by

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1. Realization of a mapping. Let $\mathfrak{C}$ be a category of pairs (X, X_0), where $X_0 \subset X$ are metric spaces and morphisms are continuous. Though many of the results of this section are valid for an arbitrary $\mathfrak{C}$ we shall pay our attention mainly to the three following categories:

a) The category $\mathfrak{S}$ of all metric pairs and all continuous mappings.

b) The category $\mathfrak{P}$ of polyhedral pairs and simplicial mappings. By a polyhedral pair (X, X_0) we understand a finite polyhedron X with a triangulation and a subpolyhedron X_0 of X in this triangulation. Simplicial mappings are considered with respect to the given triangulations. However, the same polyhedral pair may have various triangulations.

c) The category $\mathfrak{M}$ of pairs of differentiable manifolds and differentiable mappings. By a pair of manifolds (X, X_0) we understand a separable manifold X (with boundary or not) of class C^∞ and its submanifold X_0; a differentiable mapping is also of class C^∞.

As it is a frequent practice to do, we identify the pair $(X, \emptyset)$ with the space X alone. If (X, X_0) is an object of $\mathfrak{C}$, then we call X_0 a subobject of X. An isomorphism h of an object A onto a subobject B of an object X is called an *imbedding* of A into X. If such an imbedding exists, the object A is called *imbeddable* in X.

If A, B are subsets of a metric space X and $f\colon A \to B$ is a mapping, then we define $D(f) = \sup_{x\in A} \varrho\big(x, f(x)\big)$.

Let A, B and X be objects and let $f\colon A \to B$ be a morphism. Let $h\colon B \to X$ be an imbedding of B into X. We say that the morphism f is *realizable* in $X \operatorname{rel} h$ if there exists a sequence $\{h_n\}$ (called a *realization* of $f \operatorname{rel} h$), where $h_n\colon A \to X$ is an imbedding of A into X for $n = 1, 2, \ldots$, such that $\lim_n D(f_n) = 0$ for $f_n = hfh_n^{-1}$.

If an object B is imbeddable in X and if a morphism $f\colon A \to B$ is realizable in $X \operatorname{rel} h$ for any imbedding h of B into X, then we simply say that the morphism f is *realizable* in X. (1)

The definition depends on the category $\mathfrak{C}$ under consideration and we will always make it clear if a statement concerns a particular $\mathfrak{C}$. Usually,

(1) In [8] such a morphism has been called *imbeddable* in X.