[10] E. Snapper, *Cohomology of permutation representations II. Cup product*, J. Math. Mech., 13 (1964), pp. 1047-1064.

[11] — *Duality in the cohomology ring of transitive permutation representations*, J. Math. Mech., 14 (1965), pp. 323-336.

[12] — *Inflation and deflation for all dimensions*, Pacific J. Math. 15 (1965), pp. 1061-1081.

[13] — *Spectral sequences and Frobenius groups*, Trans. Amer. Math. Soc. 114 (1965), pp. 133-146.

THE UNIVERSITY OF ILLINOIS
at Chicago Circle

Reçu par la Rédaction le 6. 1. 1968

Some theorems on the embeddability of ANR-spaces into Euclidean spaces

by

H. Patkowska (Warszawa)

1. Introduction. In 1930 C. Kuratowski (see [7]) has characterized the local dendrites (i.e. connected, 1-dimensional ANR-sets) which are embeddable into the plane E^2 as those which do not contain homeomorphic images of the two graphs K_1 and K_2. K_1 and K_2 will be called the *graphs of Kuratowski*. K_1 is the 1-skelton of a 3-simplex in which the midpoints of a pair of non-adjacent edges are joined by a segment; K_2 is the 1-skelton of a 4-simplex. C. Kuratowski has also described two locally connected curves K_3 and K_4 which are not ANR-sets and he has conjectured the characterization of locally connected continua which are embeddable into the sphere S^2 as those which do not contain homeomorphic images of the four curves K_i, $i = 1, 2, 3, 4$. This was proved in 1937 by S. Claytor (see [3] and [4]). As a corollary, Claytor obtained the following result ([3], p. 632), which will be useful for us: Each cyclic locally connected continuum which does not contain homeomorphic images of the graphs of Kuratowski is embeddable into S^2. Recall that a connected space is cyclic (in the sense of Whyburn) if it is separated by no point.

In 1966 S. Mardešić and J. Segal (see [9] and [10]) showed that the connected polyhedra which are embeddable into S^2 can be characterized as those which do not contain homeomorphic images of three polyhedra, namely K_1, K_2 and $\perp$, where $\perp$ is the one-point union of a 2-simplex and of a segment relative to an interior point of the 2-simplex and an end-point of the segment. They raised the question if this characterization can be extended to the connected ANR-sets (they are always assumed to be compact) which are embeddable into S^2. We shall show in this paper that this is in fact true. Namely, we shall derive this from Claytor's result mentioned above and from the positive answer to the following question for $n = 2$: If X is a connected ANR containing no n-umbrella and if the cyclic elements of X are embeddable into E^n, is X also embeddable into E^n? By an *n-umbrella* we shall mean here a one-point union of a (topological) n-ball and of an arc relative to an interior point of the n-ball and an end-point of the arc. For the definition of cyclic elements see section 3.

I proved in an earlier paper [11] that the answer to this question is negative for $n=3$. In this paper I shall prove that positive answer in a modified form is possible in the general case and it is positive for $n=1,2$.

Given two spaces X, Y, call Y a Cartesian divisor of X if there exists a topological space Z such that the product $Y\times Z$ is homeomorphic with X. We shall prove the following

THEOREM 1. *If X is a connected* ANR *containing no n-umbrella and if the cyclic elements of X are embeddable into E^n, then X is embeddable into an n-dimensional Cartesian divisor of E^{n+1}.*

The arrangement of this paper is as follows: In section 2 we shall derive some corollaries from theorem 1 in the case $n=2$, in particular we shall give the solution of the problem of Mardešić and Segal, mentioned above. In section 3 we shall list the properties of cyclic elements and, more generally, of the sets entirely arcwise connected, which will be needed in the sequel. In section 4 we recall some properties of ANR-sets contained in Euclidean spaces and give some useful definitions. The last four sections are devoted to the proof of theorem 1. In section 5 we shall reduce this theorem to a lemma, and we shall formulate three cases into which the proof of this lemma is divided (the last case is general). The proof of the lemma in these three cases is given successively in sections 6, 7 and 8.

2. A characterization of ANR-sets which are embeddable (and quasi-embeddable) into E^2 or S^2. As mentioned in the introduction, in this section we shall assume theorem 1 to be true (in the case of $n=2$) and give some corollaries.

Let X be a connected ANR containing no 2-umbrella and assume that the cyclic elements of X are embeddable into E^2. Then, by theorem 1, X is embeddable into a 2-dimensional Cartesian divisor of E^3. It has been proved by Borsuk (see [2], p. 286) that each 2-dimensional Cartesian divisor of E^n is homeomorphic with E^2. Therefore X is embeddable into E^2 and we obtain the following

THEOREM 2. *If X is a connected* ANR *containing no 2-umbrella and is the cyclic elements of X are embeddable into E^2, then X is also embeddable into E^2.*

Notice that the theorem is false for the locally connected continua which are not ANR-sets; the counter examples are the curves K_3 and K_4. On the other hand, this theorem is true for $n=1$ (cf. the beginning of section 5).

Now, let X be a connected ANR which does not contain any 2-umbrellas and any homeomorphic images of the graphs K_1 and K_2. Then each cyclic element of X is a cyclic locally connected continuum (see section 3) which also does not contain these sets. Thus, it follows from Claytor's result mentioned in the introduction ([3], p. 632) that each cyclic element of X is embeddable into S^2. If there is a cyclic element E of X homeomorphic with S^2, then $E=X$; otherwise X would contain a 2-umbrella (since X is arcwise connected). Hence, in this case, $X \underset{\text{top}}{=} S^2$. On the other hand, if there is no such cyclic element, then each cyclic element of X is embeddable into E^2 and — by theorem 2 — X is also embeddable into E^2. Thus, in either case X is embeddable into S^2, and we obtain the following

THEOREM 3. *If X is a connected* ANR *which does not contain any 2-umbrellas and any homeomorphic images of the graphs K_1 and K_2, then X is embeddable into S^2.*

Exactly as in [9], p. 636, we can obtain from this theorem the following two corollaries:

COROLLARY 1. *If X is an* ANR *which does not contain any 2-umbrellas and any homeomorphic images of K_1 and K_2 and if no component of X different from X is homeomorphic with S^2, then X is embeddable into S^2.*

COROLLARY 2. *If X is an* ANR *which does not contain any 2-umbrellas and any homeomorphic images of K_1, K_2 and S^2, then X is embeddable into E^2.*

Let us recall that X is quasi-embeddable into Y if for every $\varepsilon>0$ there is a map $f\colon X\to Y$ such that $\operatorname{diam}(f^{-1}(y))<\varepsilon$ for every $y\in f(X)$. Using the same argument as in [9], namely that none of the sets K_1, K_2, $\perp$ and S^2 is quasi-embeddable into E^2, we obtain from theorem 3 also the following

COROLLARY 3. *An* ANR*-set is embeddable into $E^2(S^2)$ if and only if it is quasi-embeddable into $E^2(S^2)$.*

3. The cyclic elements and the sets entirely arcwise connected. The concepts of a cyclic element and of a set entirely arcwise connected are basic for the proof of theorem 1. We shall recall their definitions and the most important properties. Our general reference will be [8], § 47, and in the sequel of this section we shall give only the page and the number of the proposition in question.

In this section X will denote a fixed locally connected continuum.

A set $A\subset X$ is said to be *entirely arcwise connected* (in X) if $x,y\in A$ and $x\neq y$ imply that each arc (in X) joining x and y is contained in A.

A set $E\subset X$ is said to be a *cyclic element* of X in either of the following three cases:

1° E consists of one point which separates X.

2° E consists of one point $x \in X$ such that $\operatorname{ord}_x X = 1$, where $\operatorname{ord}_x X$ denotes the order of x in X in the sense of Menger–Urysohn.

3° E is a connected subset of X containing more than one point and maximal with respect to the property of being a cyclic space, i.e. of containing no point which separates it.

This definition of cyclic elements slightly differs from that given in [8], but it is easily seen from [8] to be equivalent to it.

In the next five propositions A will denote any closed and entirely arcwise connected subset of X. Thus we have:

(3.1) *A is a locally connected continuum* (p. 231, No. 2).

(3.2) *If C is a component of $X-A$, then $A \cap \bar{C}$ consists of only one point* (p. 232, No. 4).

(3.3) *The set of the components of $X-A$ is at most countable, and if it is infinite, then the diameters of those components converge to zero* (p. 232, No. 7).

(3.4) *A is a retract of X and, consequently, if $X \in$ ANR (AR), then also $A \in$ ANR (AR)* (p. 263, No. 15).

(3.5) *If B is another closed and entirely arcwise connected subset of X and $A \cap B \neq \emptyset$, then the set $A \cup B$ is also entirely arcwise connected* (p. 232, No. 8).

For cyclic elements we have (the cyclic elements of the form 3° will be said to be non-degenerate):

(3.6) *Each cyclic element of X is a closed and entirely arcwise connected subset of X* (p. 236, No. 6).

(3.7) *X is the union of the cyclic elements of it* (p. 235, No. 1).

(3.8) *The set of the non-degenerate cyclic elements of X is at most countable, and if it is infinite, then their diameters converge to zero* (p. 238, No 9).

(3.9) *Each connected subset of X separated by no point is contained in a cyclic element* (p. 238, No. 10).

(3.10) *A non-degenerate continuum $A \subset X$ is entirely arcwise connected if and only if it is a union of cyclic elements* (p. 239, No. 11).

The following theorem is due to Borsuk [1]:

(3.11) *X is an AR-set if and only if all cyclic elements of X are AR-sets.*

The next proposition follows easily from the preceding ones and from some elementary properties of ANR-sets, especially that each subset of an $X \in$ ANR with a sufficiently small diameter is contractible in X:

(3.12) *X is an ANR-set if and only if all cyclic elements of X are ANR-sets and almost all are AR-sets.*

(The sufficiency of the condition can be proved by embedding each cyclic element of X which is not an AR-set in a set homeomorphic with the Hilbert cube and thus embedding X in an AR-set so that X is a neighbourhood retract of it).

Now, we shall prove two properties of X, related to the subject-matter of this section, which will be useful in the sequel:

(3.13) *Given two different points $a, b \in X$, the least closed and entirely arcwise connected subset of X containing a and b is the union of an arc L joining a and b and of all the cyclic elements of X which have at least two points in common with L. Moreover, if $E_1, E_2, \ldots$ is a sequence of the cyclic elements having this property, then $E_i \cap L$ is a non-degenerate subarc L_i of L and $E_i \cap E_j = L_i \cap L_j = \dot{L}_i \cap \dot{L}_j$ for $i \neq j$. ($\dot{L}$ denotes the boundary of the arc L).*

Proof. Let Z denote the least closed and entirely arcwise connected subset of X containing a and b. Of course, there is an arc $L \subset Z$ such that $\dot{L} = (a) \cup (b)$. If two different points $x, y \in L$ are contained in a cyclic element E, then the subarc of L joining those points is also contained in E, since — by (3.6) — E is entirely arcwise connected. Consequently, if a cyclic element E contains more than one point of L, then $E \cap L$ is a non-degenerate subarc of L. Arrange into a sequence (finite or not) $E_1, E_2, \ldots$ all cyclic elements of X with this property and let $E_i \cap L = L_i$ (cf. (3.8)). Denote by Z' the union of L and of all sets E_i. We shall prove that $Z = Z'$.

Given an index i, let us observe that there is a point $a_i \in L_i$ which belongs to no cyclic element of X different from E_i (see [8], p. 238, No. 8). Since $a_i \in L \subset Z$ and Z is a union of cyclic elements by (3.10), it follows that $E_i \subset Z$. Thus $Z' \subset Z$.

To show the inclusion $Z \subset Z'$ we have to prove that Z' is a closed and entirely arcwise connected subset of X containing a and b. Indeed, $a, b \in L \subset Z'$ and Z' is a continuum by (3.8). Considering (3.10), it remains to prove that Z' is the union of cyclic elements, i.e. that each point x of L which belongs to no E_i is a cyclic element of the form 1° or 2°. Indeed, otherwise — by (3.7) — x belongs to a cyclic element E of the form 3° such that $E \cap L = (x)$. Thus, there is a component C of $X-E$ containing a component of $L-x$. By (3.2), $\bar{C} \cap E = (x)$, and therefore C is a component of $X-x$. Since $C \neq X-x$, this contradicts the assumption that x is not a cyclic element of the form 1° or 2°, and thus the inclusion $Z \subset Z'$ is proved.

In order to finish the proof of (3.13) we have to prove that $E_i \cap E_j = L_i \cap L_j = \dot{L}_i \cap \dot{L}_j$ for $i \neq j$. Since different cyclic elements have at most one point in common (see [8], p. 236, No. 4), $L_i \cap L_j = \dot{L}_i \cap \dot{L}_j$. If $(p) = E_i \cap E_j$ and $p \in X-L$, then p does not separate X between the sets $E_i - p$ and $E_j - p$, which contradicts the remark given in [8] (p. 238). Consequently, $E_i \cap E_j = L_i \cap L_j$.

(3.14) *Let $X = \overline{\bigcup_{i=1}^{\infty} A_i}$, where $A_i \subset A_{i+1}$ and $A_i = \bar{A}_i \neq X$ is a set entirely arcwise connected. If the maximum of the diameters of the components of $X - A_i$ is equal to δ_i, then $\lim_{i=\infty} \delta_i = 0$.*

Proof. Since the sequence A_i is increasing, it suffices to prove that for every $\varepsilon > 0$ there is an index $i(\varepsilon)$ such that the diameter of each component of $X - A_{i(\varepsilon)}$ is less than or equal to ε. Suppose that such an index $i(\varepsilon)$ does not exist. Therefore, for each i there is a component C_i of $X - A_i$ such that $\operatorname{diam}(C_i) > \varepsilon$. We shall define by induction a sequence of indices $i_1, i_2, \ldots$ and two sequences of points $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ such that for $k > 1$ $a_k, b_k \in A_{i_k} \cap C_{i_{k-1}}$ and $\varrho(a_k, b_k) > \varepsilon$.

Let $i_1 = 1$ and let a_1, b_1 be arbitrary points of X. Now, given an index l, suppose that the indices i_k and the points a_k, b_k for $k < l$ have been defined. Since the set $C_{i_{l-1}}$ is open and the set $\bigcup_{i=1}^{\infty} A_i$ is dense in X, it follows that the set $\bigcup_{i=1}^{\infty} A_i \cap C_{i_{l-1}}$ is dense in $C_{i_{l-1}}$, whence $\operatorname{diam}(\bigcup_{i=1}^{\infty} A_i \cap C_{i_{l-1}}) > \varepsilon$. Since the sequence of sets $A_i \cap C_{i_{l-1}}$ is increasing, there exist an index $i_l > i_{l-1}$ and two points $a_l, b_l \in A_{i_l} \cap C_{i_{l-1}}$ such that $\varrho(a_l, b_l) > \varepsilon$. Thus the required sequences have been defined.

Of course, there is a sequence $\{k_l\}$ such that $\lim_{l=\infty} a_{k_l} = a$ and $\lim_{l=\infty} b_{k_l} = b$. Since the sequence A_{i_k} is increasing, we can assume that $\lim_{k=\infty} a_k = a$ and $\lim_{k=\infty} b_k = b$. The local arcwise connectedness of X implies that there are and index k' and two arcs I, J with diameters less than $\varepsilon/2$ such that $\dot{I} = (a_{k'}) \cup (a_{k'+1})$, $\dot{J} = (b_{k'}) \cup (b_{k'+1})$. Since $C_{i_{k'}}$, as an open and connected set, is arcwise connected and $a_{k'+1}, b_{k'+1} \in C_{i_{k'}}$, it follows that the set $I \cup C_{i_{k'}} \cup J$ contains an arc K joining the points $a_{k'}$ and $b_{k'}$. Taking into consideration that $\varrho(a_{k'}, b_{k'}) > \varepsilon$ and $\operatorname{diam}(I), \operatorname{diam}(J) < \varepsilon/2$, we infer that $K \cap C_{i_{k'}} \neq \emptyset$. However, since $a_{k'}, b_{k'} \in A_{i_{k'}}$ and $C_{i_{k'}}$ is a component of $X - A_{i_{k'}}$, we obtain a contradiction to the entire arcwise connectedness of $A_{i_{k'}}$, and thus (3.14) is proved.

4. Some properties of ANR-sets contained in Euclidean spaces. The following two well-known properties will be useful for us:

(4.1) *If $A \subset E^n$ and $A \in$ ANR, then the set $E^n - A$ has only a finite number of components, which is equal to 1 whenever $n > 1$ and $A \in$ AR* (see [6], p. 192).

(4.2) *If $A \subset E^n$ and $A \in$ ANR, then every boundary point of a component C of $E^n - A$ is accessible (by an arc) from C* (see [6], p. 195).

Now, we introduce some definitions which will be useful in section 7.

Let $A \subset E^n$ be a locally connected continuum. By a *star with the core* A we shall mean every locally connected continuum $H \subset E^n$ such that A is a closed and entirely arcwise connected subset of H.

Given a point $a \in E^n$ and a cardinal number α, where $1 \leqslant \alpha \leqslant \aleph_0$, by a *necklace with α beads and the initial point a* we shall mean any set $T \subset E^n$ which is the union of an arc L and of α geometrical n-balls $Q_1, Q_2, \ldots$ such that:

1° $a \in \dot{L} - \bigcup_i Q_i$.

2° $Q_i \cap Q_j = \emptyset$ for $i \neq j$.

3° $Q_i \cap L$ is a non-degenerate subarc L_i of L.

4° If $\alpha = \aleph_0$, then $\lim_{i=\infty} \operatorname{diam}(Q_i) = 0$ and there is a sequence of points $a_i \in Q_i$, $i = 1, 2, \ldots$ convergent to the point $b \in \dot{L} - a = (b)$.

Now, let $A \subset E^n$ be a fixed locally connected continuum and suppose that there is given a sequence (finite or not) of points $a_i \in \operatorname{Bd}(A)$, $i = 1, 2, \ldots$ and also suppose that for each i there is given a cardinal number α_i, where $1 \leqslant \alpha_i \leqslant \aleph_0$. By a *star with the core A determined by $\{a_i\}_{i=1}^m$ and $\{\alpha_i\}_{i=1}^m$* (m — natural or ∞) we shall mean each star H with the core A for which there is a one-to-one correspondence between the components of $H - A$ and the points a_i such that, if C_i corresponds to a_i, then $\bar{C}_i \cap A = (a_i)$ and $\bar{C}_i$ is a necklace with α_i beads and the initial point a_i.

Let us notice that:

(4.3) *If $A \in$ ANR (AR), then each star with the core A determined by sequences $\{a_i\}_{i=1}^m$ and $\{\alpha_i\}_{i=1}^m$ is also an ANR (AR).*

Indeed, it is evident from the definition that all cyclic elements of a necklace are AR-sets. Consequently, (4.3) follows from the definition of a star with the core A determined by $\{a_i\}_{i=1}^m$ and $\{\alpha_i\}_{i=1}^m$ and from (3.11) and (3.12).

(4.4) *Given $A \in$ ANR and two sequences $\{a_i\}_{i=1}^m$ and $\{\alpha_i\}_{i=1}^m$ with the described properties, there exists a star $H \subset E^n$ with the core A determined by $\{a_i\}_{i=1}^n$ and $\{\alpha_i\}_{i=1}^n$.*

Actually, since $a_i \in \operatorname{Bd}(A)$, it follows from (4.1) and (4.2) that the points a_i are accessible from $E^n - A$. Consequently, by the use of induction, one can easily construct a sequence of sets $T_i \subset E^n$ (where $i = 1, 2, \ldots, m$

if m is natural or $i = 1, 2, \ldots$, if $m = \infty$) such that T_i is a necklace with a_i beads and the initial point a_i, $T_i \cap T_j = \emptyset$ for $i \neq j$, $T_i \cap A = (a_i)$ and such that $\lim_{i=\infty} \operatorname{diam}(T_i) = 0$ if $m = \infty$. Then, defining $H = A \cup \bigcup_i T_i$, we obtain the required star H with the core A.

5. Reduction of theorem 1 to a lemma. First, let us notice that theorem 1 is true for $n = 1$; moreover, in this case we shall show that each space X satisfying the assumptions of this theorem is embeddable into E^1. Indeed, it follows from the assumptions that X cannot contain any simple closed curve, because — in virtue of (3.9) — each simple closed curve is contained in a cyclic element of X and such cyclic element is not embeddable into E^1. Thus, X is a dendrite and, since X does not contain any 1-umbrella, we infer that X has no ramification points. Consequently, X is an arc (or a point), which proves the embeddability of X into E^1. Therefore, in the sequel we shall assume that $n > 1$.

Now, we shall show that theorem 1 derives from the following lemma (condition 3° in the lemma is not used to derive the theorem, but it is useful for the proof):

LEMMA. *If X satisfies the assumptions of theorem 1 (for $n > 1$), then there exist a locally connected continuum $X' \subset E^n$ and a map g from X' onto X such that:*

1° *For every point $x \in X$ the inverse set $g^{-1}(x)$ is either a point or an arc.*

2° *The family of all arcs of the form $g^{-1}(x)$ is at most countable and, if it is infinite, then the diameters of these arcs converge to zero.*

3° *The non-degenerate cyclic elements of X' are in a one-to-one correspondence with the non-degenerate cyclic elements of X such that for each non-degenerate cyclic element of E' of X' the map $g|E'$ is a homeomorphism of E' onto the corresponding cyclic element of X.*

Suppose that the lemma is true and consider the decomposition $\mathfrak{D}$ of E^n whose non-degenerate elements are the arcs of the form $g^{-1}(x)$. It follows from 2° that $\mathfrak{D}$ is upper semi-continuous. Since $\mathfrak{D}$ has an at most countable number of non-degenerate elements and all those elements are arcs, the theorem of Gillman and Martin [5] implies that the decomposition space $E^n/\mathfrak{D}$ is an n-dimensional Cartesian divisor of E^{n+1}. Evidently the elements of $\mathfrak{D}$ contained in X' determine an upper semi-continuous decomposition of X' such that its decomposition space Y' embeds in a natural way into $E^n/\mathfrak{D}$. Since X' is a compactum, Y' is also a compactum, and consequently the map g determines a map from Y' onto X, which is a homeomorphism. Thus theorem 1 is derived from the lemma.

Now, we shall prove four simple propositions which will be useful in the proof of the lemma. We shall assume that X is a fixed space satisfying the assumptions of the lemma (i.e. of theorem 1).

(5.1) *Each closed and entirely arcwise connected set $A \subset X$ satisfies the assumptions of the lemma (with X replaced by A).*

Since X is a connected ANR, it follows from (3.4) that A is also a connected ANR which evidently does not contain any n-umbrella. From (3.10) and from the definition of cyclic elements we infer that the cyclic elements of A are at the same time cyclic elements of X, and therefore the assumption of theorem 1 concerning cyclic elements is also satisfied for A.

(5.2) *If X satisfies the conclusion of the lemma and if X' is an appropriate subset of E^n whose existence is given by the lemma, then $X' \in$ ANR. The same is true if, instead of X, one considers any closed and entirely arcwise connected subset of X.*

Indeed, since $X \in$ ANR, it follows from (3.12) and from the condition 3° of the lemma that all cyclic elements of X' are ANR-sets and almost all are AR-sets. Using (3.12) again, we infer that $X' \in$ ANR. The second statement of (5.2) is a consequence of the first one and of (5.1).

(5.3) *If A is a closed and entirely arcwise connected subset of X satisfying the conclusion of the lemma (with X replaced by A) and if A' and g_0 are, respectively, the appropriate subset of E^n and the appropriate map of A' onto A, then $x' \in A' - \overline{E^n - A'}$ implies that $g_0(x') \in A - \overline{X - A}$.*

Since $x' \in A' - \overline{E^n - A'}$, there is an n-ball $Q \subset A'$ such that $x' \in \operatorname{Int}(Q)$. By (3.9), Q is contained in the cyclic element E' of A' containing x'. In virtue of the condition 3° of the lemma, $g_0|E'$ is a homeomorphism of E' onto the corresponding cyclic element of A, and therefore $g_0(x')$ is an interior point of the (topological) n-ball $g_0(Q)$. If $g_0(x') \in \overline{X - A}$, then, in view of (3.2), one sees that $g_0(x')$ is accessible from $X - A$. Thus, as $g_0(Q) \subset A$, X contains an n-umbrella, which contradicts the assumption.

(5.4) *Suppose that $X = \bigcup_{i=1}^{m} A_i$, where $A_i = \bar{A}_i$ is an entirely arcwise connected subset of X such that if $i > 1$ then $A_i \cap \bigcup_{j=1}^{i-1} A_j$ consists of exactly one point. If the lemma is satisfied for each A_i, then it is also satisfied for X. Moreover, given appropriate $A_i' \subset E^n$ and g_i: $A_i' \to A_i$, the set $X' \subset E^n$ and the map g: $X' \to X$ may be so chosen that, for $x \in A_i - \bigcup_{j \neq i} A_j$, $g^{-1}(x)$ is a point if and only if $g_i^{-1}(x)$ is a point.*

We shall prove (5.4) by induction with respect to m. If $m = 1$, then (5.4) is evident. Now, given $m > 1$, suppose that (5.4) is true for $m-1$. Denote $Y = \bigcup_{i=1}^{m-1} A_i$ and let $y_0 \in Y \cap A_m = (y_0)$. Then, considering (5.1) and observing that A_i, $i = 1, 2, \ldots, m-1$, are closed and entirely arcwise connected subsets of Y, we see that the induction hypothesis applies to Y. Let Y' denote the appropriate subset of E^n and g_0 the appropriate map of Y' onto Y. If $g_0^{-1}(y_0)$ is a point, let $y_0' = g_0^{-1}(y_0)$, and if $g_0^{-1}(y_0)$ is an arc, let y_0' be one of its end-points. Since we can assume that A_m contains more than one point, in view of (5.3), $y_0' \in \overline{E^n - Y'}$. Consequently, by (4.1) and (5.2), there is a component C of $E^n - Y'$ such that $y_0' \in \bar{C}$.

By assumption, for A_m there are an appropriate set $A_m' \subset E^n$ and an appriopriate map g_m: $A_m' \to A_m$. Evidently, we can assume that $A_m' \subset C$. As before, we define the point $\hat{y}_0' \in g_m^{-1}(y_0)$ and, by (5.3), we infer that $\hat{y}_0' \in \overline{E^n - A_m'}$. Replacing A_m' by a set homeomorphic with it (and g_m by the appropriate map) if necessary, we can assume that $\hat{y}_0'$ belongs to the closure of the unbounded component of $E^n - A_m'$. Thus there is a component C_0 of $E^n - (Y' \cup A_m')$ such that $y_0', \hat{y}_0' \in \bar{C}_0$. By (5.2) and (4.2), both these points are accessible from C_0. Consequently, there is an arc $I' \subset E^n$ such that $I' \cap (Y' \cup A_m') = (y_0') \cup (\hat{y}_0') = \dot{I}'$.

Now, define

$$X' = Y' \cup I' \cup A_m'$$

and define g: $X' \to X$ by:

$$g(x') = \begin{cases} g_0(x') & \text{if} \quad x' \in Y', \\ y_0 & \text{if} \quad x' \in I', \\ g_m(x') & \text{if} \quad x' \in A_m'. \end{cases}$$

It is easily verified that X' and g satisfy the conditions 1°, 2° and 3° of the lemma. In particular, one sees that $g^{-1}(y_0)$ is an arc as the union of three arcs $g_0^{-1}(y_0) \cup I' \cup g_m^{-1}(y_0)$ (one or two of which may be degenerate) such that $g_0^{-1}(y_0) \cap g_m^{-1}(y_0) = \emptyset$ and the intersection of two successive arcs is a point which belongs to the boundary of either. Since, for $x \neq y_0$, $g^{-1}(x)$ coincides with the counter-image of x under g_0 or g_m, it follows that the additional requirement of (5.4) concerning the counter-images is satisfied for g whenever it is satisfied for g_0. This completes the induction step, and therefore (5.4) is proved.

The last three sections of the paper are devoted to the proof of the lemma. As mentioned in the introduction, the proof is divided into three cases, which will be considered in turn in sections 6, 7 and 8. Now, we shall formulate these cases:

Case I. There exists a finite set of points $a_1, a_2, \ldots, a_m \in X$ such that the least closed and entirely arcwise connected subset of X containing those points is equal to X.

Case II. The least closed and entirely arcwise connected subset of X containing all the non-degenerate cyclic elements of X is equal to X.

Case III. The general one.

6. Proof of the lemma in the case I. We shall consider separately three subcases: $m = 1$, $m = 2$ and $m > 2$.

Subcase $m = 1$. This subcase is trivial, since X coincides with the point a_1.

Subcase $m = 2$. In virtue of (3.13), $X = L \cup \bigcup_{i=1}^{k} E_i$, where L is an arc joining a_1 to a_2, k is a natural number 0 or ∞, $\{E_i\}_{i=1}^k$ is a sequence of all the non-degenerate cyclic elements of X, $L \cap E_i$ is a non-degenerate subarc L_i of L and $L_i \cap L_j = \dot{L}_i \cap \dot{L}_j = E_i \cap E_j$ for $i \neq j$. Since $X \in$ ANR, there exists a non-negative integer l such that X has exactly l cyclic elements which are not AR-sets (cf. (3.12)). We can assume that $k > 0$.

First, suppose that $l > 0$ and that the lemma is true if $l = 0$ ($m = 2$). Consider the family of subsets of X consisting of the sets $E_1, E_2, \ldots, E_l$ and of the closures of the components of $X - \bigcup_{i=1}^{l} E_i$. It is easily seen that this family has at most $2l+1$ elements. Suppose that the number of those elements is equal to s and denote those elements by $A_1, A_2, \ldots, A_s$. Evidently, $X = \bigcup_{i=1}^{s} A_i$. It follows from the definition that, for each $i = 1, 2, \ldots, s$, either A_i is a cyclic element of X or there exist two points of X belonging to L such that A_i coincides with the least closed and entirely arc-wise connected subset of X containing those points and, moreover, all cyclic elements of A_i are AR-sets. Thus, by hypothesis, the lemma is satisfied for each set A_i. Suppose the arc L is ordered by a relation $\lesssim$ such that $a_1 \lesssim a_2$. Reordering the sets A_i if necessary, we can assume that, for $i < j$, $x \in A_i \cap L$, and $y \in A_j \cap L$ imply that $x \lesssim y$. Thus, if $1 < i \leqslant s$, then $A_i \cap \bigcup_{j=1}^{i-1} A_j$ consists of exactly one point. Consequently, all the assumptions of (5.4) are satisfied and therefore X satisfies the thesis of the lemma.

It remains to give a proof under the assumption that $l = 0$, i.e. that all cyclic elements E_i of X are AR-sets. We are going to construct a set $X' \subset E^n$ and a map g: $X' \to X$ satisfying the conditions 1°, 2° and 3° of the lemma. We shall prove that the set X' and the map g may be so chosen that:

(6.1) *The sets $g^{-1}(a_1)$ and $g^{-1}(a_2)$ consist of one point either.*

Denote by f an arbitrary homeomorphism from the arc L onto a segment $L' \subset E^n$. Let $f(L_i) = L'_i$. One can easily construct a sequence of geometrical n-balls $\{Q'_i\}_{i=1}^{k}$ contained in E^n such that $Q'_i \cap L' = L'_i$, $Q'_i \cap Q'_j = L'_i \cap L'_j$ if $i \neq j$ and such that $\operatorname{diam}(Q'_i)$ converges to zero if i converges to the infinity provided $k = \infty$. Since E_i is a cyclic element of X, the assumption implies that for each i there is a homeomorphism f_i from E_i onto a set E'_i contained in the interior of Q'_i. Suppose that the arc L is ordered by a relation $\prec$ such that $a_1 \prec a_2$ and let $\dot{L}_i = (b_i) \cup (c_i)$, where $b_i \prec c_i$ (certain points b_i and c_j may coincide). Let us denote

$$f_i(b_i) = b'_i\,, \quad f_i(c_i) = c'_i\,, \quad f(b_i) = \hat{b}'_i\,, \quad f(c_i) = \hat{c}'_i\,.$$

If $b_i \neq a_1$ $(c_i \neq a_2)$, then $b_i \in \overline{X - E_i}$ $(c_i \in \overline{X - E_i})$ and using (5.3) we infer that $b'_i \in \overline{E^n - E'_i}$ $(c'_i \in \overline{E^n - E'_i})$. Since each E_i is an AR-set, it follows that each E'_i is also one, and we infer from (4.1) and (4.2) that E'_i does not separate Q'_i and that the points b'_i and c'_i (except possibly the images of $b_i = a_1$ and of $c_i = a_2$) are accessible from $Q'_i - E'_i$. Consequently, there exist two disjoint arcs $I'_i, J'_i \subset Q'_i$ such that

$$\dot{I}'_i = (b'_i) \cup (\hat{b}'_i)\,, \quad \dot{J}'_i = (c'_i) \cup (\hat{c}'_i)\,, \quad I'_i \cap E'_i = (b'_i)\,, \quad J'_i \cap E'_i = (c'_i)\,.$$

(Observe that $\hat{b}'_i, \hat{c}'_i \in \dot{L}'_i \subset \operatorname{Bd}(Q'_i)$.) If $b_i = a_1$ $(c_i = a_2)$, let us define $I'_i = (b'_i)$ $\big(J'_i = (c'_j)\big)$.

Now, define X' by the formula:

$$X' = \Big(L' - \bigcup_{i=1}^{k} L'_i\Big) \cup \bigcup_{i=1}^{k} (I'_i \cup E'_i \cup J'_i)\,.$$

Since $\dot{L}_i = f(\dot{L}_i) = (\hat{b}'_i) \cup (\hat{c}'_i)$, it is easily seen from the construction that $X' \subset E^n$ is a locally connected continuum and that the sets E'_i are non-degenerate cyclic elements of X'. Define a function $g\colon X' \to X$ by the formulas:

$$g(x') = \begin{cases} f^{-1}(x') & \text{if} \quad x' \in L' - \bigcup_{i=1}^{k} L'_1\,, \\ b_i & \text{if} \quad x' \in I'_i\,, \\ f_i^{-1}(x') & \text{if} \quad x' \in E'_i\,, \\ c_i & \text{if} \quad x' \in J'_i\,. \end{cases}$$

Since f and f_i are homeomorphisms, $f^{-1}(\hat{b}'_i) = f_i^{-1}(b'_i) = b_i$ and $f^{-1}(\hat{c}'_i) = f_i^{-1}(c'_i) = c_i$, one may easily verify that g is a map from X' onto X. Observing that, for each i, $g^{-1}(L) \cap Q'_i$ is an arc as the union of three arcs $I'_i \cup f_i(L_i) \cup J'_i$, we infer that $g^{-1}(L)$ is an arc. The non-degenerate counter-images $g^{-1}(x)$, $x \in X$, are subarcs of $g^{-1}(L)$ of the form $I'_i = g^{-1}(b_i)$ (where $b_i \neq a_1$), $J'_i = g^{-1}(c_i)$ (where $c_i \neq a_2$) or of the form $J'_j \cup I'_i$, provided $c_j = b_i$. Consequently, all conditions 1°, 2° and 3° of the lemma (and also (6.1)) are satisfied, which completes the proof in the subcase $m = 2$.

Subcase $m > 2$. We shall define inductively a sequence $A_1, A_2, \ldots, A_s$ consisting of closed und entirely arcwise connected subsets of X, such that, for every $1 \leqslant j \leqslant s$, $\bigcup_{i=1}^{j} A_i$ is also entirely arcwise connected.

Let A_1 denote the least closed and entirely arcwise connected subset of X containing the points a_1 and a_2. Now, consider an index j and suppose that the sets A_i for $i < j$ have been defined. If $B_{j-1} = \bigcup_{i=1}^{j-1} A_i$ contains all points a_i, $i = 1, 2, \ldots, m$, let $s = j-1$, i.e. let A_{j-1} be the last element of our sequence. Otherwise, let k denote the minimal index i such that a_i does not belong to B_{j-1}. Consider the component C of $X - B_{j-1}$ containing a_k. In virtue of (3.2), there is a point c such that $c \in \overline{C} \cap B_{j-1} = (c)$. Define A_j as the least closed and entirely arcwise connected subset of X containing the points c and a_k. We infer from (3.5) that $B_{j-1} \cup A_j$ is entirely arcwise connected.

Evidently, the process of defining A_i's ends after at most $m-1$ steps, i.e. $s \leqslant m-1$. It follows from the definition that for each i there exist two points such that A_i coincides with the least closed and entirely arcwise connected subset of X containing those points. Thus, since the subcase $m = 2$ has been previously considered, each set A_i satisfies the thesis of the lemma. If $1 < i \leqslant s$, then $A_i \cap \bigcup_{j=1}^{i-1} A_j$ consists of exactly one point. Since $\bigcup_{i=1}^{s} A_i$ is a closed and entirely arcwise connected subset of X containing all points a_i, $i = 1, 2, \ldots, m$, it follows from the assumptions of Case I that $\bigcup_{i=1}^{s} A_i = X$. Consequently, all assumptions of (5.4) are satisfied, and therefore X satisfies the conclusion of the lemma.

7. Proof of the lemma in case II. Evidently, we can assume that X contains more than one non-degenerate cyclic element (cf. (3.6)). Since $X \in \mathrm{ANR}$, we infer from (3.12) that there is a non-negative integer l such that X has exactly l cyclic elements which are not AR-sets. Arrange the non-degenerate cyclic elements of X into a sequence $\{E_i\}_{i=-l}^{k}$ (k is an integer or ∞, cf. (3.8)) such that the elements with the negative indices are those which are not AR-sets. Choose from each set E_i a point a_i which does not belong to any other cyclic element (this is possible by (3.1) and by [8], p. 238, No. 8). Observe that:

(7.1) *If A is a closed and entirely arcwise connected subset of X containing the sequence of points $\{a_i\}_{i=-l}^{k}$, then $A = X$.*

Indeed, it follows from (3.10) that A contains all sets E_i, and therefore, by the assumptions of case II, $A = X$.

Since case I has been previously considered and in view of (7.1) we shall assume in the sequal that $k = \infty$.

Now, we shall define inductively (similarly as in section 6 for the subcase $m > 2$) two families consisting of closed and entirely arcwise connected sets $A_s, B_s \subset X$, $s \geqslant 0$. Define A_0 and B_0 as follows:

(7.2) *A_0 is the least closed and entirely arcwise connected subset of X containing the points a_i for $i = -l, -l+1, \ldots, 0$ and $B_0 = A_0$.*

Next, consider an $s_0 > 0$ and suppose that for $s < s_0$ the sets A_s and B_s have been defined. If B_{s_0-1} contains all points a_i, then A_{s_0-1} and B_{s_0-1} will be the last elements of our families. Otherwise denote by b_{s_0} the first element of the sequence $\{a_i\}_{i=-l}^{\infty}$ which does not belong to B_{s_0-1}. Then:

(7.3) *A_{s_0} ($s_0 > 0$) is the least closed and entirely arcwise connected subset of X containing b_{s_0} and c_{s_0}, where c_{s_0} is the point which bounds the component of $X - B_{s_0-1}$ containing b_{s_0}. $B_{s_0} = B_{s_0-1} \cup A_{s_0}$.*

It follows from the definition that for $s > 0$ all cyclic elements of A_s are AR-sets, whence by (3.11):

(7.4) *If $s > 0$ then $A_s \in$ AR.*

It follows from (3.5) and (7.3) that B_{s_0} is entirely arcwise connected. Moreover, (7.2) and (7.3) imply that for every $s \geqslant 0$ (provided B_s is defined) there exists a finite set of points such that the least closed and entirely arcwise connected subset of X containing those points is equal to B_s. Therefore and with regard to (7.1), we shall assume that A_s and B_s are defined for all $s \geqslant 0$; otherwise the considerations reduce to case I. Now, let us show that

(7.5) $$\overline{\bigcup_{s=0}^{\infty} B_s} = X .$$

Let $B = \overline{\bigcup_{s=0}^{\infty} B_s}$. Since $B_0 \subset B_1 \subset \ldots$, one concludes that B is a continuum which contains all points a_i. We infer from the definition of these points that B contains all non-degenerate cyclic elements of X, and therefore, by (3.10), B is entirely arcwise connected. Hence (7.1) yields (7.5).

From (3.14) we infer:

(7.6) *For every $\varepsilon > 0$ there is an index s such that the diameter of each component of $X - B_s$ is less than ε.*

It follows from (7.3) and (7.6) that:

(7.7) *If $x \in X - \bigcup_{s=0}^{\infty} B_s = X - \bigcup_{s=0}^{\infty} A_s$, then there is a sequence s_i such that $\lim_{i=\infty} s_i = \infty$ and such that $x = \bigcap_{i=0}^{\infty} C_i$, where C_i is a component of $X - A_{s_i}$ bounded by a point different from c_{s_i}.*

Now, for every $s \geqslant 0$, arrange the points belonging to $A_s - c_s$ (to A_s, if $s = 0$) which bound the components of $X - A_s$ in a sequence (finite or not) $\{a_{st}\}_{t=1}^{T(s)}$. For each t between 1 and $T(s)$ denote by α_{st} the cardinal number of the set of the components of $X - A_s$ bounded by a_{st}. By (3.2) and (3.3) this is possible and we have $1 \leqslant \alpha_{st} \leqslant \aleph_0$. It follows from this definition and from (7.2) and (7.3) that:

(7.8) *If $x \in B_{s_0}$ bounds a component C of $X - B_{s_0}$, then x belongs to exactly one sequence $\{a_{st}\}_{t=1}^{T(s)}$ with $0 \leqslant s \leqslant s_0$. Moreover, $x = a_{st}$ implies that C is a component of $X - A_s$. In particular, c_{s_0+1} determines the indices $s(s_0+1)$, $t(s_0+1)$ (where $s(s_0+1) \leqslant s_0$) according to the formula $c_{s_0+1} = a_{s(s_0+1),t(s_0+1)}$.*

Indeed, $B_{s_0} = A_0 \cup (A_1 - c_1) \cup \ldots \cup (A_{s_0} - c_{s_0})$, where the terms are disjoint. If $x = a_{st}$, then C is a both closed and open subset of a component D of $X - A_s$, whence $C = D$.

Now, we are in position to construct the required set $X' \subset E^n$ and the map $g\colon X' \to X$. The subsets of E^n which will appear will be labelled with "primes". We begin with the construction concerning the sets A_s.

Suppose first that for each $s \geqslant 0$ there is given an n-ball $Q_s' \subset E^n$ (it will be defined later on). If follows from (7.2), (7.3) and from the result of section 6 that each set A_s satisfies the conclusion of the lemma. Thus, we can construct a suitable set $A_s' \subset \mathrm{Int}(Q_s')$ and a suitable map g_s from A_s' onto A_s. In view of (7.3), (7.4) and (6.1), we may require that:

(7.9) *If $s > 0$, then $g_s^{-1}(c_s)$ consists of only one point c_s'.*

For every point a_{st} defined before, choose a point

$$a_{st}' \in g_s^{-1}(a_{st})$$

such that, if $g_s^{-1}(a_{st})$ is an arc, then a_{st}' is one of its end-points. From (5.3) and from the definition of a_{st} we infer that $a_{st}' \in \mathrm{Bd}(A_s')$. By (5.2) and (4.4), there exists a star $H_s' \subset \mathrm{Int}(Q_s')$ with the core A_s' determined by the sequences $\{a_{st}'\}_{t=1}^{T(s)}$ and $\{\alpha_{st}\}_{t=1}^{T(s)}$. We can assume that:

(7.10) $$\mathrm{diam}(H_s') < \frac{1}{s} \quad \textit{for every} \quad s > 0 .$$

According to the definitions of $\{a_{st}\}_{t=1}^{T(s)}$ and of the star H'_s (see section 4), we can choose a one-to-one correspondence φ_{st} between the components of $X-A_s$ bounded by a_{st} and the beads of the necklace, which is the closure of the component of $H'_s-A'_s$ bounded by a'_{st}.

Now, define Q'_0 as an arbitrary n-ball in E^n. Next, consider an $s_0>0$ and assume inductively that for $s<s_0$ the n-balls Q'_s have been defined. Suppose also that for $s<s_0$ the constructions of A'_s, g_s, $\{a'_{st}\}_{t=1}^{T(s)}$, H'_s and of $\{\varphi_{st}\}_{t=1}^{T(s)}$ have been performed as described before. Then:

(7.11) $Q'_{s_0}=\varphi_{s(s_0)t(s_0)}(C)$, *where* $s(s_0)$, $t(s_0)$ *are determined according to* (7.8) *and where* C *denotes the component of* $X-A_{s(s_0)}$ (*and also of* $X-B_{s_0-1}$), *whose closure contains* A_{s_0}.

Thus, if $s_0>0$, then $Q'_{s_0}\subset H'_{s(s_0)}$ and it follows from the definition of the star $H'_{s(s_0)}$ that the set $Q'_{s_0}\cap\overline{H'_{s(s_0)}-Q'_{s_0}}$ consists of two points, which will be denoted by p'_{s_0} and q'_{s_0}. (7.4) and (4.3) imply that $H'_{s_0}\in\mathrm{AR}$ and — by definition — $H'_{s_0}\subset\mathrm{Int}(Q'_{s_0})$. It follows from (7.9), (7.3) and (5.3) that $c'_{s_0}\in\mathrm{Bd}(A'_{s_0})$ and therefore also $c'_{s_0}\in\mathrm{Bd}(H'_{s_0})$. We conclude by (4.1) and (4.2) that there exist two arcs $I'_{s_0}, J'_{s_0}\subset Q'_{s_0}$ such that:

$$\dot I'_{s_0}=(c'_{s_0})\cup(p'_{s_0}),\qquad \dot J'_{s_0}=(c'_{s_0})\cup(q'_{s_0}),$$
$$I'_{s_0}\cap J'_{s_0}=H'_{s_0}\cap(I'_{s_0}\cup J'_{s_0})=(c'_{s_0}). \tag{7.12}$$

Finally, define inductively some sets B'_s, $s\geqslant 0$, as follows:

(7.13) $B'_0=H'_0$ and $B'_s=(B'_{s-1}-Q'_s)\cup(I'_s\cup H'_s\cup J'_s)$ for $s>0$.

Let
$$X'=\bigcap_{s=0}^{\infty}B'_s.$$

We shall show that X' is the set required in the lemma. It follows easily from (7.12) and (7.13) that $\{B'_s\}_{s=0}^{\infty}$ is a decreasing sequence of continua, and therefore X' is also a continuum. One can observe by means of (7.11) that each n-ball of the form $\varphi_{s_0t_0}(C)$ (where C is a component of $X-A_{s_0}$) appears once in the sequence $\{Q'_s\}_{s=s_0+1}^{\infty}$; namely $\varphi_{s_0t_0}(C)=Q'_{s_1}$, where $s(s_1)=s_0$, $t(s_1)=t_0$ and $\bar C\supset A_{s_1}$. Since, for every $s\geqslant 0$, $X'\cap Q'_s$ is a continuum, it may be verified by (7.10) that X' is locally connected.

Now, we are going to define the required map $g\colon X'\to X$. For this purpose we shall define a sequence of maps $f_s\colon X'\to X$. Since H'_s is a star with the core A'_s, there exists a retraction r_s of H'_s onto A'_s such that, for each component C' of $H'_s-A'_s$, $r_s(\bar C')=\bar C'\cap A'_s$ ($\bar C'\cap A'_s$ is a point by the definition of H'_s). Let

$$f_0=g_0r_0|X'$$

and, if $s_0>0$, let

$$f_{s_0}(x')=\begin{cases} f_{s_0-1}(x') & \text{for } x'\in X'-Q'_{s_0},\\ g_{s_0}r_{s_0}(x') & \text{for } x'\in X'\cap H'_{s_0},\\ c_{s_0} & \text{for } x'\in I'_{s_0}\cup J'_{s_0}.\end{cases}$$

By (7.9) and (7.12) the definitions of f_{s_0} agree on the set $(I'_{s_0}\cup J'_{s_0})\cap H'_{s_0}=(c'_{s_0})\subset A'_{s_0}$. In order to show that they agree on the set $Q'_{s_0}\cap\overline{X'-Q'_{s_0}}=(p'_{s_0})\cup(q'_{s_0})$ we shall prove that $f_{s_0-1}(Q'_{s_0}\cap X')=c_{s_0}$. By (7.11) and by the definition of φ_{st}, $Q'_{s_0}=\varphi_{s(s_0)t(s_0)}(C)$ is contained in the component of $H'_{s(s_0)}-A'_{s(s_0)}$ bounded by the point $a'_{s(s_0)t(s_0)}\in g_{s(s_0)}^{-1}(a_{s(s_0)t(s_0)})$, whence $f_{s(s_0)}(Q'_{s_0}\cap X')=g_{s(s_0)}r_{s(s_0)}(Q'_{s_0}\cap X')=a_{s(s_0)t(s_0)}$. Consequently, (7.8) and the definitions of f_s yield $f_{s_0-1}(Q'_{s_0}\cap X')=f_{s(s_0)}(Q'_{s_0}\cap X')=c_{s_0}$. Thus f_{s_0} is a map.

We infer from the definition of f_{s_0} and from the formula $g_{s_0}(A'_{s_0})=A_{s_0}$ that $f_{s_0}(Q'_{s_0}\cap X')=A_{s_0}$ for every $s_0\geqslant 0$. Since, by (7.2) and (7.3), $B_{s_0}=A_0\cup\ldots\cup A_{s_0}$, it follows that:

$$f_{s_0}(X')=B_{s_0}=f_{s_0}(A'_0\cup\ldots\cup A'_{s_0}). \tag{7.14}$$

Taking into consideration (7.11) and the definition of f_s, one can show by an inductive argument that:

(7.15) *If* $s_1>s_0$, *then* f_{s_1} *and* f_{s_0} *agree on the set* $(X'-\bigcup_{s>s_0}Q'_s)\supset\bigcup_{0\leqslant s\leqslant s_0}A'_s$, *and for* $s>s_0$ *the sets* $f_{s_0}(Q'_s\cap X')$ *and* $f_{s_1}(Q'_s\cap X')$ *are contained in the closure of the same component of* $X-B_{s_0}$.

Thus, we conclude from (7.6) that the sequence f_s is uniformly convergent and $g=\lim_{s=\infty}f_s$ is a map from X' into X. By (7.14) and (7.15) we have $g(X')\supset\bigcup_{s\geqslant 0}B_s$ and therefore, by (7.5), g is onto.

We shall prove that g is the map required in the lemma. For this purpose, taking into consideration (7.15) and the definition of f_s let us notice that:

(7.16) $g|A'_s=g_s$ *for every* $s\geqslant 0$.

It follows from the definitions of f_s and r_s that, for every point a_{st}, $f_s^{-1}(a_{st})$ is the union of $g_s^{-1}(a_{st})$ and of the necklace $r_s^{-1}(a'_{st})\subset H'_s$. If C is a component of $X-A_s$ bounded by a_{st} and $\varphi_{st}(C)=Q'_{s_0}$, then, by (7.11) and (7.9), $c_{s_0}=a_{st}$, $f_{s_0}^{-1}(c_{s_0})\cap Q'_{s_0}=I'_{s_0}\cup J'_{s_0}$, $f_{s_0}^{-1}(C)=Q'_{s_0}\cap X'-(I'_{s_0}\cup J'_{s_0})$. We conclude from (7.15) that:

(7.17) $g^{-1}(a_{st})$ *is the union of* $g_s^{-1}(a_{st})$ *and of the arc which arises if one replaces each bead* Q'_{s_0} *of the necklace* $r_s^{-1}(a'_{st})$ *by the arc* $I'_{s_0}\cup J'_{s_0}$; *moreover, if* $Q'_{s_0}=\varphi_{st}(C)$, *then* $g^{-1}(C)=Q'_{s_0}\cap X'-(I'_{s_0}\cup J'_{s_0})$ $=H'_{s_0}\cap X'-c'_{s_0}$.

Now, consider a point $x \in X$. By (7.8), there are three possibilities: $x \in A_{s_0} - c_{s_0}$ (for some $s_0 \geqslant 0$) and it is different from every point $a_{s_0 t}$, $x = a_{st}$ for some s and t, or $x \in X - \bigcup_{s=0}^{\infty} A_s$. If the first possibility holds and $s_0 > 0$, then $x \in C$, where C is the component of $X - A_{s(s_0)}$ such that $\bar{C} \supset A_{s_0}$. Consequently, (7.17) implies that $g^{-1}(x) \subset g^{-1}(C) \subset H'_{s_0}$ and, since $x \neq a_{s_0 t}$, we infer that $g^{-1}(x) \subset A'_{s_0}$. We conclude by (7.16) that $g^{-1}(x) = g_{s_0}^{-1}(x)$ (also if $s_0 = 0$). If the second possibility holds, then (7.17) and the definition of a'_{st} imply that $g^{-1}(x)$ is an arc. If the third possibility holds, then (7.7) and (7.17) imply that there is a sequence s_i (not the same as in (7.7)) such that $\lim_{i=\infty} s_i = \infty$ and such that $g^{-1}(x) \subset \bigcap_{i=1}^{\infty} Q'_{s_i}$. Since, by (7.11), each Q'_{s_i} (except Q'_0) is contained in some H'_s, it is easy to show, by (7.10) and by the definition of a star (see section 4), that $\lim_{i=\infty} \operatorname{diam}(Q'_{s_i}) = 0$. Thus, $g^{-1}(x)$ is a point. Since each g_s satisfies condition 1° of the lemma (with respect to A_s), we conclude from these considerations that so does g.

Since each non-degenerate inverse image $g^{-1}(x)$ is contained in some H'_s, using (7.10), the definition of a star and again the properties of g_s, we infer that g satisfies condition 2° of the lemma.

To prove that it also satisfies 3°, first observe that, by (7.2) and (7.3), each non-degenerate cyclic element of X is a cyclic element of a set A_s. Next, the construction of X' implies that each non-degenerate cyclic element E' of X' is contained in a set A'_s. Indeed, $\lim_{s=\infty} \operatorname{diam}(Q'_s) = 0$, and therefore there is an index s_0 such that $E' \subset Q'_{s_0} - \bigcup_{s > s_0} Q'_s$, whence $E' \subset A'_{s_0}$. Consequently, (7.16) and the respective properties of g_s imply that g satisfies 3°. Thus, the proof of the lemma in case II is completed.

8. Proof of the lemma in case III. Let X be any space satisfying the assumptions of the lemma with $n > 1$ (see section 5). Denote by A the least closed and entirely arcwise connected subset of X containing all non-degenerate cyclic elements of X. By the result of section 7, we can assume that $X - A \neq \emptyset$ (however, it may happen that $A = \emptyset$, and then, in view of (8.1), the proof is trivial). Let us notice that:

(8.1) *If C is a component of X-A, then $\bar{C}$ is a dendrite.*

Indeed, it is clear in view of (3.2) that $\bar{C}$ is a locally connected continuum. By (3.9) and by the definition of A, $\bar{C}$ does not contain any simple closed curve, and therefore it is a dendrite.

From (5.1) and from the result of section 7, we infer that there exist a set $A' \subset E^n$ and a map g_0 from A' onto A satisfying the conclusion of the lemma with respect to A. In virtue of (3.3), the points of A which bound some components of $X - A$ can be ordered in a sequence $\{a_i\}_{i=1}^{k}$, where k is a natural number or ∞. Let $\{C_{ij}\}_{j=1}^{J(i)}$ be a sequence consisting of all components of $X - A$ bounded by a_i. For each point a_i, choose a point

$$a'_i \in g_0^{-1}(a_i)$$

such that, if $g_0^{-1}(a_i)$ is an arc, then a'_i is one of its end-points. By (5.3), $a'_i \in \operatorname{Bd}(A')$. Consequently, in virtue of (5.2), (4.1) and (4.2), every point a'_i is accessible from $E^n - A'$. Since $n > 1$ and in view of (8.1), one can easily construct by induction a sequence of sets $D'_{ij} \subset E^n$ such that:

(8.2) *$D'_{ij} \cap A' = (a'_i)$, the pair (D'_{ij}, a'_i) is homeomorphic with the pair $(\bar{C}_{ij}, a_i)$, $(D'_{i_1 j_1} - a'_{i_1}) \cap (D'_{i_2 j_2} - a'_{i_2}) = \emptyset$ if $(i_1, j_1) \neq (i_2, j_2)$; $\operatorname{diam}(D'_{ij}) < \frac{1}{i}$ and $\lim_{j=\infty} \operatorname{diam}(D'_{ij}) = 0$ if $J(i) = \infty$.*

Let

$$X' = A' \cup \bigcup_{i=1}^{k} \bigcup_{j=1}^{J(i)} D'_{ij}.$$

Since A' is a locally connected continuum, it follows from (8.2) that X' is also one. In virtue of (8.2) there is a homeomorphism g_{ij} from D'_{ij} onto $\bar{C}_{ij}$ such that $g_{ij}(a'_i) = a_i$. Define a function $g\colon X' \to X$ as follows:

$$g(x') = \begin{cases} g_0(x') & \text{if} \quad x' \in A', \\ g_{ij}(x') & \text{if} \quad x' \in D'_{ij}. \end{cases}$$

One can easily prove using (8.2) and the formula $g_0(A') = A$ that g is a map from X' onto X. It follows from the definition of g that $g^{-1}(x) = g_0^{-1}(x)$ if $x \in A$ and $g^{-1}(x)$ is a point if $x \in X - A$. Since the map g_0 satisfies conditions 1° and 2° of the lemma, it follows that so does g. From (8.2) and from the definition of X' we infer that the non-degenerate cyclic elements of X' coincide with the non-degenerate cyclic elements of A'. Evidently, the non-degenerate cyclic elements of X coincide with the non-degenerate cyclic elements of A. Since A' and g_0 satisfy to condition 3° of the lemma, we conclude that so does X' and g. Thus, the proof of the lemma, and therefore also of theorem 1, is completed.

References

[1] K. Borsuk, *Einige Sätze über stetige Streckenbilder*, Fund. Math. 18 (1932), pp. 198-213.

[2] — *On the decomposition of manifolds into products of curves and surfaces*, ibidem 33 (1945), pp. 273-298.

[3] S. Claytor, *Peanian continua not imbeddable in a spherical surface*, Ann. of Math. 38 (1937), pp. 631-646.

[4] — *Topological inversion of peanian continua in a spherical surface*, ibidem 35 (1934), pp. 809-835.

[5] D. S. Gillman and J. M. Martin, *Countable decompositions of E^3 into points and pointlike arcs*, Amer. Math. Soc. Notices 10 (1963), pp. 74-75.
[6] S. T. Hu, *Theory of retracts*, Detroit 1965.
[7] C. Kuratowski, *Sur le problème des courbes gauches en topologie*, Fund. Math. 15 (1930), pp. 271-283.
[8] — *Topologie II*, Monografie Matematyczne 21, Warszawa—Wrocław 1950.
[9] S. Mardešić and J. Segal, *A note on polyhedra embeddable in the plane*, Duke Math. J. 33 (1966), pp. 633-638.
[10] — *On polyhedra embeddable in the 2-sphere*, Glasnik Matem. 1 (21) (1966), pp. 167-175.
[11] H. Patkowska, *On the one-point union of crumpled cubes*, Bull. Acad. Polon. Sci. 15 (1967), pp. 849-853.

Reçu par la Rédaction le 10. 1. 1968

Structure of self-dual torsion-free metric LCA groups*

by

M. Rajagopalan** (Madurai, India)
and T. Soundararajan (Urbana, Ill.)

Since Pontrjagin [3] and Van Kampen [5] introduced the notion of the dual of a locally compact Abelian group, many examples of self-dual LCA groups have been given in the literature. However, the structure of all self-dual LCA groups has been an open problem till to-day (see [1]). As a matter of fact, there is even no conjecture about how a self-dual LCA group should look like. In this paper we give the structure of all metric self-dual LCA groups which are torsion-free as abstract groups.

Notations and Conventions. All topological spaces occuring in this paper are taken to be Hausdorff ones. We usually follow [7] for notations and concepts related to topological groups which are not defined here. We write LCA group as an abbreviation for a locally compact Abelian group. The dual of the LCA group G with the usual topology is denoted by $\hat{G}$. We use the additive notation for groups. If $H \subset G$ is a subgroup of the LCA group G, then $H^{\perp}$ denotes the annihilator of H in $\hat{G}$. R^n denotes the usual Euclidean group ($n \geqslant 0$). If p is a prime, then J_p denotes the group of all p-adic numbers and I_p the group of all p-adic integers with the usual topology. (We use the symbol $\oplus$ for topological direct sums). The definition of a local direct sum of LCA groups is given in [1], [6] and [4]. But we prefer to repeat this definition here for the sake of completeness.

Definition 1. Let (G_α) be a family of LCA groups indexed by a set A. Let $H_\alpha \subset G_\alpha$ be a compact and open subgroup of G_α for each $\alpha \in A$. We define the local direct sum $\sum_{\alpha \in A} G_\alpha$ of the family (G_α) with respect to (H_α) of subgroups as follows:

$$\sum_{\alpha \in A} G_\alpha = \Big\{(x_\alpha) \Big| \; (x_\alpha) \in \prod_{\alpha \in A} G_\alpha;\; x_\alpha \in H_\alpha \text{ for all } \alpha \in A \text{ except possibly for a finite number of indices}\Big\}.$$

* An announcement of the result presented here appeared in [4]. The main theorem there should have been only for the metric case instead of for all the groups.

** This author was partially supported by a NSF grant No. NSFGP5370 while this work was being done.