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Reçu par la Rédaction le 10. 1. 1968

Structure of self-dual torsion-free metric LCA groups*

by

M. Rajagopalan** (Madurai, India)
and **T. Soundararajan** (Urbana, Ill.)

Since Pontrjagin [3] and Van Kampen [5] introduced the notion of the dual of a locally compact Abelian group, many examples of self-dual LCA groups have been given in the literature. However, the structure of all self-dual LCA groups has been an open problem till to-day (see [1]). As a matter of fact, there is even no conjecture about how a self-dual LCA group should look like. In this paper we give the structure of all metric self-dual LCA groups which are torsion-free as abstract groups.

Notations and Conventions. All topological spaces occuring in this paper are taken to be Hausdorff ones. We usually follow [7] for notations and concepts related to topological groups which are not defined here. We write LCA group as an abbreviation for a locally compact Abelian group. The dual of the LCA group G with the usual topology is denoted by $\hat{G}$. We use the additive notation for groups. If $H \subset G$ is a subgroup of the LCA group G, then $H^{\perp}$ denotes the annihilator of H in $\hat{G}$. R^n denotes the usual Euclidean group ($n \geqslant 0$). If p is a prime, then J_p denotes the group of all p-adic numbers and I_p the group of all p-adic integers with the usual topology. (We use the symbol $\oplus$ for topological direct sums). The definition of a local direct sum of LCA groups is given in [1], [6] and [4]. But we prefer to repeat this definition here for the sake of completeness.

Definition 1. Let (G_α) be a family of LCA groups indexed by a set A. Let $H_\alpha \subset G_\alpha$ be a compact and open subgroup of G_α for each $\alpha \in A$. We define the local direct sum $\sum_{\alpha \in A} G_\alpha$ of the family (G_α) with respect to (H_α) of subgroups as follows:

$$\sum_{\alpha\in A} G_\alpha = \Big\{(x_\alpha) \Big| \; (x_\alpha) \in \prod_{\alpha \in A} G_\alpha;\; x_\alpha \in H_\alpha \text{ for all } \alpha \in A \text{ except possibly for a finite number of indices}\Big\}.$$

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* An announcement of the result presented here appeared in [4]. The main theorem there should have been only for the metric case instead of for all the groups.

** This author was partially supported by a NSF grant No. NSFGP5370 while this work was being done.

The group addition in $\sum G_a$ is taken to be coordinatewise addition. We note that $\prod H_a \subset G$. By declaring $\prod H_a$ with its product topology to be open in $\sum_{a\in A} G_a$ we get a natural topology τ for $\sum_{a\in A} G_a$, so that $\big(\sum_{a\in A} G_a, \tau\big)$ is an LCA group.

Definition 2. An element x_0 of an LCA group G is said to be *compact* if the closed subgroup generated by x_0 in G is compact.

Definition 3. An element x_0 of an LCA group G is called *topologically p-primary* if $p^n x_0 \to 0$ as $n \to \infty$.

Definition 4. A map $T\colon G_1 \to G_2$ between two LCA groups G_1 and G_2 is called a *topological isomorphism* if it is a group isomorphism and a homeomorphism. In this case we write $G_1 \simeq G_2$. An LCA group G is said to be self-dual if there exists a topological isomorphism between G and $\hat{G}$.

Remark 5. Let $\mathfrak{I}$ be a collection of primes. For each $p \in \mathfrak{I}$ let K_p be an index set. For each $p \in \mathfrak{I}$ and $i \in K_p$ let J_p^i denote an LCA group which is topologically isomorphic to J_p. Let H_p^i be a compact and open subgroup of J_p^i for all $p \in \mathfrak{I}$ and $i \in K_p$. Then we can form the LCA group $S_p = \sum_{i\in K_p} J_p^i$ with respect to (H_p^i). Now each S_p contains $A_p = \prod_{i\in K_p} H_p^i$ as a compact and open subgroup. So we can again form the local direct sum $\sum_{p\in\mathfrak{I}} S_p$ with respect to (A_p). We call this group $\sum_{p\in\mathfrak{I}} S_p$ as $\sum_{p\in\mathfrak{I}}\big(\sum_{i\in K_p} J_p^i\big)$. We call such groups the local direct sum of J_p's. By 25.34 (b) of [1] on page 422 we see that $\sum_{p\in\mathfrak{I}}\big(\sum_{i\in K_p} J_p^i\big)$ is always a self-dual LCA group. Moreover such groups are torsion-free.

Lemma 6. *Let G be an LCA group of the form $R^n \oplus D \oplus \hat{D} \oplus \sum_{p\in\mathfrak{I}}\big(\sum_{i\in K_p} J_p^i\big)$ where $\mathfrak{I}$, J_p^i and K_p are as in Remark 5 and D is a torsion-free divisible group with the discrete topology. Then G is self-dual and torsion-free. If D and K_p are countable for all p, then G is metrisable.*

Proof. This follows from Remark 5 and the standard theorems on duality.

The rest of the paper is devoted to proving the converse of Lemma 6 in order to get the structure of self-dual torsion-free metric LCA groups.

Lemma 7. *Let G be an LCA group which contains a compact, open subgroup H. Let G be torsion-free and let G/H be isomorphic to the group $C(p^\infty)$ for some prime p. Then there exists a closed subgroup $A \subset G$ such that $A+H = G$ and $A \simeq J_p$.*

Proof: Consider $H^\perp \subset \hat{G}$. Then $H^\perp$ is the dual of G/H and hence is topologically isomorphic to I_p (see [1]). Let x_0 be a monothetic generator of $H^\perp$ (see [1] for the definition of a monothetic generator). We claim first of all that $\hat{G}$ is a divisible group. Indeed, suppose that $q\hat{G} \neq \hat{G}$ for some prime q. Now $q\hat{G} \supset qH^\perp$ and so it is an open subgroup of $\hat{G}$. Thus the annihilator of qG in G (being the dual of $\hat{G}/q\hat{G}$) contains a non-zero element of finite order. This is a contradiction. Thus $\hat{G}$ is divisible, and hence we have a sequence $x_0, x_0/p, x_0/p^2, \ldots$ in $\hat{G}$. Then the group J generated by $H^\perp$ and $x_0/p, x_0/p^2 \ldots$ is topologically isomorphic to the p-adic number group. J is now open and divisible and hence is a topological direct summand. Consequently G contains a p-adic number group A. This is obviously not contained in H and its image in G/H, being divisible, is $C(p^\infty)$. Thus $A+H = G$.

Lemma 8. *Let G be a torsion-free LCA group and let p be a fixed prime. Let G contain an open subgroup J which is a local direct sum $\sum_{i\in S} J_p^i$ of a family (J_p^i) of LCA groups indexed by a set S. For each $i \in S$ let $J_p^i \simeq J_p$. Suppose further that G contains a family (D_a) of closed subgroups D_a indexed by a set K such that the following hold:*

(1) $|K| \leqslant |S|$.

(2) $D_a \simeq J_p$ *for every* $a \in K$.

(3) *The algebraic subgroup generated by the set* $J \cup \big(\bigcup_{a\in K} D_a\big)$ *is* G.

Then G is $\simeq$ to a local direct sum of p-adic numbers.

Proof: Let $\sum_{i\in S} J_p^i$ be the local direct sum with respect to the compact open subgroups H_p^i of J_p^i and let $H = \prod_{i\in S} H_p^i$. Then H is compact and open in J and hence in G. Thus, for any D_a, $H \cap D_a$ is compact and open in D_a, so that for any $y_a \in D_a$ there is an integer n such that $ny_a \in H$. Thus for any $y \neq 0$ of G there is an m such that $my \in J$ and $my \neq 0$. Now $\prod_{i\in S} J_p^i$ is a divisible torsion-free group containing J and so there is a homomorphism of G into $\prod_{i\in S} J_p^i$ which is an identity on J. This homomorphism can easily be verified to be an isomorphism and also unique. Thus we can consider G to be a subgroup of $\prod_{i\in S} J_p^i$ (as an abstract group only).

For each J_p^i we shall suppose that J_p^i is generated topologically by $a^i, a^i/p, a^i/p^2 \ldots$ with a^i a monothetic generator of H_p^i.

We can then define a continuous character χ^i in $\prod J_p^i$ with its product topology as follows: χ^i on $\prod_{j\neq i} J_p^j = 0$ and $\chi^i(a^i) = 0$ and $\chi^i(a^i/p^n) = 1/p^n$, $n = 1, 2, 3 \ldots$ We can define $(1/p^m)\chi^i$ for each m by $(1/p^m)\chi^i(a^i/p^n) = a^i/p^{m+n}$. This collection $(1/p^m)\chi^i$, $m = 0, 1, 2 \ldots$ distinguishes any two points of J_p^i. Each $(1/p^m)\chi^i$ is a continuous character of G when restricted to G. We denote this restriction by $\big((1/p^m)\chi^i\big)G$. Let us consider the group generated in $\hat{G}$ by all $\big((1/p^m)\chi^i\big)G$ for all $i \in S$ and all non-negative integers m. This group distinguishes any two points of G. If K_i is the closed group generated by the set of all $\big\{\big((1/p^m)\chi^j\big)G \,\big|\, j \neq i,\ m \text{ non-negative integers}\big\}$, then the closed subgroup L_i generated by $\{((1/p^n)\chi^i)G \mid n \text{ varying}\}$ and K_i are topologically independent. Let M_i be the closed subgroup generated by χ^i in L_i. Then M_i is compact and open in L_i and $H^{\perp} = \prod M_i$. Also each L_i is $\simeq$ to J_p. It follows that $\sum L_i$ with respect to the groups M_i is a closed subgroup of $\hat{G}$. Since it is dense in $\hat{G}$, we get $\hat{G}$ is a local direct sum of p-adic numbers. Consequently G is a local direct sum of p-adic numbers.

LEMMA 9. *Let G be a torsion-free* LCA *metric group with a compact and open subgroup $H = \prod^{\infty} E_n$ such that each $E_n \simeq I_p$. Let $\{D_n\}$ be a sequence of closed subgroups in G, each isomorphic to J_p, such that the family $F_n = D_n \cap H$ generate a dense subgroups of H. Then we can find closed subgroups H_n such that*

(1) *each $H_n \simeq I_p$;*

(2) *H_n are topologically independent,*

(3) $H = \prod H_n$,

(4) *each H_n is contained in a group generated by a finite number of the F_n.*

Proof. Let us consider F_1. Since $F_1 = D_1 \cap H$, F_1 is a pure subgroup of H, and so H/F_1 is torsion-free. Thus the annihilator of F_1 in $\hat{H}$ is a divisible subgroup and so splits $\hat{H}$. So F_1 is a topological direct summand of H. We can assume that $H = \prod E_n$ with $E_1 = F_1 = H_1$. Thus H_1 has a generator $h_1 = (e_1, 0, 0, \ldots)$, where e_1 is a generator of E_1. The open set $(pE_1, E_2 \sim pE_2, E_3, E_4, E_5, \ldots)$ in H has to intersect F, the algebraic group generated by $F_1, F_2, \ldots$ Let $(pa_{21}, e_2, a_{23}, \ldots)$ be an element in the intersection. Since $H_1 = F_1 \subset H$, we can take this element to be $(0, e_2, a_{23}, \ldots)$. Let us write $h_2 = (0, e_2, a_{23}, \ldots)$. Here $(0, e_2, 0, 0, \ldots)$ is a generator for E_2. Let H_2 be the closed subgroup generated by h_2. Since $h_2 \in \{F_{i_1}, \ldots, F_{i_m}\}$, H_2 satisfies condition (4) of the Lemma.

Again the open set $(pE_1, pE_2, E_3 \sim pE_3, E_4, E_5, \ldots)$ intersects F, and so we get an element $(pa_{31}, pa_{32}, e_3', a_{34}, \ldots)$ We can take a_{31} to be 0, so that we have $(0, pa_{32}, e_3', a_{34}, \ldots)$. Now H_2 is a compact subgroup of H and is $\simeq I_p$. The projection of pH_2 on E_2 is a compact (closed) subgroup of E_2 containing pe_2 and hence contains pE_2. Thus there is an element in pH_2 whose projection on E_2 is $(0, pa_{32}, 0, 0, \ldots)$. This element has the form $(0, pa_{32}, pb_{33}, pb_{34}, \ldots)$ since it belongs to pH_2. Subtracting this element from $(0, pa_{32}, e_3', a_{34}, \ldots)$, we get an element $h_3 = (0, 0, e_3, c_{34}, \ldots)$ where $e_3 = e_3' - pb_{33}$, $c_{34} = a_{34} - pb_{34}, \ldots$ Now $e_3 \in E_3 \sim pE_3$, and hence $(0, 0, e_3, 0, 0, \ldots)$ is a monothetic generator for E_3. Let H_3 be the closed subgroup generated by h_3. Since $h_3 \in F$, H_3 satisfies conditions (1) and (4).

Proceeding thus by induction, we get a sequence of elements

$$h_1 = (e_1, 0, 0, \ldots),$$
$$h_2 = (0, e_2, a_{23}, \ldots), \text{ (Let us write } c_{23} = a_{23} \text{ for uniform notation),}$$
$$h_3 = (0, 0, e_3, c_{34}, \ldots),$$
$$\vdots$$
$$h_n = (0, 0, \ldots, 0, e_n, c_{n,n+1}, \ldots),$$
$$\vdots$$

and a sequence of groups $H_1, H_2, \ldots$ such that each $H_n \simeq I_p$ and each H_n satisfies condition (4) of the Lemma.

We assert the following:

(a) *The groups $H_1, H_2, \ldots$ are topologically independent.*

Suppose that H_i is not independent of the remaining, i.e. $H_i \cap \mathrm{Cl}\{H_1, \ldots, H_{i-1}, H_{i+1}, \ldots\} \neq 0$, i.e. their intersection is some $p^k H_i$. In particular, the element $x = p^k h_i$ is a limit from the other groups. Let (x_n) be a sequence from $\{H_1, \ldots, H_{i-1}, H_{i+1}, \ldots\}$ converging to $p^k h_i$. Then $x_n = y_n + z_n$, $y_n \in \{H_1, \ldots, H_{i-1}\}$ and $z_n \in \{H_{i+1}, \ldots\}$. Since $\{H_1, \ldots, H_{i-1}\}$ is a compact subgroup, we can have a subsequence (y_{n_m}) converging to some $y \in \{H_1, \ldots, H_{i-1}\}$. So the sequence (z_{n_m}) converges to $x - y \in \{H_1, \ldots, H_i\}$. Now every element in (z_{n_m}) has the first i coordinates equal to zero. Hence in $x - y$ the first i coordinates are zero. But $x = (0, 0, \ldots, p^k e_i, c_{i,i+1}, \ldots)$. So $y = (0, 0, \ldots, p^k e_i, d_{i\,i+1}, \ldots)$. Now $(y_{n_m}) \to y$.

Consider the open set

$$(p^{k+1}E_1, p^{k+1}E_2, \ldots, p^{k+1}E_{i-1}, E_i \sim p^{k+1}E_i, E_{i+1}, \ldots).$$

This must ultimately contain all the (y_{n_m}). So this open set intersects $\{H_1, \ldots, H_{i-1}\}$ for which the algebraic group generated by $h_1, \ldots, h_{i-1}$ is dense. So this open set contains an element $n_1 h_1 + n_2 h_2 + \ldots + n_{i-1} h_{i-1}$. Hence

$$\begin{aligned} n_1 h_1 + n_2 h_2 + \ldots + n_{i-1} h_{i-1} \\ &= \big(n_1 e_1, n_2 e_2, (n_3 e_3 + n_2 a_{23}), \ldots, n_{i-1} a_{i-1} + n_{i-2} c_{i-2,i-1} + \ldots, \ldots\big) \\ &\in (p^{k+1}E_1, p^{k+1}E_2, \ldots, p^{k+1}E_{i-1}, E_i \sim p^{k+1}E_i, \ldots). \end{aligned}$$

We conclude that each n_i is a multiple of p^{k+1}, so that the ith term in $n_1 h_1 + n_2 h_2 + \ldots + n_{i-1} h_{i-1}$ is also a multiple of p^{k+1} and thus does not belong to $E_i \sim p^{k+1} E_i$. This is a contradiction.

(b) *The algebraic group $\{H_1, \ldots, H_n, \ldots\}$ generated by $H_1, \ldots$ is dense in H.*

We show that every element $(0, 0, \ldots, e_n, 0, 0, \ldots)$ is a limit point of this group. Consider an open set $(p^{k_1}E_1, p^{k_2}E_2, \ldots, e_n + p^{k_n}E_n, p^{k_{n+1}}E_{n+1}, \ldots, p^{k_i}E_i, E_{i+1}, \ldots)$ containing $(0, 0, \ldots, e_n, 0, 0, 0, \ldots)$. It is enough to show that this intersects $\{H_1, \ldots\}$.

Start with the element $h_n = (0, 0, \ldots, e_n, c_{n,n+1}, \ldots)$. Then consider $h_{n+1} = (0, 0, \ldots, e_{n+1}, \ldots)$. By suitably choosing m_1 we get $c_{n,n+1} - m_1 e_{n+1} \in p^{k_{n+1}} E_{n+1}$. Then $h_n - m_1 h_{n+1} = (0, 0, \ldots, e_n, (c_{n,n+1} - m_1 e_{n+1}), \ldots)$. Now we can start with this element, consider h_{n+2} and repeat the process. Proceeding thus, we prove our assertion.

Now $H_1, \ldots, H_n, \ldots$ is an independent family and $\{H_1, \ldots, H_n, \ldots\}$ is a dense subgroup of H. It follows that $H = \prod H_n$.

This completes the Lemma.

Lemma 10. *Let G be a torsion-free self-dual* LCA *group. Then $G = R^n \oplus \oplus H \oplus D \oplus G_1$ where R^n $(n \geqslant 0)$ is the Euclidean vector group with usual topology, D is a divisible torsion-free Abelian group with discrete topology, H is the dual of D, and G_1 is a self-dual* LCA *group which is totally disconnected and torsion-free.*

Proof. Now let $\hat{G}$ be the dual of G. Let $\sigma\colon G \to \hat{G}$ be a topological isomorphism. Let G_0 be the connected component of identity of G. Then $G = G_0 \oplus S$ where S is a totally disconnected group (see 25.30(c), p. 418 of [1]). Now $G_0 = E \oplus H$ where E is a closed subgroup of G isomorphic to some R^n $(n \geqslant 0)$ and H is a compact and connected group. Then $\hat{G} = \hat{E} \oplus \hat{H} \oplus \hat{S}$. Now let $\hat{G}_0$ be the connected component of identity of $\hat{G}$. Then $\hat{G}_0 = \hat{E} \oplus \hat{F}$ where $\hat{F}$ is a compact connected group. By considering the projection of $\hat{F}$ on $\hat{E} \oplus \hat{H}$ we get $\hat{F} \subset \hat{S}$. Since $\hat{S} \subset \hat{G}$ and since $\hat{G}$ is torsion-free, we find that $\hat{S}$ is torsion-free. Therefore $\hat{S} = \hat{F} \oplus \hat{G}_1$. So $S = F \oplus G_1$ where G_1 is the dual of $\hat{G}_1$ and F is the dual of $\hat{F}$. Therefore $G = E \oplus H \oplus F \oplus G_1$. Now $G_1 \subset S$ and hence is totally disconnected. $\hat{G}_1$ is clearly totally disconnected. Hence all elements of G_1 are compact. From this it follows that $H \oplus G_1$ is exactly the set of all compact elements of G. In the same way $\hat{F} \oplus \hat{G}_1$ is the set of all compact elements of $\hat{G}$. So the isomorphism σ maps $H \oplus G_1$ onto $\hat{F} \oplus \hat{G}_1$. Since σ is also a homeomorphism, we must have $\sigma(H) = \hat{F}$. (Note that H is the connected component of 0 in $H \oplus G_1$.) Hence σ induces a natural topological isomorphism between $H \oplus G_1/H$ and $\hat{F} \oplus \hat{G}_1/\hat{F}$. But $H \oplus G_1/H \simeq G_1$ and $\hat{F} \oplus \hat{G}_1/\hat{F} \simeq \hat{G}_1$. Hence it follows that G_1 is self-dual and already G_1 is totally disconnected. This establishes the Lemma.

Lemma 11. *Let G be a self-dual totally disconnected torsion-free* LCA *group. For each prime p let G_p be the closed subgroup of all topologically p-primary elements of G. Let H be a compact open subgroup of G and $H_p = G_p \cap H$ for any prime p. Then G is the local direct sum $\sum_{p \in \mathfrak{I}} G_p$ with respect to the compact open subgroups H_p; where $\mathfrak{I}$ is the set of primes for which $G_p \neq 0$. Moreover, each G_p is self-dual.*

Proof. We observe that every element of G is compact. Now, adopting the proof of Theorem 1 on p. 86 in [6], we get $G = \sum G_p$. Therefore $\hat{G} = \sum \hat{G}_p$. By adopting the proof of Theorem 17 on p. 145 in [6] we get $\hat{G}_p = (\hat{G})_p$. If σ is a topological isomorphism from G to $\hat{G}$, then it is clear that σ maps G_p onto $(\hat{G})_p$. Hence G_p is self-dual.

Lemma 12. *Let G be a totally disconnected self-dual torsion-free metric* LCA *group. Then G is isomorphic to $\sum_{p \in \mathfrak{I}} \sum_{i \in K_p} J_p^i$ where $\mathfrak{I}$ is a collection of primes and K_p is a countable index set for each $p \in \mathfrak{I}$ and $J_p^i \simeq J_p$ for each p in $\mathfrak{I}$ and i in K_p.*

Proof: Now, by Lemma 11, $G = \sum_{p \in \mathfrak{I}} G_p$ where $\mathfrak{I}$ is a collection of primes, and G_p is topologically p-primary and self-dual for all $p \in \mathfrak{I}$. Since G_p is torsion-free, totally disconnected, and self-dual, we conclude that the compact (metric) open subgroup H_p is of the form $\prod E_\alpha$ where α varies over a countable index set K_p and each $E_\alpha \simeq I_p$. Moreover, G_p/H_p is a direct sum $\sum_{i \in S} L_i$ where each $L_i \simeq C(p^\infty)$ and $|S| = |K_p|$. We claim that G_p is a local direct sum of p-adic numbers: By Lemma 7, to each L_i we can find a closed subgroup $J_i \simeq J_p$ such that the image of J_i is L_i under the canonical map Φ from G_p to G_p/H_p. Let $F_i = H_p \cap J_i$. Clearly the family (J_i) is independent (algebraically only). If J is the subgroup generated by all the J_i, then $\bar{J} = G_p$ itself. Otherwise $G_p/\bar{J}$ is compact (since it is the image of H_p), which implies that $\hat{G}_p$ has elements of finite order. Since H_p is compact and open and J is dense, we have $H_p = \overline{H_p \cap J}$. If $|S| < \infty$, it is easy to see that G_p is a topological direct sum of a finite number of p-adic number groups. So we can assume $|S| = \aleph_0$, and then the conditions of Lemma 9 are satisfied by G_p. Hence $H = \prod H_n$, and we find from the proof of Lemma 9 that each of these H_n can be extended to a p-adic number group J_n' in G_p and that G_p contains the local direct sum $\sum J_n'$ with respect to H_n. This local direct sum $\sum J_n'$ is an open subgroup of G_p since it contains H_p.

Now G_p/H_p is a direct sum $\sum \Phi(J_n') + \sum_{\beta \in M} D_\beta'$ where $|M| \leqslant |K_p|$ and where each $D_\beta' \simeq C(p^\infty)$. To each D_β' we can find by Lemma 7 a closed subgroup $D_\beta \simeq J_p$ such that the image of D_β is D_β'. Now Lemma 8 completes the proof considering $\sum J_n'$ and the D_β.

Theorem. *Let G be a torsion-free metric* LCA *group. Then G is self-dual if and only if G is of the form $R^n \oplus D \oplus \hat{D} \oplus \sum_{p \in \mathfrak{I}} \big(\sum_{i \in K_p} J_p^i\big)$, where $\mathfrak{I}$ is a subset of primes and for each $p \in \mathfrak{I}$, K_p is a certain countable index set and $J_p^i \simeq J_p$ for all $i \in K_p$ and $p \in \mathfrak{I}$ and D is a torsion-free divisible* countable *discrete group.*

Proof. The sufficiency has been established in Lemma 6. The necessity follows from Lemmas 10 and 12.

Addendum (August 14, 1969)

THEOREM. *Let G be a torsion-free* LCA *group satisfying any one of the following conditions:*

(a) *G is separable.*

(b) *G satisfies countable chain condition, i.e. any family of disjoint open sets in G is countable.*

(c) *G is σ-compact.*

(d) *G is Lindelöf.*

(e) *Any uncountable family of open sets has an uncountable subfamily with non empty intersection.*

Then G is self-dual if and only if G is of the form mentioned in the preceding theorem.

Proof. If G is of the form mentioned in the preceding theorem then it is self-dual by Lemma 6. Suppose now G is self-dual. Then $G = R^n \oplus A$ where A has a compact open subgroup H. $\hat{G} \simeq \hat{R}^n \oplus \hat{A}$ where $\hat{A}$ has a compact open subgroup $H^{\perp}$ which is the dual of the discrete A/H. Let G satisfy anyone of the conditions (a)–(e). Observe that (a) $\Rightarrow$ (b) and (c) $\Rightarrow$ (d). We assert that A/H is countable. If not, $A = \bigcup_{a \in I} Hx_a$, a set union of disjoint cosets and I is an uncountable set. Each of these cosets is open in A. If we now consider $\{R^n + Hx_a\}_{a \in I}$ we easily arrive at a contradiction. So A/H is countable. Now $\hat{G} \simeq \hat{R}^n \oplus \hat{A}$ where $\hat{A}$ has a compact open subgroup $H^{\perp}$ which is now the dual of the countable discrete group A/H. Hence $H^{\perp}$ is metrizable. Since G is isomorphic to $\hat{G}$, by a similar reasoning we get $\hat{A}/H^{\perp}$ is countable. Hence H, the dual of $\hat{A}/H^{\perp}$ is metrizable. Since H is metrizable and A/H is countable discrete and hence metrizable we get A is metrizable. Already R^n is metric. Hence G is metrizable. Then the preceding theorem completes the proof.

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UNIVERSITY OF ILLINOIS, Urbana, Illinois
MADURAI UNIVERSITY, Madurai, India

Reçu par la Rédaction le 23. 1. 1968

Results on ω_μ-metric spaces

by

F. W. Stevenson (Oberlin, Ohio) and **W. J. Thron** (Boulder, Colorado)

§ 1. Introduction and preliminary results. A linearly ordered abelian group is a set A, together with a binary operation $\cdot$, and an order relation $>$, such that $(A, \cdot)$ is an abelian group $(A, >)$ is a linearly ordered set and the following condition is satisfied: if $a > b$ then $ac > bc$. The group A has character ω_μ iff there exists a decreasing ω_μ-sequence converging to 0 in the order topology on A. Here ω_μ denotes the μth infinite cardinal number. Cardinal numbers are considered as initial ordinal numbers and each ordinal coincides with the set of all smaller ordinals. The power of ω_μ is denoted by $\aleph_\mu$. We will be concerned with only that ω_μ which represents the least character of A and it is easily shown that such an ω_μ must be a regular cardinal number.

Let X be a set and ϱ a function from $X \times X$ to $(A, \cdot, >)$ such that

(i) $\varrho(x, y) = 0$ iff $x = y$,

(ii) $\varrho(x, y) = \varrho(y, x) > 0$ if $x \neq y$,

(iii) $\varrho(x, y) \leqslant \varrho(x, z) + \varrho(z, y)$,

then ϱ is called an ω_μ-metric and (X, ϱ) is an ω_μ-metric space. Sikorski [10] has done the most extensive study of ω_μ-spaces; other references include Hausdorff [3], Cohen and Goffman [1] and [2], Parovicenko [6], and most recently, Shu-Tang [7].

The ω_μ-metric ϱ on X induces a topology $\mathcal{T}_\varrho$ on X; a base for the topology consisting of sets of the form $N_a(x)$ where $N_a(x) = \{y \in X : \varrho(x, y) < a\}$, $a \in A$, and $a > 0$. Also ϱ induces a uniformity $\mathcal{U}_\varrho$ on X: a base for the uniformity consisting of sets U_a where $U_a = \{(x, y) : \varrho(x, y) < a\}$, $a \in A$, $a > 0$. It is easily shown that the ω_μ-metric topology and the ω_μ-uniform topology are identical.

An ω_μ-additive space is a topological space $(X, \mathcal{T})$ which satisfies the condition that for any family of open sets $\mathcal{F}$, of power $< \aleph_\mu$ it follows that $\bigcap \mathcal{F}$ is an open set. Clearly every topological space is an ω_0-additive space. It is easily shown that if (X, ϱ) is an ω_μ-metric space then $(X, \mathcal{T}_\varrho)$ is an ω_μ-additive space. Sikorski defines the following concepts on an ω_μ-additive space $(X, \mathcal{T})$. The space $(X, \mathcal{T})$ has a basis iff it has a base