Some theorems on the embeddability of ANR-spaces into Euclidean spaces

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1. Introduction. In 1930 C. Kuratowski (see [7]) has characterized the local dendrites (i.e. connected, 1-dimensional ANR-sets) which are embeddable into the plane $\mathbb{E}^2$ as those which do not contain homeomorphic images of the two graphs $K_1$ and $K_2$. $K_1$ and $K_2$ will be called the graphs of Kuratowski. $K_i$ is the 1-skeleton of a 3-simplex in which the midpoints of a pair of non-adjacent edges are joined by a segment; $K_2$ is the 1-skeleton of a 4-simplex. C. Kuratowski has also described two locally connected curves $K_3$ and $K_4$ which are not ANR-sets and he has conjectured the characterization of locally connected continua which are embeddable into the sphere $S^2$ as those which do not contain homeomorphic images of the four curves $K_i$, $i = 1, 2, 3, 4$. This was proved in 1937 by S. Claytor (see [3] and [4]). As a corollary, Claytor obtained the following result ([5], p. 632), which will be useful for us: Each cyclic locally connected continuum which does not contain homeomorphic images of the graphs of Kuratowski is embeddable into $S^2$. Recall that a connected space is cyclic (in the sense of Whyburn) if it is separated by no point.

In 1966 S. Mardesić and J. Segal (see [9] and [10]) showed that the connected polyhedra which are embeddable into $S^2$ can be characterized as those which do not contain homeomorphic images of three polyhedra, namely $K_1$, $K_2$, and $\bot$, where $\bot$ is the one-point union of a 2-simplex and of a segment relative to an interior point of the 2-simplex and an end-point of the segment. They raised the question if this characterization can be extended to the connected ANR-sets (they are always assumed to be compact) which are embeddable into $S^2$. We shall show in this paper that this is in fact true. Namely, we shall derive this from Claytor’s result mentioned above and from the positive answer to the following question for $n = 2$: If $X$ is a connected ANR containing no $n$-umbrella and if the cyclic elements of $X$ are embeddable into $\mathbb{E}^n$, is $X$ also embeddable into $\mathbb{E}^{n+1}$? By an $n$-umbrella we shall mean here a one-point union of a (topological) $n$-ball and of an arc relative to an interior point of the $n$-ball and an end-point of the arc. For the definition of cyclic elements see section 3.
I proved in an earlier paper [11] that the answer to this question is negative for \( n = 3 \). In this paper I shall prove that positive answer in a modified form is possible in the general case and it is positive for \( n = 1, 2 \).

Given two spaces \( X, Y \), call \( Y \) a Cartesian divisor of \( X \) if there exists a topological space \( Z \) such that the product \( X \times Z \) is homeomorphic with \( X \). We shall prove the following

**Theorem 1.** If \( X \) is a connected ANR containing no \( n \)-umbrella and if the cyclic elements of \( X \) are embeddable into \( E^n \), then \( X \) is embeddable into an \( n \)-dimensional Cartesian divisor of \( E^{n+1} \).

The arrangement of this paper is as follows: In section 2 we shall derive some corollaries from theorem 1 in the case \( n = 2 \), in particular we shall give the solution of the problem of Mardešić and Segal, mentioned above. In section 3 we shall list the properties of cyclic elements and, more generally, of the sets entirely arcwise connected, which will be needed in the sequel. In section 4 we recall some properties of ANR-sets contained in Euclidean spaces and give some useful definitions. The last four sections are devoted to the proof of theorem 1. In section 5 we shall reduce this theorem to a lemma, and we shall formulate three cases into which the proof of this lemma is divided (the last case is general). The proof of the lemma in these three cases is given successively in sections 6, 7, and 8.

2. A characterization of ANR-sets which are embeddable into \( E^2 \) or \( S^3 \). As mentioned in the introduction, in this section we shall assume theorem 1 to be true (in the case of \( n = 2 \)) and give some corollaries.

Let \( X \) be a connected ANR containing no 2-umbrella and assume that the cyclic elements of \( X \) are embeddable into \( E^n \). Then, by theorem 1, \( X \) is embeddable into a 2-dimensional Cartesian divisor of \( E^n \). It has been proved by Bočan (see [2], p. 286) that each 2-dimensional Cartesian divisor of \( E^n \) is homeomorphic with \( E^n \). Therefore, \( X \) is embeddable into \( E^n \) and we obtain the following

**Theorem 2.** If \( X \) is a connected ANR containing no 2-umbrella and is the cyclic elements of \( X \) are embeddable into \( E^n \), then \( X \) is also embeddable into \( E^n \).

Notice that the theorem is false for the locally connected continua which are not ANR-sets; the counter examples are the curves \( K_2 \) and \( K_4 \). On the other hand, this theorem is true for \( n = 1 \) (cf. the beginning of section 5).

Now, let \( X \) be a connected ANR which does not contain any 2-umbrellas and any homeomorphic images of the graphs \( K_3 \) and \( K_4 \). Then each cyclic element of \( X \) is a cyclic locally connected continuum (see section 3) which also does not contain these sets. Thus, it follows from Claytor's result mentioned in the introduction ([3], p. 633) that each cyclic element of \( X \) is embeddable into \( S^n \). If there is a cyclic element \( E \) of \( X \) homeomorphic with \( S^n \), then \( E = X \); otherwise \( X \) would contain a 2-umbrella (since \( X \) is arcwise connected). Hence, in this case, \( X = S^n \).

On the other hand, if there is no such cyclic element, then each cyclic element of \( X \) is embeddable into \( E^n \) and — by theorem 2 — \( X \) is also embeddable into \( E^n \). Thus, in either case \( X \) is embeddable into \( S^n \), and we obtain the following

**Theorem 3.** If \( X \) is a connected ANR which does not contain any 2-umbrellas and any homeomorphic images of the graphs \( K_3 \) and \( K_4 \), then \( X \) is embeddable into \( S^n \).

Exactly as in [9], p. 636, we can obtain from this theorem the following two corollaries:

**Corollary 1.** If \( X \) is an ANR which does not contain any 2-umbrellas and any homeomorphic images of \( K_1 \) and \( K_4 \) and if no component of \( X \) different from \( X \) is homeomorphic with \( S^n \), then \( X \) is embeddable into \( S^n \).

**Corollary 2.** If \( X \) is an ANR which does not contain any 2-umbrellas and any homeomorphic images of \( K_3 \), \( K_4 \) and \( S^3 \), then \( X \) is embeddable into \( E^n \).

Let us recall that \( X \) is quasi-embeddable into \( Y \) if for every \( \varepsilon > 0 \) there is a map \( f: X \to Y \) such that \( \text{diam}(f^{-1}(y)) < \varepsilon \) for every \( y \in f(X) \). Using the same argument as in [9], namely that none of the sets \( K_2 \), \( K_4 \), and \( S^3 \) is quasi-embeddable into \( E^n \), we obtain from theorem 3 also the following

**Corollary 3.** An ANR-set is embeddable into \( E^n(S^n) \) if and only if it is quasi-embeddable into \( E^n(S^n) \).

3. The cyclic elements and the sets entirely arcwise connected. The concepts of a cyclic element and of a set entirely arcwise connected are basic for the proof of theorem 1. We shall recall their definitions and the most important properties. Our general reference will be [8], § 47, and in the sequel of this section we shall give only the page and the number of the proposition in question.

In this section we will denote a fixed locally connected continuum. A set \( A \subset X \) is said to be entirely arcwise connected (in \( X \)) if \( x, y \in A \) and \( x \neq y \) imply that each arc \( (x, y) \) joining \( x \) and \( y \) is contained in \( A \). A set \( E \subset X \) is said to be a cyclic element of \( X \) in either of the following three cases:

1. \( E \) consists of one point which separates \( X \).
2° $\mathcal{E}$ consists of one point $x \in \mathcal{X}$ such that $ord_x = 1$, where $ord_x$ denotes the order of $x$ in $\mathcal{X}$ in the sense of Menger-Urysohn.

3° $\mathcal{E}$ is a connected subset of $\mathcal{X}$ containing more than one point and maximal with respect to the property of being a cyclic space, i.e. of containing no point which separates it.

This definition of cyclic elements slightly differs from that given in [8], but it is easily seen from [8] to be equivalent to it.

In the next five propositions $A$ will denote any closed and entirely arcwise connected subset of $\mathcal{X}$. Thus we have:

(3.1) $A$ is a locally connected continuum (p. 231, No. 2).

(3.2) If $\mathcal{C}$ is a component of $\mathcal{X} - A$, then $A \cap \mathcal{C}$ consists of only one point (p. 233, No. 4).

(3.3) The set of the components of $\mathcal{X} - A$ is at most countable, and if it is infinite, then the diameters of those components converge to zero (p. 232, No. 7).

(3.4) $A$ is a retract of $\mathcal{X}$ and, consequently, if $\mathcal{X} \epsilon \text{ANR (AR)}$, then also $\mathcal{A} \epsilon \text{ANR (AR)}$ (p. 263, No. 15).

(3.5) If $B$ is another closed and entirely arcwise connected subset of $\mathcal{X}$ and $A \cap B \neq 0$, then the set $A \cup B$ is also entirely arcwise connected (p. 232, No. 8).

For cyclic elements we have (the cyclic elements of the form 3° will be said to be non-degenerate):

(3.6) Each cyclic element of $\mathcal{X}$ is a closed and entirely arcwise connected subset of $\mathcal{X}$ (p. 236, No. 6).

(3.7) $\mathcal{X}$ is the union of the cyclic elements of it (p. 235, No. 1).

(3.8) The set of the non-degenerate cyclic elements of $\mathcal{X}$ is at most countable, and if it is infinite, then their diameters converge to zero (p. 238, No. 9).

(3.9) Each connected subset of $\mathcal{X}$ separated by no point is contained in a cyclic element (p. 238, No. 10).

(3.10) A non-degenerate continuum $A \subset \mathcal{X}$ is entirely arcwise connected if and only if it is a union of cyclic elements (p. 239, No. 11).

The following theorem is due to Borsuk [1]:

(3.11) $\mathcal{X}$ is an AR-set if and only if all cyclic elements of $\mathcal{X}$ are AR-sets.

The next proposition follows easily from the preceding ones and from certain elementary properties of ANR-sets, especially that each subset of an $\mathcal{X} \epsilon \text{ANR}$ with a sufficiently small diameter is contractible in $\mathcal{X}$:

(3.12) $\mathcal{X}$ is an ANR-set if and only if all cyclic elements of $\mathcal{X}$ are ANR-sets and almost all are AR-sets.

(The sufficiency of the condition can be proved by embedding each cyclic element of $\mathcal{X}$ which is not an AR-set in a set homeomorphic with the Hilbert cube and thus embedding $\mathcal{X}$ in an AR-set so that $\mathcal{X}$ is a neighborhood retract of it).

Now, we shall prove two properties of $\mathcal{X}$, related to the subject-matter of this section, which will be useful in the sequel:

(3.13) Given two different points $a, b \in \mathcal{X}$, the least closed and entirely arcwise connected subset of $\mathcal{X}$ containing $a$ and $b$ is the union of an arc $\mathcal{L}$ joining $a$ and $b$ and of all the cyclic elements of $\mathcal{X}$ which have at least two points in common with $\mathcal{L}$. Moreover, if $\mathcal{E}_1, \mathcal{E}_2, \ldots$ is a sequence of the cyclic elements having this property, then $\mathcal{E}_1 \cap \mathcal{L} = \mathcal{L} \cap \mathcal{L}_i = \mathcal{L}_i \cap \mathcal{L}_j$ for $i \neq j$. ($\mathcal{L}$ denotes the boundary of the arc $\mathcal{L}$).

Proof. Let $\mathcal{Z}$ denote the least closed and entirely arcwise connected subset of $\mathcal{X}$ containing $a$ and $b$. Of course, there is an arc $\mathcal{L} \subset \mathcal{Z}$ such that $\mathcal{L} = \mathcal{Z} - (a \cup b)$. If there are two different points $x, y \epsilon \mathcal{L}$ contained in a cyclic element $\mathcal{E}_1$, then the subarc of $\mathcal{L}$ joining those points is also contained in $\mathcal{E}_1$, since — by (3.6) — $\mathcal{E}_1$ is entirely arcwise connected. Consequently, if a cyclic element $\mathcal{E}_1$ contains more than one point of $\mathcal{L}$, then $\mathcal{E}_1 \cap \mathcal{L}$ is a non-degenerate subarc of $\mathcal{L}$. Arrange into a sequence (finite or not) $\mathcal{E}_1, \mathcal{E}_1, \ldots$ all cyclic elements of $\mathcal{X}$ with this property and let $\mathcal{L}_1 \cap \mathcal{L} = \mathcal{L}_1$. (cf. (3.8)). Denote by $\mathcal{Z}'$ the union of $\mathcal{L}$ and all sets $\mathcal{E}_i$. We shall prove that $\mathcal{Z}_1 = \mathcal{L}'$.

Given an index $i$, let us observe that there is a point $a_i \epsilon \mathcal{L}_i$ which belongs to no cyclic element of $\mathcal{X}$ different from $\mathcal{E}_i$ (see [8], p. 238, No. 8). Since $a_i \epsilon \mathcal{L}_i \subset \mathcal{Z}$ and $\mathcal{Z} = \mathcal{L}_1$ is a union of cyclic elements by (3.10), it follows that $\mathcal{E}_1 \subset \mathcal{Z}$. Thus $\mathcal{L}_1 \subset \mathcal{Z}$.

To show the inclusion $\mathcal{Z} \subset \mathcal{L}'$, we have to prove that $\mathcal{Z}'$ is a closed and entirely arcwise connected subset of $\mathcal{X}$ containing $a$ and $b$. Indeed, $a, b \epsilon \mathcal{L}'$ and $\mathcal{Z}'$ is a continuum by (3.8). Considering (3.10), it remains to prove that $\mathcal{Z}'$ is the union of cyclic elements, i.e. that each point $x$ of $\mathcal{L}'$ which belongs to no $\mathcal{E}_i$ is a cyclic element of the form $1^0$ or $2^0$. Indeed, otherwise — by (3.7) — $x$ belongs to a cyclic element $\mathcal{E}$ of the form $3^a$ such that $\mathcal{L} \cap \mathcal{L}_i = \mathcal{L}_i$. Thus, there is a component $C$ of $\mathcal{X} - \mathcal{E}$ containing a point of $\mathcal{L} - \mathcal{L}$. By (3.2), $\mathcal{L} - \mathcal{L}_i = \mathcal{L}_i$, and therefore $C$ is a component of $\mathcal{X} - \mathcal{L}$. Since $C \neq \mathcal{X} - \mathcal{L}$, this contradicts the assumption that $x$ is not a cyclic element of the form $1^a$ or $2^a$, and thus the inclusion $\mathcal{Z} \subset \mathcal{L}'$ is proved.

In order to finish the proof of (3.13) we have to prove that $\mathcal{E}_i \cap \mathcal{E}_j = \mathcal{L}_i \cap \mathcal{L}_j$ for $i \neq j$. Since different cyclic elements have at most one point in common (see [8], p. 236, No. 4), $\mathcal{L}_i \cap \mathcal{L}_j = \mathcal{L}_i \cap \mathcal{L}_j$. If $p = \mathcal{E}_i \cap \mathcal{E}_j$ and $p \epsilon \mathcal{X} - \mathcal{L}_i$, then $p$ does not separate $\mathcal{X}$ between the
sets $E_i = p$ and $E_0 = p$, which contradicts the remark given in [8] (p. 238).
Consequently, $E_0 \cap E_1 = L_0 \cap L_1$.

(3.14) Let $X = \bigcup_{i=1}^{\infty} A_i$, where $A_i \subset C_{i+2}$ and $A_1 = \hat{A} \neq X$ is a set entirely arcwise connected. If the maximum of the diameters of the components of $X - A_0$ is equal to $\delta_i$, then \(\lim_{i \to \infty} \delta_i = 0\).

Proof. Since the sequence $A_i$ is increasing, it suffices to prove that for every $\varepsilon > 0$ there is an index $i(\varepsilon)$ such that the diameter of each component of $X - A_{i(\varepsilon)}$ is less than or equal to $\varepsilon$. Suppose that such an index $i(\varepsilon)$ does not exist. Therefore, for each $i$ there is a component $C_i$ of $X - A_i$ such that $\text{diam}(C_i) > \varepsilon$. We shall define by induction a sequence of indices $i_1, i_2, \ldots$ and two sequences of points $a_i, a_{i+1}, \ldots$ and $b_i, b_{i+1}, \ldots$ such that for $k > 1$ $a_k, b_k \in A_k \cap C_{i_k}$ and $\text{diam}(C_{i_k}) > \varepsilon$.

Let $i = 1$ and let $a_1, b_1$ be arbitrary points of $X$. Now, given an index $i$, suppose that the indices $i_1$ and the points $a_i, b_i$ for $k < l$ have been defined. Since the set $C_{i_k}$ is open and the set $\bigcup_{i=1}^{\infty} A_i$ is dense in $X$, it follows that the set $\bigcup_{i=1}^{m} A_i \cap C_{i_k} = C_{i_k}$ is dense in $C_{i_k}$, whence $\text{diam}(\bigcup_{i=1}^{m} A_i \cap C_{i_k}) > \varepsilon$.

Since the sequence of sets $A_i \cap C_{i_k}$ is increasing, there exist an index $i_k > i_{k-1}$ and two points $a_k, b_k \in A_k \cap C_{i_k}$ such that $\text{diam}(C_{i_k}) > \varepsilon$. Thus the required sequences have been defined.

Of course, there is a sequence $b_k$ such that $\lim_{k \to \infty} a_k = a$ and $\lim_{k \to \infty} b_k = b$.

Since the sequence $A_k$ is increasing, we can assume that $\lim_{k \to \infty} a_k = a$ and $\lim_{k \to \infty} b_k = b$. The local arcwise connectedness of $X$ implies that there are $I, J$ and index $k'$ and two arcs $I, J$ with diameters less than $\varepsilon/2$ such that $I = (a_{k'} - \varepsilon/2, a_{k'} + \varepsilon/2)$, $J = (b_{k'} - \varepsilon/2, b_{k'} + \varepsilon/2)$. Since $A_{i_k}$, as an open and connected set, is arcwise connected and $a_k, b_k \in C_{i_k}$, it follows that the set $I \cap C_{i_k} \cap J$ contains an arc $K$ joining the points $a_k$ and $b_k$. Taking into consideration that $\text{diam}(C_{i_k}) > \varepsilon$ and $\text{diam}(I), \text{diam}(J) < \varepsilon/2$, we infer that $K \cap C_{i_k} \neq \emptyset$. However, since $a_k, b_k \in A_{i_k}$ and $C_{i_k}$ is a component of $X - A_{i_k}$, we obtain a contradiction to the entire arcwise connectedness of $A_{i_k}$, and thus (3.14) is proved.

4. Some properties of ANR-sets contained in Euclidean spaces. The following two well-known properties will be useful for us:

(4.1) If $A \subset \mathbb{R}^n$ and $A$ is ANR, then the set $\mathbb{R}^n - A$ has only a finite number of components, which is equal to 1 whenever $n > 1$ and $A$ is AR (see [5], p. 180).

(4.2) If $A \subset \mathbb{R}^n$ and $A$ is ANR, then every boundary point of a component $C$ of $\mathbb{R}^n - A$ is accessible (by an arc) from $C$ (see [9], p. 196).

Now, we introduce some definitions which will be useful in section 7.
Let $A \subset \mathbb{R}^n$ be a locally connected continuum. By a star with the core $A$ we shall mean every locally connected continuum $H \subset \mathbb{R}^n$ such that $A$ is a closed and entirely arcwise connected subset of $H$.

Given a point $a \in \mathbb{R}^n$ and a cardinal number $n$, where $1 \leq n < \aleph_1$, by a necklace with $n$ beads and the initial point $a$ we shall mean any set $T \subset \mathbb{R}^n$ which is the union of an arc $L$ and of a geometrical $n$-balls $Q_1, Q_2, \ldots$, such that:

$1^o$ $a \in L$, $\bigcup_{i=1}^{n} Q_i$.
$2^o$ $Q_i \cap Q_j = \emptyset$ for $i \neq j$.
$3^o$ $Q_1 \cap L$ is a non-degenerate subspace $L_1$ of $L$.
$4^o$ If $a = \aleph_1$, then $\lim_{i \to \infty} \text{diam}(Q_i) = 0$ and there is a sequence of points $a_i \in Q_i$, $i = 1, 2, \ldots$, convergent to the point $b \in L - a = (8)$.

Now, let $A \subset \mathbb{R}^n$ be a fixed locally connected continuum and suppose that there is given a sequence (finite or not) of points $a_i \in \text{Bd}(A)$, $i = 1, 2, \ldots$, and also suppose that for each $i$ there is given a cardinal number $n_i$, where $1 \leq n_i < \aleph_1$. By a star with the core $A$ determined by $(a_i)_{i=1}^{n_i}$ and $(n_i)_{i=1}^{n_i}$ (m — natural or $\infty$) we shall mean each star $H$ with the core $A$ for which there is a one-to-one correspondence between the components of $H - A$ and the points $a_i$ such that, if $C_i$ corresponds to $a_i$, then $C_i \cap A = (a_i)$ and $C_i$ is a necklace with $a_i$ beads and the initial point $a_i$.

Let us notice that:

(4.3) If $A$ is ANR (AR), then each star with the core $A$ determined by sequences $(a_i)_{i=1}^{n_i}$ and $(n_i)_{i=1}^{n_i}$ is also an ANR (AR).

Indeed, it is evident from the definition that all cyclic elements of a necklace are AR-sets. Consequently, (4.3) follows from the definition of a star with the core $A$ determined by $(a_i)_{i=1}^{n_i}$ and $(n_i)_{i=1}^{n_i}$ and from (3.11) and (3.12).

(4.4) Given $A$ is ANR and two sequences $(a_i)_{i=1}^{n_i}$ and $(n_i)_{i=1}^{n_i}$ with the described properties, there exists a star $H \subset \mathbb{R}^n$ with the core $A$ determined by $(a_i)_{i=1}^{n_i}$ and $(n_i)_{i=1}^{n_i}$.

Actually, since $a_i \in \text{Bd}(A)$, it follows from (4.1) and (4.2) that the points $a_i$ are accessible from $\mathbb{R}^n - A$. Consequently, by the use of induction, one can easily construct a sequence of sets $T_i \subset \mathbb{R}^n$ (where $i = 1, 2, \ldots$, $m$...
if \( m \) is natural or \( i = 1, 2, \ldots \), if \( m = \infty \) such that \( T_1 \) is a necklace with \( a_i \) beads and the initial point \( a_i \), \( T_1 \cap T_1 = 0 \) for \( i \neq j \), \( T_1 \cap A = (a_i) \) and such that \( \lim \text{diam}(T_i) = 0 \) if \( m = \infty \). Then, defining \( H = A \cup \bigcup_{i=1}^{\infty} T_i \), we obtain the required star \( H \) with the core \( A \).

5. Reduction of theorem 1 to a lemma. First, let us notice that theorem 1 is true for \( n = 1 \); moreover, in this case we shall show that each space \( X \) satisfying the assumptions of this theorem is embeddable into \( E^n \). Indeed, it follows from the assumptions that \( X \) cannot contain any simple closed curve, because in virtue of (3.9) — each simple closed curve is contained in a cyclic element of \( X \) and such cyclic element is not embeddable into \( E^n \). Thus, \( X \) is a dendrite and, since \( X \) does not contain any 1-umbrella, we infer that \( X \) has no ramification points. Consequently, \( X \) is an arc (or a point), which proves the embeddability of \( X \) into \( E^n \). Therefore, in the sequel we shall assume that \( n > 1 \).

Now, we shall show that theorem 1 derives from the following lemma (condition 3\( ^o \) in the lemma is not used to derive the theorem, but it is useful for the proof):

**Lemma.** If \( X \) satisfies the assumptions of theorem 1 (for \( n > 1 \)), then there exists a locally connected continuum \( X' \subseteq E^n \) and a map \( g : X \to X' \) onto \( X \) such that:

1. For every point \( x \in X \) the inverse set \( g^{-1}(x) \) is either a point or an arc.
2. The family of all arcs of the form \( g^{-1}(x) \) is at most countable and, if it is infinite, then the diameters of these arcs converge to zero.
3. The non-degenerate cyclic elements of \( X' \) are in a one-to-one correspondence with the non-degenerate cyclic elements of \( X \) such that for each non-degenerate cyclic element of \( E^n \) of \( X' \) the map \( g : E^n \to E^n \) is a homeomorphism of \( E^n \) onto the corresponding cyclic element of \( X \).

Suppose that the lemma is true and consider the decomposition \( D \) of \( E^n \) whose non-degenerate elements are the arcs of the form \( g^{-1}(x) \). It follows from 2\( ^o \) that \( D \) is upper semi-continuous. Since \( D \) has an at most countable number of non-degenerate elements and all those elements are arcs, the theorem of Gillman and Martin [5] implies that the decomposition space \( E^n/D \) is an \( n \)-dimensional Cartesian divisor of \( E^n \). Evidently the elements of \( D \) contained in \( X' \) determine an upper semi-continuous decomposition of \( X' \) such that its decomposition space \( X' \) embeds in a natural way into \( E^n/D \). Since \( X' \) is a compactum, \( X' \) is also a compactum, and consequently the map \( g \) determines a map from \( X' \) onto \( X \), which is a homeomorphism. Thus theorem 1 is derived from the lemma.

Now, we shall prove four simple propositions which will be useful in the proof of the lemma. We shall assume that \( X \) is a fixed space satisfying the assumptions of the lemma (i.e. of theorem 1).

(5.1) Each closed and entirely arcwise connected set \( A \subseteq X \) satisfies the assumptions of the lemma (with \( X \) replaced by \( A \)).

Since \( X \) is a connected ANR, it follows from (3.4) that \( A \) is also a connected ANR which evidently does not contain any \( n \)-umbrella. From (3.10) and from the definition of cyclic elements we infer that the cyclic elements of \( A \) are at the same time cyclic elements of \( X \), and therefore the assumption of theorem 1 concerning cyclic elements is also satisfied for \( A \).

(5.2) If \( X \) satisfies the conclusion of the lemma and if \( X' \) is an appropriate subset of \( E^n \) whose existence is given by the lemma, then \( X' \subseteq \text{ANR} \).

The same is true if, instead of \( X \), one considers any closed and entirely arcwise connected subset of \( X \).

Indeed, since \( X \subseteq \text{ANR} \), it follows from (3.12) and from the condition 3\( ^o \) of the lemma that all cyclic elements of \( X' \) are ANR-sets and almost all are AR-sets. Using (3.12) again, we infer that \( X' \subseteq \text{ANR} \). The second statement of (5.2) is a consequence of the first one and of (5.1).

(5.3) If \( A \) is a closed and entirely arcwise connected subset of \( X \) satisfying the conclusion of the lemma (with \( X \) replaced by \( A \)) and if \( A' \) and \( g \) are, respectively, the appropriate subset of \( E^n \) and the appropriate map of \( A' \) onto \( A \), then \( A' \subseteq E^n \), \( A' \subseteq A \), implies that \( g(x) \in A \).

Since \( x \subseteq A' \subseteq E^n \), there is an \( n \)-ball \( Q \subseteq A' \) such that \( x \subseteq \text{Int}(Q) \). By (3.9), \( Q \) is contained in the cyclic element \( E^n \) of \( A' \) containing \( x \). In virtue of the condition 3\( ^o \) of the lemma, \( g(x) \) is a homeomorphism of \( E^n \) onto the corresponding cyclic element of \( A' \), and therefore \( g(x) \) is an interior point of the (topological) \( n \)-ball \( g(Q) \). If \( g(x) \subseteq E^n \), then, in view of (3.2), one sees that \( g(x) \) is accessible from \( X \) to \( X \). Thus, as \( g(Q) \subseteq A \), \( Q \) contains an \( n \)-umbrella, which contradicts the assumption.

(5.4) Suppose that \( X = \bigcup_{i=1}^{\infty} A_i \), where \( A_i = \overline{A_i} \) is an entirely arcwise connected subset of \( X \) such that if \( i > 1 \) then \( A_i \cap \bigcup_{j<i} A_j \) consists of exactly one point. If the lemma is satisfied for each \( A_i \), then it is also satisfied for \( X \). Moreover, given appropriate \( A_i \subseteq E^n \) and \( g_i : A_i \to A_i \), the set \( X' \subseteq E^n \) and the map \( g : X' \to X \) may be so chosen that, for \( x \in A_i \), \( g^{-1}(x) \) is a point if and only if \( g_i^{-1}(x) \) is a point.
We shall prove (5.4) by induction with respect to \( m \). If \( m = 1 \), then (5.4) is evident. Now, given \( m > 1 \), suppose that (5.4) is true for \( m-1 \). Denote
\[ Y = \bigcup_{i=1}^{m-1} A_i \] and let \( y_i \in Y \cap A_m = \{y_1\} \). Then, considering (5.1) and observing that \( A_i \), \( i = 1, 2, \ldots, m-1 \), are closed and entirely arcwise connected subsets of \( Y \), we see that the induction hypothesis applies to \( X \). Let \( Y' \) denote the appropriate subset of \( E^p \) and \( g_i \) the appropriate map of \( Y' \) unto \( Y \). If \( g_i(y_i) \) is a point, let \( y_i = g_i^{-1}(y_i) \), and if \( g_i^{-1}(y_i) \) is an arc, let \( y_i \) be one of its end-points. Since we can assume that \( A_m \) contains more than one point, in view of (5.3), \( y_i \in E^p - Y' \). Consequently, by (4.1) and (5.2), there is a component \( C \) of \( E^p - Y' \) such that \( y_i \in C \).

By assumption, for \( A_m \) there are an appropriate set \( A_m \subset E^p \) and an appropriate map \( g_m: A_m \rightarrow A_m \). Evidently, we can assume that \( A_m \subset C \).

As before, we define the point \( y_i \in g_m^{-1}(y_1) \) and, by (5.3), we infer that \( y_i \in E^p - A_m \). Replacing \( A_m \) by a set homeomorphic with it (and \( g_m \) by the appropriate map) if necessary, we can assume that \( y_i \) belongs to the closure of the unbounded component of \( E^p - A_m \). Thus there is a component \( C_i \) of \( E^p - (Y' \cup A_m) \) such that \( y_i, y_i \in C_i \). By (5.2) and (4.2), both these points are accessible from \( C_i \). Consequently, there is an arc \( Y' \subset E^p \) such that \( Y' \cap (Y' \cup A_m) = (y_i) \cup (y_i) = \bar{I} \).

Now, define
\[ X' = Y' \cup \bar{I} \subset A_m \] and define \( g: X' \rightarrow X \) by:
\[ g(x') = \begin{cases} g_m(x') & \text{if } x' \in Y', \\ y_1 & \text{if } x' = e, \\ g_m(x') & \text{if } x' \in A_m. \end{cases} \]

It is easily verified that \( X' \) and \( g \) satisfy the conditions 1°, 2° and 3° of the lemma. In particular, one sees that \( g^{-1}(y_1) \) is an arc as the union of three arcs \( g_m^{-1}(y_1) \cup \bar{I} \cup g_m^{-1}(y_1) \) (one or two of which may be degenerate) such that \( g_m(y_i) \cap g_m^{-1}(y_1) = \emptyset \) and the intersection of two successive arcs is a point which belongs to the boundary of either. Since, for \( x \neq y_1 \), \( g^{-1}(x) \) coincides with the counter-image of \( x \) under \( g_m \) or \( g_m \), it follows that the additional requirement of (5.4) concerning the counter-images is satisfied for \( g \) whenever it is satisfied for \( g_m \). This completes the induction step, and therefore (5.4) is proved.

The last three sections of the paper are devoted to the proof of the lemma. As mentioned in the introduction, the proof is divided into three cases, which will be considered in turn in sections 6, 7 and 8. Now, we shall formulate these cases:

**Case I.** There exists a finite set of points \( a_1, a_2, \ldots, a_m \in X \) such that the least closed and entirely arcwise connected subset of \( X \) containing those points is equal to \( X \).

**Case II.** The least closed and entirely arcwise connected subset of \( X \) containing all the non-degenerate cyclic elements of \( X \) is equal to \( X \).

**Case III.** The general one.

6. **Proof of the lemma in the case I.** We shall consider separately three subcases: \( m = 1, m = 2 \) and \( m > 2 \).

Subcase \( m = 1 \). This subcase is trivial, since \( X \) coincides with the point \( a_1 \).

Subcase \( m = 2 \). In virtue of (3.13), \( X = L \cup \bigcup_{i=1}^{k} B_i \), where \( L \) is an arc joining \( a_1 \) to \( a_2 \), \( k \) is a natural number 0 or \( \infty \), \( \bigcap_{i=1}^{k} B_i \) is a sequence of all the non-degenerate cyclic elements of \( X \), \( L \subset B_i \) is a non-degenerate subarc \( L_1 \subset L_2 \subset \cdots \subset L_k \subset L \) for \( i \neq j \). Since \( X \in ANR \), there exists a non-negative integer \( l \) such that \( X \) has exactly \( l \) cyclic elements which are not AR-sets (cf. (3.12)). We can assume that \( k > 0 \).

First, suppose that \( l > 0 \) and that the lemma is true if \( l = 0 \) (m = 2).

Consider the family of subsets of \( X \) consisting of the sets \( E_1, E_2, \ldots, E_l \) and of the closures of the components of \( X - \bigcup_{i=1}^{l} B_i \). It is easily seen that this family has at most \( 2l+1 \) elements. Suppose that the number of those elements is equal to \( l \) and denote those elements by \( A_1, A_2, \ldots, A_l \).

Evidently, \( X = \bigcup_{i=1}^{l} A_i \). It flows from the def;inition that, for each \( i = 1, 2, \ldots, l \), \( A_i \) is a cyclic element of \( X \) or there exist two points of \( X \) belonging to \( L \) such that \( A_i \) coincides with the least closed and entirely arc-wise connected subset of \( X \) containing those points and, moreover, all cyclic elements of \( A_i \) are AR-sets. Thus, by hypothesis, the lemma is satisfied for each set \( A_i \). Suppose the arc \( L \) is ordered by a relation \( \leq \) such that \( a_1 \leq a_2 \). Reordering the sets \( A_i \) if necessary, we can assume that, for \( i < j \), \( a_i \leq A_j \cap L \) and \( y \in A_j \cap L \) imply that \( x \leq y \).

Thus, if \( 1 < i < l \), then \( A_i \cap \bigcup_{j=1}^{n} A_j \) consists of exactly one point. Consequently, all the assumptions of (5.4) are satisfied and therefore the lemma satisfies the hypothesis of the lemma.

It remains to give a proof under the assumption that \( l > 0 \), i.e. that all cyclic elements \( E_i \) of \( X \) are AR-sets. We are going to construct a set \( X' \subset E^p \) and a map \( g: X' \rightarrow X \) satisfying the conditions 1°, 2° and 3° of the lemma. We shall prove that the set \( X' \) and the map \( g \) may be so chosen that:

(6.1) The sets \( g^{-1}(a_i) \) and \( g^{-1}(a_j) \) consist of one point either.
Denote by \( f \) an arbitrary homeomorphism from the arc \( L \) onto a segment \( L' \subset \mathbb{R}^2 \). Let \( f (L) = L' \). One can easily construct a sequence of geodesic \( \mathcal{q} \)-balls \( \{ Q_i \}_{i=1}^{\infty} \) contained in \( \mathbb{R}^2 \) such that \( Q_i \cap L' = L_i \), \( Q_i \cap Q_j = L_i \cap L_j \) if \( i \neq j \) and such that diam \( Q_i \) converges to zero if \( i \) converges to the infinity provided \( h = \infty \). Since \( E \) is a cyclic element of \( X \), the assumption implies that for each \( i \) there is a homeomorphism \( f_i \) from \( E \) onto \( E_i \) contained in the interior of \( Q_i \). Suppose that the arc \( L \) is ordered by a relation \( \prec \) such that \( a_i \prec a_j \) and let \( L_i = (b_i) \cup (c_i) \), where \( b_i \prec b_j \) and \( c_i \) coincide. Let us denote
\[
 f (b_i) = b_i, \quad f (c_i) = c_i, \quad f (a_i) = a_i.
\]

If \( b_i \neq a_i \) (\( a_i \neq a_j \)), then \( b_i \in \mathbb{X} - \mathbb{E}_i \) and using (5.3) we infer that \( b_i \in \mathbb{X} - \mathbb{E}_i \) and that the points \( b_i \) and \( c_i \) (except possibly the images of \( b_i = a_i \) of \( a_i = a_j \)) are accessible from \( \mathbb{Q}_i \). Consequently, there exist two disjoint arcs \( I_i, J_i \subset \mathbb{Q}_i \) such that
\[
 I_i = (b_i) \cup (b_i), \quad J_i = (c_i) \cup (c_i), \quad I_i \cap E_i = (b_i), \quad J_i \cap E_i = (c_i).
\]

(Observe that \( b_i, c_i \in L_i \subset \mathbb{Q}_i \).) If \( b_i = a_i \) (\( a_i = a_j \)), let us define \( I_i = (b_i) \cup (a_i) \). Now, define \( X \) by the formulas:
\[
 X = (L - \bigcup_{i=1}^{\infty} I_i) \cup (\bigcup_{i=1}^{\infty} (J_i \cup E_i)).
\]

Since \( L = f (L) \), it is easily seen from the construction that \( X \subset \mathbb{X} \) is a locally connected continuum and that the sets \( E_i \) are non-degenerate cyclic elements of \( X \). Define a function \( g: X \to X \) by the formulas:
\[
 g (a_i) = \begin{cases} \mathcal{f}^{-1} (a_i) & \text{if } \mathcal{f}^{-1} (a_i) \cap I_i \neq \emptyset, \\
 b_i & \text{if } a_i \in I_i, \\
 f^{-1} (a_i) & \text{if } a_i \in J_i, \\
 c_i & \text{if } a_i \in J_i. \end{cases}
\]

Since \( f \) and \( f_i \) are homeomorphisms, \( f^{-1} (b_i) = f_{i}^{-1} (b_i) = b_i \) and \( f^{-1} (c_i) = f_{i}^{-1} (c_i) = c_i \), one may easily verify that \( g \) is a map from \( X \) onto \( X \). Observing that, for each \( i \), \( g^{-1} (L) \cap Q_i \) is an arc as the union of three arcs \( I_i \cup f_i (L) \cup J_i \), we infer that \( g^{-1} (L) \) is an arc. The non-degenerate counter-images \( g^{-1} (a), a \in X \), are subarcs of \( g^{-1} (L) \) of the form \( I_i = g^{-1} (b_i) \) (where \( b_i \neq a_i \)), \( J_i = g^{-1} (c_i) \) (where \( c_i \neq a_i \)) or of the form \( J_i \cup J_i \), provided \( a_i = b_i \). Consequently, all conditions \( 1, 2, 3 \) of the lemma (and also (6.1)) are satisfied, which completes the proof in the case \( m = 2 \).

When \( m > 2 \), we shall define inductively a sequence \( C_1, A_1, \ldots, A_m \) consisting of closed and entirely arcwise connected subsets of \( X \), such that, for every \( 1 \leq j \leq m \), \( \bigcup_{i=1}^{m} A_i \) is entirely arcwise connected.

Let \( A_i \) denote the least closed and entirely arcwise connected subset of \( X \) containing the points \( a_i \) and \( a_i \). Now, consider an index \( j \) and suppose that the sets \( A_i \) for \( i < j \) have been defined. If \( B_{j-1} = \bigcup_{i=1}^{j} A_i \) contains all points \( a_i \), \( i = 1, 2, \ldots, m \), let \( s = j - 1 \), i.e. let \( A_{j-1} \) be the last element of our sequence. Otherwise, let \( k \) denote the minimal index \( i \) such that \( a_i \) does not belong to \( B_{j-1} \). Consider the component \( C \) of \( X - B_{j-1} \) containing \( a_i \). In virtue of (3.3), there is a point \( c \in C \) such that \( c \in C \cap B_{j-1} \). Define \( A_j \) as the least closed and entirely arcwise connected subset of \( X \) containing the points \( c \) and \( a_i \). We refer from (3.9) that \( B_{j-1} \cup A_j \) is entirely arcwise connected.

Evidently, the process of defining \( A_i \)'s ends after at most \( m-1 \) steps, i.e. \( s < m - 1 \). It follows from the definition that for each \( i \) there exist two points such that \( A_i \) coincides with the least closed and entirely arcwise connected subset of \( X \) containing those points. Thus, since the subcase \( m = 2 \) has been previously considered, each set \( A_i \) satisfies the thesis of the lemma. If \( 1 < i < s \), then \( A_i \cap \bigcup_{i=1}^{m} A_i \) consists of exactly one point. Since \( \bigcup_{i=1}^{m} A_i \) is a closed and entirely arcwise connected subset of \( X \) containing all points \( a_i \), \( i = 1, 2, \ldots, m \), it follows from the assumptions of Case I that \( \bigcup_{i=1}^{m} A_i = X \). Consequently, all assumptions of (5.4) are satisfied, and therefore \( X \) satisfies the conclusion of the lemma.

7. Proof of the lemma in case II. Evidently, we can assume that \( X \) contains more than one non-degenerate cycle element (cf. (3.6)). Since \( X \subset \mathcal{D} \), we infer from (3.12) that there is a non-negative integer \( l \) such that \( X \) has exactly \( l \) cyclic elements which are not \( \mathcal{D} \)-sets. Arrange the non-degenerate cyclic elements of \( X \) into a sequence \( \{ E_i \}_{i=1}^{l} \) (\( b \) is an integer or \( \infty \), cf. (3.8)) such that the elements with the negative indices are those which are \( \mathcal{D} \)-sets. Choose from each set \( E_i \) a point \( a_i \) which does not belong to any other cyclic element (this is possible by (3.1) and by [8], p. 238, No. 8). Observe that:

(7.1) If \( A \) is a closed and entirely arcwise connected subset of \( X \) containing the sequence of points \( \{ a_i \}_{i=1}^{m} \), then \( A = X \).
Indeed, it follows from (3.10) that $A$ contains all sets $B_i$, and therefore, by the assumptions of case $II$, $A = X$.

Since case I has been previously considered and in view of (7.1) we shall assume in the sequel that $k = \infty$.

Now, we shall define inductively (similarly as in section 6 for the subcase $m > 2$) two families consisting of closed and entirely arcwise connected sets $A_i, B_i \subset X, i > 0$. Define $A_0$ and $B_0$ as follows:

(7.2) $A_0$ is the least closed and entirely arcwise connected subset of $X$ containing the points $a_i$ for $i = -1, -1 + 1, \ldots, 0$ and $B_0 = A_0$.

Next, consider an $s_0 > 0$ and suppose that for $s < s_0$ the sets $A_s$ and $B_s$ have been defined. If $B_{n-1}$ contains all points $a_i$, then $A_{n-1}$ and $B_{n-1}$ will be the last elements of our families. Otherwise denote by $b_n$ the first element of the sequence $\{a_i\}_{i=1}^{n}$ which does not belong to $B_{n-1}$.

Then:

(7.3) $A_n (s_0 > 0)$ is the least closed and entirely arcwise connected subset of $X$ containing $b_n$ and $A_{n-1}$, where $b_n$ is the point which bounds the component of $X - B_{n-1}$ containing $b_n$. $B_n = B_{n-1} \cup A_{n-1}$.

It follows from the definition that for $s > 0$ all cyclic elements of $A_s$ are AR-sets, whence by (3.11):

(7.4) If $s > 0$ then $A_s \subset AR$.

It follows from (3.5) and (7.3) that $B_n$ is entirely arcwise connected. Moreover, (7.2) and (7.3) imply that for every $s > 0$ (provided $B_s$ is defined) there exists a finite set of points such that the least closed and entirely arcwise connected subset of $X$ containing those points is equal to $B_s$. Therefore and with regard to (7.1), we shall assume that $A_s$ and $B_s$ are defined for all $s > 0$; otherwise the considerations reduce to case I. Now, let us show that

(7.5) $\bigcup_{s=0}^{\infty} B_s = X$.

Let $B = \bigcup_{s=0}^{\infty} B_s$. Since $B_0 \supset B_1 \supset B_2 \ldots$, one concludes that $B$ is a continuum which contains all points $a_i$. We infer from the definition of these points that $B$ contains all non-degenerate cyclic elements of $X$, and therefore, by (3.10), $B$ is entirely arcwise connected. Hence (7.1) yields (7.5).

From (3.14) we infer:

(7.6) For every $\varepsilon > 0$ there is an index $s$ such that the diameter of each component of $X - B_s$ is less than $\varepsilon$.

It follows from (7.5) and (7.6) that:

(7.7) If $x \in X - \bigcup_{s=0}^{\infty} B_s = X - \bigcup_{s=0}^{\infty} A_s$, then there is a sequence $s_i$ such that $\lim s_i = \infty$ and such that $x = \bigcap_{i=1}^{\infty} C_i$, where $C_i$ is a component of $X - A_{s_i}$ bounded by a point different from $a_i$.

Now, for every $s > 0$, arrange the points belonging to $A_s - a_i$ (to $A_s$ if $s = 0$) which bound the components of $X - A_s$ in a sequence (finite or not) $(a_{s_1,n}^{(1)}), \ldots, (a_{s_1,n}^{(m)})$. For each $i$ between 1 and $T(s)$ denote by $c_{s_i,n}$ the cardinal number of the set of the components of $X - A_s$ bounded by $a_{s_i,n}$. By (3.2) and (3.3) this is possible and we have $1 \leq c_{s_i,n} \leq s_0$. It follows from this definition and from (7.2) and (7.7) that:

(7.8) If $x \in B_s$, then $x$ belongs to exactly one sequence $(a_{s_i,n}^{(1)})$ with $0 \leq s \leq s_0$. Moreover, $x = a_{s_0}$ implies that $x$ is a component of $X - A_s$. In particular, $c_{s_i,n}$ determines the indices $s_{i_0} + 1, t_{i_0} + 1$ (where $s_{i_0} + 1 \leq s_0$) according to the formula $s_{i_0} + 1 = a_{s_{i_0} + 1, t_{i_0} + 1}$.

Indeed, $B_s = A_s \cup (A_s - a_i) \cup \ldots \cup (A_s - a_{s_0})$, where the terms are disjoint. If $a = a_{s_0}$, then $C$ is a closed and open subset of a component $D$ of $X - A_s$, whence $C = D$.

Now, we are in position to construct the required set $X' \subset E^p$ and the map $g: X' \hookrightarrow X$. The subsets of $E^p$ which will appear will be labelled with "primes". We begin with the construction concerning the sets $A_s$.

Suppose first that for each $s > 0$ there is given an $n$-ball $Q_s' \subset E^p$ (it will be defined later on). If follows from (7.2), (7.3) and from the result of section 6 that each set $A_s$ satisfies the conclusion of the lemma. Thus, we can construct a suitable set $A'_s \subset Int(Q_s')$ and a suitable map $g_s$ from $A'_s$ onto $A_s$. In view of (7.3), (7.4) and (6.1), we may require that:

(7.9) If $s > 0$, then $g_s^{-1}(a_i)$ consists of only one point $a_i$.

For every point $a_{s_0}$ defined before, choose a point

$a_{s_0} \in g_{s_0}^{-1}(a_{s_0})$

such that, if $g_{s_0}^{-1}(a_{s_0})$ is an arc, then $a_{s_0}$ is one of its end-points. From (5.3) and from the definition of $a_{s_0}$ we infer that $\overline{a_{s_0}} \subset Bd(A_1)$. By (5.2) and (4.4), there exists a star $H_{a_{s_0}} \subset Inter(Q_s')$ with the core $A_1$ determined by the sequences $(a_{s_0}^{(1)}, a_{s_0}^{(m)})$. We can assume that:

(7.10) $\text{diam}(H_{a_{s_0}}) < \frac{1}{s}$ for every $s > 0$. 
According to the definitions of $\{a_n\}_n$ and of the star $H'$ (see section 4), we can choose a one-to-one correspondence $a_n$ between the components of $X - A_n$ bounded by $a_n$ and the heads of the necklace, which is the closure of the component of $H' \cap A_n$ bounded by $a_n$.

Now, define $Q_n$ as an arbitrary $n$-ball in $B'$. Next, consider an $n > 0$ and assume inductively that for $t < n$, the $n$-balls $Q_t$ have been defined. Suppose also that for $s < t$, the constructions of $A_s$, $g_s$, $\{a_s\}_{s=1}^n$, $H'_s$ and of $\{\alpha_s\}_{s=1}^n$ have been performed as described before. Then:

$$Q_n = \cap_{s=1}^n (Q_s),$$

where $s(a_n)$, $t(a_n)$ are determined according to (7.8) and $C$ denotes the component of $X - A_{n+1}$ (and also of $X - B_n$), whose closure contains $A_n$.

Thus, if $i > 0$, then $Q_n \subset C H_{n+1}$ and it follows from the definition of the star $H_{n+1}$ that the set $Q_n \cap H'_{n+1} - Q_n$ consists of two points, which will be denoted by $p_n$ and $q_n$. (4.4) and (4.3) imply that $H_n^s \cap A_n$ and $B_n^s \cap C H_{n+1}$. It follows from (7.9), (7.3) and (5.3) that $q_n \in B_n^s$. Therefore also $q_n \in B_n^s \cap H_{n+1}$. We conclude by (4.1) and (4.2) that there exist two arcs $H_{n+1}, H'_{n+1} \subset C H_{n+1}$ such that:

$$I_n = (q_n) \cup (p_n), \quad J_n = (q_n) \cup (p_n),$$

$$H_n = H_n \cap J_n = (H_n \cap J_n) \cup (H_n \cap J_n) = (q_n).$$

Finally, define inductively some sets $B_n, s > 0$, as follows:

$$B_n = B_n \cap B_n \cap (B_n \cup B_n \cup B_n)$$

and, if $s > 0$, let

$$f_n(x) = \begin{cases} f_n(x) & \text{for } x \in X - Q_n, \\ g_n(x) & \text{for } x \in X \cap B_n, \\ q_n & \text{for } x \in I_n \cup J_n. \end{cases}$$

By (7.9) and (7.12) the definitions of $f_n$ agree on the set $(U_n \cup V_n) \cap C H_{n+1} = (Q_n \cap C H_{n+1})$. In order to show that they agree on the set $Q_n \cap C H_{n+1} - Q_n = (p_n) \cup (q_n)$ we shall prove that $f_n(x) - f_n(y) = q_n$.

By (7.11) and by the definition of $p_n$, $Q_n = \cap_{t=1}^n (Q_t)$ is contained in the component of $H_{n+1} \cap A_{n+1}$ bounded by the point $a_n$, $a_n \in A_{n+1}$ (where $a_n = (a_n) \cup (a_n)$). Consequently, (7.8) and the definitions of $f_n$ yield $f_n(x) = f_n(y) = (q_n)$. Thus $f_n$ is a map.

We infer from the definition of $f_n$ and from the formula $d_n(A_n) = A_n$ that $f_n(Q_n \cap C H_{n+1} - Q_n = A_n$ for every $s > 0$. Since, by (7.2) and (7.3), $B_n = A_n \cup \cdots \cup A_n$, it follows that:

$$f_n(x) = B_n = f_n(A_n \cup \cdots \cup A_n).$$

Taking into consideration (7.11) and the definition of $f_n$, one can show by an inductive argument that:

$$f_n(x) = f_n(x) \cup f_n(x) \cup f_n(x) \cup f_n(x),$$

and, if $s > 0$, then $f_n$ and $f_n$ agree on the set $(X - \cup Q_n) \cup (X, \cup A_n)$ for $s > 0$ the sets $f_n(Q_n \cap C H_{n+1})$ and $f_n(Q_n \cap C H_{n+1})$ are contained in the closure of the same component of $X - A_n$.

Thus, we conclude from (7.6) that the sequence $g$ is uniformly convergent and $g = \lim_{n \to \infty} f_n$ is a map from $X$ into $Y$. By (7.14) and (7.15) we have $g(x) \cap B_n$, and therefore, by (7.5), $g$ is onto.

We shall prove that $g$ is the map required in the lemma. For this purpose, taking into consideration (7.15) and the definition of $f_n$ let us notice that:

$$g(A_n) = q_n \quad \text{for every } s > 0.$$

It follows from the definitions of $f_n$ and $r_n$ that, for every point $a_n$, $f_n^{-1}(a_n)$ is the union of $g_n(a_n)$ and of the necklace $r_n^{-1}(a_n) \subset H_n$. If $G$ is a component of $X - A_n$ bounded by $a_n$ and $q_n(a_n) = Q_n$, then, by (7.11) and (7.9), $a_n = a_n f_n^{-1}(a_n) \cap Q_n \cap C H_{n+1} = (Q_n \cap C H_{n+1})$. We conclude from (7.15) that:

$$g^{-1}(a_n) = \cup_{a_n} g_n(a_n) \cap C H_{n+1}$$

and, if $s > 0$, let

$$f_n(x) = \begin{cases} f_n(x) & \text{for } x \in X - Q_n, \\ g_n(x) & \text{for } x \in X \cap B_n, \\ q_n & \text{for } x \in I_n \cup J_n. \end{cases}$$

By (7.9) and (7.12) the definitions of $f_n$ agree on the set $(U_n \cup V_n) \cap C H_{n+1} = (Q_n \cap C H_{n+1})$. In order to show that they agree on the set $Q_n \cap C H_{n+1} - Q_n = (p_n) \cup (q_n)$ we shall prove that $f_n(x) - f_n(y) = q_n$.

By (7.11) and by the definition of $p_n$, $Q_n = \cap_{t=1}^n (Q_t)$ is contained in the component of $H_{n+1} \cap A_{n+1}$ bounded by the point $a_n$, $a_n \in A_{n+1}$ (where $a_n = (a_n) \cup (a_n)$). Consequently, (7.8) and the definitions of $f_n$ yield $f_n(x) = f_n(y) = (q_n)$. Thus $f_n$ is a map.

We infer from the definition of $f_n$ and from the formula $d_n(A_n) = A_n$ that $f_n(Q_n \cap C H_{n+1} - Q_n = A_n$ for every $s > 0$. Since, by (7.2) and (7.3), $B_n = A_n \cup \cdots \cup A_n$, it follows that:

$$f_n(x) = B_n = f_n(A_n \cup \cdots \cup A_n).$$

Taking into consideration (7.11) and the definition of $f_n$, one can show by an inductive argument that:

$$f_n(x) = f_n(x) \cup f_n(x) \cup f_n(x) \cup f_n(x),$$

and, if $s > 0$, then $f_n$ and $f_n$ agree on the set $(X - \cup Q_n) \cup (X, \cup A_n)$ for $s > 0$ the sets $f_n(Q_n \cap C H_{n+1})$ and $f_n(Q_n \cap C H_{n+1})$ are contained in the closure of the same component of $X - A_n$.

Thus, we conclude from (7.6) that the sequence $g$ is uniformly convergent and $g = \lim_{n \to \infty} f_n$ is a map from $X$ into $Y$. By (7.14) and (7.15) we have $g(x) \cap B_n$, and therefore, by (7.5), $g$ is onto.

We shall prove that $g$ is the map required in the lemma. For this purpose, taking into consideration (7.15) and the definition of $f_n$ let us notice that:

$$g(A_n) = q_n \quad \text{for every } s > 0.$$

It follows from the definitions of $f_n$ and $r_n$ that, for every point $a_n$, $f_n^{-1}(a_n)$ is the union of $g_n^{-1}(a_n)$ and of the necklace $r_n^{-1}(a_n) \subset H_n$. If $G$ is a component of $X - A_n$ bounded by $a_n$ and $g_n^{-1}(a_n) = Q_n$, then, by (7.11) and (7.9), $a_n = a_n f_n^{-1}(a_n) \cap Q_n \cap C H_{n+1} = (Q_n \cap C H_{n+1})$. We conclude from (7.15) that:

$$g^{-1}(a_n) = \cup_{a_n} g_n(a_n) \cap C H_{n+1}$$

and, if $s > 0$, let

$$f_n(x) = \begin{cases} f_n(x) & \text{for } x \in X - Q_n, \\ g_n(x) & \text{for } x \in X \cap B_n, \\ q_n & \text{for } x \in I_n \cup J_n. \end{cases}$$
Now, consider a point \( x \in X \). By (7.8), there are three possibilities: 
\( x \in A_k = A_\infty \) (for some \( k \geq 0 \)) and it is different from every point \( a_s \), 
\( x = a_s \) for some \( s \) and \( t \), or \( x \in X - \bigcup_{s=0}^\infty A_s \). If the first possibility holds 
and \( s_k > 0 \), then \( x \in \mathcal{O} \), where \( \mathcal{O} \) is the component of \( X - A_\infty \) such that 
\( \mathcal{O} \supset A_\infty \). Consequently, (7.17) implies that 
\( g^{-1}(x) \in \mathcal{O} \supset A_\infty \) and, 
since \( x \neq a_\infty \), we infer that 
\( g^{-1}(x) \subset A_\infty \). We conclude by (7.16) that 
\( g^{-1}(x) = g^{-1}_s(x) \) (also if \( s_k = 0 \)). If the second possibility holds, then 
(7.17) and the definition of \( a_s \) imply that 
\( g^{-1}(\infty) \) is an arc. If the third possibility holds, then 
(7.7) and (7.17) imply that there is a sequence \( s_k \) (not the 
same as in (7.7)) such that \( \lim s_k = \infty \) and such that 
\( g^{-1}(x) \subset \bigcap_{k=1}^\infty H_k \). Since, 
by (7.11), each \( Q_k \) (except \( Q_1 \)) is contained in some \( H_k \), it is easy to show, 
by (7.10) and by the definition of a star (see section 4), that 
\( \lim \text{diam}(Q_k) = 0 \). Thus, \( g^{-1}(x) \) is a point. Since each \( g_s \) satisfies condition 1\( ^* \) of 
the lemma (with respect to \( A_\infty \)), we conclude from these considerations that 
such \( g \) is a point.

Since each non-degenerate inverse image \( g^{-1}(x) \) is contained in some \( H_k \) 
using (7.10), the definition of a star and again the properties of \( g_s \), we infer 
that \( g \) satisfies condition 2\( ^* \) of the lemma.

To prove that it also satisfies 3\( ^* \), first observe that, by (7.2) and (7.3), 
each non-degenerate cyclic element of \( X \) is a cyclic element of a set \( A \). 
Next, the construction of \( X' \) implies that each non-degenerate cyclic element 
\( K' \) of \( X' \) is contained in a set \( A \). Indi\( \text{diam}(Q_k) = 0 \), and 
therefore there is an index \( k \) such that \( E' \subset Q_k \). Since \( E' \subset A_k \). 
Consequently, (7.16) and the respective properties of \( g_s \) imply that 
\( g \) satisfies 3\( ^* \). Thus, the proof of the lemma in case II is completed.

8. Proof of the lemma in case III. Let \( X \) be any space satisfying 
the assumptions of the lemma with \( n > 1 \) (see section 5). Denote 
by \( A \) the least closed and entirely arcwise connected subset of \( X \) 
containing all non-degenerate cyclic elements of \( X \). By the result of section 7, 
we can assume that \( X - A = \varnothing \) (however, it may happen that 
\( A = \varnothing \), and then, in view of (8.1), the proof is trivial). Let us notice that:

(8.1) If \( \mathcal{O} \) is a component of \( X - A \), then \( \mathcal{O} \) is a dendrite.

Indeed, it is clear in view of (3.2) that \( \mathcal{O} \) is a locally connected 
continuum. By (3.9) and by the definition of \( A \), \( \mathcal{O} \) does not contain any 
simple closed curve, and therefore it is a dendrite.

From (8.1) and from the result of section 7, we infer that there exist 
a set \( A' \subset E' \) and a map \( g \) from \( A' \) onto \( A \) satisfying the conclusion 
of the lemma with respect to \( A \). In virtue of (3.3), the points of \( A \) which

bound some components of \( X - A \) can be ordered in a sequence \( \{a_i\}_i \), 
where \( k \) is a natural number or \( \infty \). Let \( \{\mathcal{O}_i\}_i \) be a sequence consisting 
of all components of \( X - A \) bounded by \( a_i \). For each point \( a_i \), choose a 
point \( a_i^* \in g^{-1}_i(a_i) \) such that, if \( g^{-1}_i(a_i) \) is an arc, then \( a_i^* \) is one of its end-points. By (5.3), 
\( a_i^* \in Bd(A') \). Consequently, in virtue of (5.2), (4.1) and (4.2), every point \( a_i^* \) 
is accessible from \( E' - A' \). Since \( n > 1 \) and in view of (8.1), one can easily 
conclude by induction a sequence of sets \( D_i \subset E' \) such that:

(8.2) \( D_i \cap A' = \{a_i^*\} \), the pair \( (D_i, a_i^*) \) is homeomorphic with the 
pair \( (\mathcal{O}_i, a_i) \), \( (D_i - a_i^*) \cap (D_{i} - a_i^*) = \varnothing \) if \( (i_1, j_1) \neq (i_2, j_2) \); 
\( \text{diam}(D_i) < \frac{1}{I} \) and \( \text{lim diam}(D_0) = 0 \) if \( J(k) = \infty \).

Let 
\[ X' = A' \cup \bigcup_{i=0}^{\infty} D_i \].

Since \( A' \) is a locally connected continuum, it follows from (8.2) 
that \( X' \) is also one. In virtue of (8.2) there is a homeomorphism \( g \) from \( D_i \)
onto \( \mathcal{O}_i \) such that \( g(a_i) = a_i^* \). Define a function \( g' : X' \to X \) as follows:

\( g'(x') = \begin{cases} 
g(x') & \text{if } x' \in A' \\
g_0(x') & \text{if } x' \in D_i \end{cases} \)

One can easily prove using (8.2) and the formula \( g(A') = A \) that \( g \) 
is a map from \( X' \) onto \( X \). It follows from the definition of \( g \) that \( g^{-1}(x) \) 
is a point if \( x \in A \) and \( g^{-1}(x) \) is a point if \( x \in X - A \). Since the map \( g \) satisfies conditions 1\( ^* \) and 2\( ^* \) of the lemma, it follows that \( g \) is a neat point of \( X \). By (8.2) and from the definition of \( X' \) we infer that the non-degenerate cyclic elements of \( X' \) coincide with the 
non-degenerate cyclic elements of \( A' \). Evidently, the non-degenerate cyclic elements of \( X \) coincide with the 
non-degenerate cyclic elements of \( A \). Since \( A' \) and \( A \) satisfy to condition 
3\( ^* \) of the lemma, we conclude that so does \( X' \) and \( g \). Thus, the proof 
of the lemma, and therefore also of theorem 1, is completed.

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Structure of self-dual torsion-free metric LCA groups
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Since Pontrjagin [3] and Van Kampen [5] introduced the notion of the dual of a locally compact Abelian group, many examples of self-dual LCA groups have been given in the literature. However, the structure of all self-dual LCA groups has been an open problem till to-day (see [1]). As a matter of fact, there is even no conjecture about how a self-dual LCA group should look like. In this paper we give the structure of all metric self-dual LCA groups which are torsion-free as abstract groups.

Notations and Conventions. All topological spaces occurring in this paper are taken to be Hausdorff ones. We usually follow [7] for notations and concepts related to topological groups which are not defined here. We write LCA group as an abbreviation for a locally compact Abelian group. The dual of the LCA group $\hat{G}$ with the usual topology is denoted by $\hat{G}$. We use the additive notation for groups. If $H \subset G$ is a subgroup of the LCA group $G$, then $H^+$ denotes the annihilator of $H$ in $\hat{G}$, $K^n$ denotes the usual Euclidean group (n > 0). If $p$ is a prime, then $J_p$ denotes the group of all $p$-adic numbers and $L_p$ the group of all $p$-adic integers with the usual topology. (We use the symbol $\otimes$ for topological direct sums). The definition of a local direct sum of LCA groups is given in [1], [6] and [4]. But we prefer to repeat this definition here for the sake of completeness.

Definition 1. Let $(G_\alpha)$ be a family of LCA groups indexed by a set $\mathcal{A}$. Let $H_\alpha \subset G_\alpha$ be a compact and open subgroup of $G_\alpha$ for each $\alpha \in \mathcal{A}$. We define the local direct sum $\sum_{\alpha \in \mathcal{A}} G_\alpha$ of the family $(G_\alpha)$ with respect to $(H_\alpha)$ of subgroups as follows:

$$\sum_{\alpha \in \mathcal{A}} G_\alpha = \left\{ (x_\alpha) \in \prod_{\alpha \in \mathcal{A}} G_\alpha : x_\alpha \in H_\alpha \right\}$$

for all $\alpha \in \mathcal{A}$ except possibly for a finite number of indices.

* An announcement of the result presented here appeared in [4]. The main theorem there should have been only for the metric case instead of for all the groups.

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