[4] H. Hermes, *Einführung in die Verbandstheorie*, Springer 1955.

[5] J. Łoś, *Linear equations and pure subgroups*, Bull. Acad. Polon. Sci. 7 (1959), pp. 13-18.

[6] B. H. Neumann and S. Yamamuro, *Boolean powers of simple groups*, J. Australian M. S. 5 (1965), pp. 315-324.

[7] P. Ribenboim, *Sur quelques constructions de groupes réticulés et l'équivalence logique entre l'affinement de filtres et d'ordres*, Summa Brasil Math. 4 (1958), pp. 65-89.

[8] — *Fonctions modulo antifiltres*, Math. Z. 77 (1961), pp. 195-206.

[9] — *Théorie des groupes ordonnés*, Univ. Nac. del Sur, Bahía Blanca 1963.

Reçu par la Rédaction le 17. 6. 1967

Generalized group cohomology*

by

Morton E. Harris (Chicago, Ill.)

Introduction

A permutation representation (G, X) of a group G will consist of a non-empty set X with G acting on the left such that $(\varrho\sigma)x = \varrho(\sigma x)$ for all $\varrho, \sigma \in G$ and all $x \in X$ and such that $ex = x$ for all $x \in X$ where e denotes the identity element of G.

When (G, X) is a finite permutation representation (i.e., when X is a finite set) a cohomology theory is defined and investigated in a series of papers by Snapper ([9], [10], [11], [12], [13]). The results of [13] are an application of this cohomology theory to the study of Frobenius groups.

When the finite permutation representation (G, X) is fixed point free (i.e., $\sigma x = x$ for $x \in X$ and $\sigma \in G$ implies $\sigma = e$) then this cohomology theory is just the ordinary cohomology theory for finite groups.

This cohomology theory of (finite) permutation representations is a generalization to not necessarily transitive permutation representations of the cohomology theory of [1].

These cohomology theories of permutation representations are defined by means of a "standard complex". The cohomology theory of [1] has been investigated in terms of relative homological algebra in [5].

Using recent developments in relative homological algebra, we investigate the cohomology theory of finite permutation representations of [9], [10], [11], [12] and [13]. This investigation generalizes that of [5] and the well known homological algebraic foundations of the ordinary cohomology theory of finite groups.

Our investigation will permit straightforward (standard categorical) derivations of all of the results of [9] and [10], some generalizations of these results, some new results, as well as generalizations to not necessarily

* This research was supported in part by the National Science Foundation Grants NSF-GP5276 and NSF-GP6539. The author wishes to express his gratitude to Professor Ernst Snapper for his advice and suggestions.

transitive permutation representations of the results of [11]. In fact, most of the results of [3], Chapter XII will be generalized to this theory of cohomology of permutation representations. Thus the cohomology theory of permutation representations (which includes the ordinary cohomology theory) has been put into an axiomatic framework which elucidates the entire theory.

However, in this paper we will only present the relative homological algebraic background, the results of [9] in this setting and some new results. In a future paper entitled *Cup product, duality and periodicity for generalized group cohomology* we shall complete the above mentioned program.

In Chapter I of this paper we describe the relative homological algebra background, explain its connection with permutation representations, and interpret the $G|K$-regular and $G|K$-special modules of [9].

In Chapter II, we define a general cohomology theory and quote [9] to show its existence in two cases. When the (finite) permutation representation (G, X) is fixed point free then the cohomology theory is just the ordinary group cohomology theory and the axiomatizations become the absolute homological and relative Z-split homological axiomatizations, as they should be.

These relative homological algebra axiomatizations are used, in the final sections of this chapter, to study the passage to a quotient group and to study the relations with a subgroup.

In Chapter III, the connection between the cohomology theory of permutation representations and the relative deriveds of the functor Hom_G is studied. Finally, we indicate how the spectral sequences of [9], Chapters 2 and 3 are consequences.

For any group K, we shall denote the category of left K-modules by ${}_K\mathfrak{C}$ and for any $A \in {}_K\mathfrak{C}$, we shall let $A^K = \{a \in A \mid ka = a \text{ for all } k \in K\}$. The category of Abelian groups will be denoted by Ab and Z will denote the rational integers. If Z is considered as a module over a group, it will always be considered to have trivial group action.

I. Foundations

§ 1. The relative homological algebra background. Let $\mathfrak{H} = \{H\}$ be an arbitrary collection of subgroups of the group G. For each $H \in \mathfrak{H}$, we have the "forgetful" functor T_H: ${}_G\mathfrak{C} \to {}_H\mathfrak{C}$ and the functor S_H: ${}_H\mathfrak{C} \to {}_G\mathfrak{C}$ defined by $S_H(B) = Z[G] \otimes_H B$. As observed in [4], Chapter III, § 4, T_H is the adjoint of S_H. Moreover, if the class of all split exact sequences (as defined in [4], pp. 3-4) of ${}_H\mathfrak{C}$ is denoted by $\mathcal{E}_0(H)$, then, by [4], Chapter II, Theorem 3.1 $\mathcal{E}_0(\mathfrak{H}) = \bigcap_{H \in \mathfrak{H}} T_H^{-1}(\mathcal{E}_0(H))$ is a projective class in ${}_G\mathfrak{C}$ and the $\mathcal{E}_0(\mathfrak{H})$-projective objects are all retracts (direct summands) of all direct sums of the form:

$$\bigoplus_{H \in \mathfrak{H}} \big(Z[G] \otimes_H ({}_H B)\big) \quad \text{where} \quad {}_H B \in {}_H\mathfrak{C}\,.$$

Similarly, for each $H \in \mathfrak{H}$, the functor S'_H: ${}_H\mathfrak{C} \to {}_G\mathfrak{C}$ given by $S'_H(B) = \mathrm{Hom}_H(Z[G], B)$ is the adjoint of T_H. If $\mathcal{E}^0(H)$ denotes the class of all cosplit coexact sequences of ${}_H\mathfrak{C}$, then $\mathcal{E}^0(\mathfrak{H}) = \bigcap_{H \in \mathfrak{H}} T_H^{-1}(\mathcal{E}_0(H))$ is an injective class in ${}_G\mathfrak{C}$ and the injective objects are all retracts of all direct products of the form:

$$\prod_{H \in \mathfrak{H}} \mathrm{Hom}_H(Z[G], {}_H B) \quad \text{where} \quad {}_H B \in {}_H\mathfrak{C}\,.$$

Also, $\mathcal{E}_0(\mathfrak{H})$ and $\mathcal{E}^0(\mathfrak{H})$ are complementary classes (i.e., they both contain the same sequences which are unlimited in both directions).

The short exact sequences of $\mathcal{E}_0(\mathfrak{H})$ and $\mathcal{E}^0(\mathfrak{H})$ comprise the same class $P(\mathfrak{H})$. These are the short exact sequences E of ${}_G\mathfrak{C}$ such that $T_H(E)$ is split exact for all $H \in \mathfrak{H}$. It is obvious that $P(\mathfrak{H})$ is a proper class of short exact sequences in ${}_G\mathfrak{C}$ in the sense of [7], Chapter XII, § 4. Since $\mathcal{E}_0(\mathfrak{H})$ has enough projectives, it follows that the class $P(\mathfrak{H})$ has enough projectives and similarly for injectives.

If $\mathfrak{H}, \mathfrak{H}_1$ are collections of subgroups of G such that for every $H \in \mathfrak{H}$ there exists an $x \in G$ and a $K \in \mathfrak{H}_1$, such that $H \subseteq xKx^{-1}$, then write $\mathfrak{H} \leqslant \mathfrak{H}_1$. This relation is clearly a quasi-ordering. If both $\mathfrak{H} \leqslant \mathfrak{H}_1$ and $\mathfrak{H}_1 \leqslant \mathfrak{H}$, then we write $\mathfrak{H} \cong \mathfrak{H}_1$; this is an equivalence relation.

The following lemma is straightforward:

LEMMA 1.1.1. *If K, H are subgroups of the group G such that $H \subseteq xKx^{-1}$ for some $x \in G$, then if E: $A \to B \to C$ is a sequence in ${}_G\mathfrak{C}$ such that $T_K(E)$ is split exact in ${}_K\mathfrak{C}$, then $T_H(E)$ is split exact in ${}_H\mathfrak{C}$ and similarly for a cosplit coexact sequence.*

Thus if $\mathfrak{H} \leqslant \mathfrak{H}_1$, then $\mathcal{E}_0(\mathfrak{H}_1) \subseteq \mathcal{E}_0(\mathfrak{H})$, $\mathcal{E}^0(\mathfrak{H}_1) \subseteq \mathcal{E}^0(\mathfrak{H})$ and $P(\mathfrak{H}_1) \subseteq P(\mathfrak{H})$ with equality for all three inclusions if $\mathfrak{H} \cong \mathfrak{H}_1$.

There is yet another class of short exact sequences which we shall need. Let H be a subgroup of G and let Z denote the abelian group of rational integers with trivial H-action. Then applying [4], Chapter II, Proposition 5.1, the sequences of ${}_H\mathfrak{C}$ which are exact in Ab when the functor $\mathrm{Hom}_H(Z, *)$ is applied form a projective class $\mathfrak{F}(H)$ in ${}_H\mathfrak{C}$ with projective objects all free abelian groups with trivial H-action. If $\mathfrak{H} = \{H\}$ is a collection of subgroups of G, then $\mathfrak{F}(\mathfrak{H}) = \bigcap_{H \in \mathfrak{H}} T_H^{-1}(\mathfrak{F}(H))$ is a projective class in ${}_G\mathfrak{C}$ with projectives being all retracts of all direct sums of the form:

$$\bigoplus_{H \in \mathfrak{H}} \big(Z[G] \otimes_H ({}_H B)\big)$$

where ${}_H B \in {}_H\mathfrak{C}$ is a free abelian group with trivial H-action. A sequence $A \to B \to C$ in ${}_G\mathfrak{C}$ is in $\mathfrak{F}(\mathfrak{H})$ if and only if the sequence $A^H \to B^H \to C^H$ is exact in Ab for all $H \in \mathfrak{H}$.

Let $Q(\mathfrak{H})$ denote the short exact sequences of $\mathfrak{F}(\mathfrak{H})$. We claim that $Q(\mathfrak{H})$ is a proper class. For, a short exact sequence E: $0 \to A \to B \to C \to 0$ of ${}_G\mathfrak{C}$ is in $Q(\mathfrak{H})$ if and only if $T_H(E) \in \mathfrak{F}(H)$ for all $H \in \mathfrak{H}$. But the short exact sequences of $\mathfrak{F}(H)$ form a proper class in ${}_H\mathfrak{C}$, as is well known (see [7], p. 371, Exercise 6). Hence, $Q(\mathfrak{H})$ is a proper class in ${}_G\mathfrak{C}$. Moreover, we have $\mathcal{E}_0(\mathfrak{H}) \subseteq \mathfrak{F}(\mathfrak{H})$ and $P(\mathfrak{H}) \subseteq Q(\mathfrak{H})$.

For example, if $G \in \mathfrak{H}$, then $P(\mathfrak{H})$ is the class of all split exact sequences of ${}_G\mathfrak{C}$. If $\mathfrak{H}$ consists only of the identity subgroup, then $P(\mathfrak{H})$ is the class of Z-split short exact sequences of ${}_G\mathfrak{C}$ and $Q(\mathfrak{H})$ is the class of all short exact sequences of ${}_G\mathfrak{C}$.

§ 2. The relationship with permutation representations. As before, $\mathfrak{H}$ denotes any collection of subgroups of G; let $\mathfrak{A}$ denote the set of all such $\mathfrak{H}$. Let $\mathfrak{B}$ denote the class of all permutation representations (G, X) of G. Consider the mapping f: $\mathfrak{B} \to \mathfrak{A}$ such that $f\big((G, X)\big)$ is the set of all subgroups of G which fix a point of X. Clearly, f maps $\mathfrak{B}$ onto the set of all $\mathfrak{H} \in \mathfrak{A}$ which are closed under conjugation. Moreover, if X is finite, then $f\big((G, X)\big)$ is a finite set of subgroups of finite index in G which is closed under conjugation. Conversely, given any such set of subgroups $\mathfrak{H}$ then there exists a finite permutation representation (G, X) such that $f\big((G, X)\big) = \mathfrak{H}$.

To say that a permutation representation (G, X) is fixed point free is equivalent to saying that $f\big((G, X)\big) = \mathfrak{H}$ consists of just the identity subgroup. Also, X has an element fixed by every element of G if and only if $G \in f\big((G, X)\big)$.

§ 3. $G|K$-regular and $G|K$-special modules. For any subgroup K of G of finite index, the notion of a $G|K$-regular G-module has been defined in [9], § 11. It is easy to see that (when K is of finite index) this is equivalent to:

Definition 1.3.1. If K is any subgroup of G, then a G-module A is said to be *$G|K$-regular* if A is isomorphic in ${}_G\mathfrak{C}$ to a module of the form $Z[G] \otimes_K B$ for some $B \in {}_K\mathfrak{C}$.

Thus, the $G|H$-regular modules for $H \in \mathfrak{H}$ have been mentioned in § 1.

Proposition 1.3.1 *Let L be a subgroup of K; then any $G|L$-regular module is $G|K$-regular.*

Proof. $Z[G] \otimes_L B \cong Z[G] \otimes_K (Z[K] \otimes_L B)$ in ${}_G\mathfrak{C}$.

This implies [9], Proposition 11.1.

For the rest of this section, we shall assume that $\mathfrak{H} \cong \mathfrak{H}^*$ where $\mathfrak{H}^*$ consists of only one subgroup K.

Lemma 1.3.1 *$A \in {}_G\mathfrak{C}$ is $\mathcal{E}_0(\mathfrak{H})$-projective if and only if the G-mapping $\alpha_K(A)$: $Z[G] \otimes_K A \to A$ as defined in* [4], *p. 27 has a G-coretraction.*

Proof. Assume that A is $\mathcal{E}_0(\mathfrak{H})$-projective. Since $T_K(A) \xrightarrow{1} T_K(A) \to 0$ is in $\mathcal{E}_0(K)$, the proof of [4], Chapter II, Theorem 2.1 may be applied to demonstrate that $\alpha_K(A)$ has a G-coretraction. The reverse implication has already been mentioned.

Dually we have:

Lemma 1.3.2. *$A \in {}_G\mathfrak{C}$ is $\mathcal{E}^0(\mathfrak{H})$-injective if and only if the G-mapping $\alpha'_K(A)$: $A \to \mathrm{Hom}_K(Z[G], A)$ as defined in* [4], *p. 27 has a G-retraction.*

For the rest of this section we shall further assume that K is a subgroup of G of finite index.

Let $G = \bigcup_{i=1}^{n} x_i K$ be the corresponding left coset decomposition of G where $x_1, \ldots, x_n$ is a left coset representative choice. As usual, for any $A \in {}_G\mathfrak{C}$ the trace mapping $S_{G|K}$: $A^K \to A^G$ is defined by $S_{G|K}(a) = \sum_{i=1}^{n} x_i a$ and is independent of the K-coset representative choice. If $A, B \in {}_G\mathfrak{C}$ and if $\mathrm{Hom}_Z(A, B)$ is viewed as a left G-module in the usual way, then for $u \in \mathrm{Hom}_Z(A, B)^K = \mathrm{Hom}_K(A, B)$, $\big(S_{G|K}(u)\big)(a) = \sum_{i=1}^{n} x_i u(x_i^{-1} a)$ for all $a \in A$.

Using this trace mapping, we define, as in [9], § 10:

Definition 1.3.2. A G-module A is said to be *$G|K$-special* if there exists a K-homomorphism u: $A \to A$ such that $S_{G|K}(u) = 1_A$.

Under these hypotheses on $\mathfrak{H}$ (i.e. $\mathfrak{H} \cong \mathfrak{H}^*$ where $\mathfrak{H}^*$ consists only of one subgroup K of finite index in G):

Theorem 1.3.1. *For any $A \in {}_G\mathfrak{C}$, the following are equivalent:*

1) *A is $G|K$-special.*

2) *A is $\mathcal{E}_0(\mathfrak{H})$-projective.*

3) *A is $\mathcal{E}^0(\mathfrak{H})$-injective.*

The following five lemmas provide a proof of this theorem:

Lemma 1.3.3. *If $B \in {}_K\mathfrak{C}$, then the G-modules $Z[G] \otimes_K B$ and $\mathrm{Hom}_K(Z[G], B)$ are G-isomorphic.*

Proof. Let $G = \bigcup_{i=1}^{n} x_i K$ be as above. It is easily seen that the G-mapping β: $\mathrm{Hom}_K(Z[G], B) \to Z[G] \otimes_K B$ given by $\beta(f) = \sum_{i=1}^{n} x_i \otimes_K f(x_i^{-1})$ for $f \in \mathrm{Hom}_K(Z[G], B)$ has an inverse in ${}_G\mathfrak{C}$.

Lemma 1.3.4. *If $A \in {}_G\mathfrak{C}$, then $Z[G] \otimes_K A$ is $G|K$-special.*

Proof. Let $G = \bigcup_{i=1}^{n} x_i K$ where $x_1 = 1$; then

$$Z[G] = \bigoplus_{i=1}^{n} (x_i Z[K]) \cong \bigoplus_{i=1}^{n} Z[K] \quad \text{as a right } K\text{-module}.$$

Let $\pi\colon Z[G] \to Z[K]$ denote the projection onto the first component. It is easy to see that π is a homomorphism in ${}_K\mathfrak{C}$ and viewing $\pi\colon Z[G] \to Z[G]$, we have

$$S_{G|K}(\pi \otimes_K 1_A)\Big(\sum_{i=1}^{n} x_i \otimes_K a_i\Big) = \sum_{i,j=1}^{n} \big(x_j(\pi(x_j^{-1}x_i)) \otimes_K a_i\big) = \sum_{i=1}^{n} x_i \otimes_K a_i$$

for all $a_i \in A$.

Lemma 1.3.5. *A G-direct summand of a $G|K$-special module is $G|K$-special.*

Proof. Standard.

Lemma 1.3.6. *If $A \in {}_G\mathfrak{C}$ is $G|K$-special, then the mapping $\alpha_K(A)\colon Z[G] \otimes_K A \to A$ in ${}_G\mathfrak{C}$ as defined in* [4], p. 27 *has a coretraction in ${}_G\mathfrak{C}$.*

Proof. Let $u \in \mathrm{Hom}_K(A, A)$ be such that $S_{G|K}(u) = 1_A$. Defined $a\colon A \to Z[G] \otimes_K A$ by $a(a) = \sum_{i=1}^{n} x_i \otimes_K u(x_i^{-1}a)$ for $a \in A$; then, a is independent of the coset representative choice for K in G and $\alpha_K(A)\,a = 1_A$. Also, if $g \in G$, then $\{x_i' = gx_i \mid 1 \leqslant i \leqslant n\}$ is also a left K-coset representative choice and hence

$$ga(a) = \sum_{i=1}^{n} gx_i \otimes_K u(x_i^{-1}a) = \sum_{i=1}^{n} x_i' \otimes_K u\big((x_i')^{-1}ga\big) = a(ga).$$

Thus, a is a G-homomorphism.

Lemma 1.3.7. *If $A \in {}_G\mathfrak{C}$ is $G|K$-special, then the mapping $\alpha_k'(A)\colon A \to \mathrm{Hom}_K(Z[G], A)$ in ${}_G\mathfrak{C}$ as defined in* [4], p. 27 *has a retraction in ${}_G\mathfrak{C}$.*

Proof. Let $u \in \mathrm{Hom}_K(A, A)$ be such that $S_{G|K}(u) = 1_A$. Define $\delta\colon \mathrm{Hom}_K(Z[G], A) \to A$ by $\delta(f) = \sum_{i=1}^{n} x_i\big((uf)(x_i^{-1})\big)$ for $f \in \mathrm{Hom}_K(Z[G], A)$. An argument similar to that in Lemma 1.3.6 shows that δ is the required map in ${}_G\mathfrak{C}$.

This last lemma is implied by Lemmas 1.3.2, 1.3.3 and 1.3.6, but is included for completeness.

Corollary 1.3.1. *A $G|K$-regular module is $G|K$-special.*

This is Proposition 11.2 of [9].

Corollary 1.3.2. *If A, E are two G-modules such that A is $G|K$-regular, then the G-module $\mathrm{Hom}_Z(A, E)$ is $G|K$-regular.*

Proof. Let $A = Z[G] \otimes_K B$ in ${}_G\mathfrak{C}$. Then, the isomorphism $s'\colon \mathrm{Hom}_K(Z[G], \mathrm{Hom}_Z(B, E)) \to \mathrm{Hom}_Z(Z[G] \otimes_K B, E)$ of [3], Chapter II, Proposition 5.21 is a G-isomorphism and the result follows from Lemma 1.3.3.

This is Proposition 11.4 of [9].

II. A general theory of group cohomology

§ 1. Definitions. Let G be a group, let $A \in {}_G\mathfrak{C}$ and let H be a subgroup of finite index in G. In Chapter I, § 3, the trace mapping $S_{G|H}\colon A^H \to A^G$ has been defined. If $\mathfrak{H} = \{H\}$ is any collection of subgroups of finite index, then for any $A \in {}_G\mathfrak{C}$ we define:

$$U_{\mathfrak{H}}(A) = A^G \Big/ \sum_{H \in \mathfrak{H}} S_{G|H}(A^H).$$

Thus $U_{\mathfrak{H}}$ is a covariant functor from ${}_G\mathfrak{C}$ into Ab.

The following lemma is easy to prove.

Lemma 1.1.1. *If H, K are subgroups of finite index in G such that $H \subseteq xKx^{-1}$ for some $x \in G$ and if $A \in {}_G\mathfrak{C}$, then $S_{G|H}(A^H) \subseteq S_{G|K}(A^K)$.*

Now assume that $\mathfrak{H}$, $\mathfrak{H}_1$ are collections of subgroups of G consisting solely of subgroups of finite index in G such that $\mathfrak{H} \leqslant \mathfrak{H}_1$, then for any $A \in {}_G\mathfrak{C}$ there is a cannonical epimorphism $U_{\mathfrak{H}}(A) \to U_{\mathfrak{H}_1}(A)$ and hence, there is a natural transformation of functors $U_{\mathfrak{H}} \to U_{\mathfrak{H}_1}$. If $\mathfrak{H} \cong \mathfrak{H}_1$, then clearly $U_{\mathfrak{H}} = U_{\mathfrak{H}_1}$.

Let P be a proper class of short exact sequences of ${}_G\mathfrak{C}$ in the sense of [7], Chapter XII, § 4 and let $\mathfrak{H}$ be any collections of subgroups of G of finite index.

Definition 2.1.1. A $(P, \mathfrak{H})$ *cohomology theory for* ${}_G\mathfrak{C}$ is a sequence of covariant additive functors $\{F^n \mid n \in Z\}$ from ${}_G\mathfrak{C}$ into Ab, together with functions which assign to each proper short exact sequence $E\colon 0 \to A \to B \to C \to 0$, morphisms $E_*^n\colon F^n(C) \to F^{n+1}(A)$ in Ab such that (F^n, E_*^n, F^{n+1}) is a P-connected pair which is both left P-couniversal and right P-universal in the sense of [7], Chapter XII, § 7 for all $n \in Z$ and such that F^0 is naturally equivalent with $U_{\mathfrak{H}}$.

It follows from universality that if a $(P, \mathfrak{H})$ cohomology theory exists for ${}_G\mathfrak{C}$, then it is unique up to isomorphism of doubly infinite P-connected sequences of functors. Moreover, if we define the index, $d(\mathfrak{H})$, of $\mathfrak{H}$ to be the greatest common divisor of $\{|G:H| \mid H \in \mathfrak{H}\}$, then since $|G:H|\ A^G \subseteq S_{G|H}(A^H)$, we have by the usual universality argument:

Theorem 2.1.1. *If $\{F^n \mid n \in Z\}$ is a $(P, \mathfrak{H})$ cohomology theory, then $d(\mathfrak{H})F^n(A) = 0$ for all $n \in Z$ and all $A \in {}_G\mathfrak{C}$.*

Observe that if G is a finite group and $\mathfrak{H}$ consists solely of the identity subgroup, then $U_{\mathfrak{H}}(A) = A^G/NA$ for any $A \in {}_G\mathfrak{C}$ ($N = S_{G|1}$ denotes the usual norm). If, further, P consists of all short exact sequences of ${}_G\mathfrak{C}$, then the ordinary cohomology groups form a $(P, \mathfrak{H})$ cohomology theory. The same is true if P consists of all short exact sequences of ${}_G\mathfrak{C}$ which are Z-split (as we shall prove).

§ 2. Existence. Now, we prove the existence of two cohomology theories which will turn out to be the cohomology theory of [9]. These methods will generate new results in this theory and will shed new light on this theory.

For the cohomology theories in which we are interested, we restrict ourselves to a finite permutation representation (G, X) (i.e. X is a finite set). Thus, $f\big((G, X)\big) = \mathfrak{H}$ is a finite set of subgroups of finite index in G and is closed under conjugation. On the other hand, if we assume that such a set $\mathfrak{H}$ of subgroups is given, we can then construct a permutation representation (G, X) such that $f\big((G, X)\big) = \mathfrak{H}$. In either case, we assume (G, X) and $\mathfrak{H}$ with the above properties such that $f\big((G, X)\big) = \mathfrak{H}$.

If $\bar{\mathfrak{H}}$ denotes the set of subgroups obtained by adjoining to $\mathfrak{H}$ all finite intersections of its elements and if $\mathfrak{H}_0$ denotes the set of subgroups formed by choosing one subgroup from each conjugacy class of $\mathfrak{H}$, then $\mathfrak{H}_0 \cong \mathfrak{H} \cong \bar{\mathfrak{H}}$ and $\mathfrak{H}_0, \bar{\mathfrak{H}}$ are finite sets of subgroups of finite index in G. Thus, cohomology theories for $\big(P(\mathfrak{H}), \mathfrak{H}\big)$; $\big(P(\bar{\mathfrak{H}}), \bar{\mathfrak{H}}\big)$ and $\big(P(\mathfrak{H}_0), \mathfrak{H}_0\big)$ are the same if they exist.

LEMMA 2.2.1. *Let (G, T) be a transitive permutation representation of G. If $e \in T$ and $H = \{g \in G|\ ge = e\}$, then $Z[G]\otimes_H Z$ and $Z[T]$ are isomorphic in ${}_G\mathfrak{C}$. (Here, Z is a trivial H-module and $Z[T]$ is as defined on p. 135 of [9].)*

Proof. Let $G = \bigcup_{i\in I} x_i H$ be the left coset decomposition of H in G where $\{x_i|\ i \in I\}$ is a left coset representative choice. Every element $a \in Z[G]\otimes_H Z$ can be written uniquely in the form $a = \sum_{i\in J} x_i \otimes_H z_i$ for a unique finite subset J of I. It is easy to see that the G-map $f\colon Z[G]\otimes_H Z \to Z[T]$ given by $f(a) = f\big(\sum_{i\in J} x_i\otimes_H z_i\big) = \sum_{i\in J} z_i x_i e$ has an inverse in ${}_G\mathfrak{C}$.

THEOREM 2.2.1. *A $\big(P(\mathfrak{H}), \mathfrak{H}\big)$ cohomology theory exists.*

Proof. Let r be a positive integer and let $\{T_1, \ldots, T_u\}$ denote the domains of transitivity of the permutation representation (G, X^r). Then it is clear that the left G-module $Z[X^r]$ (as defined on p. 135 of [9]) is isomorphic in ${}_G\mathfrak{C}$ to $\bigoplus_{i=1}^{u} Z[T_i]$. But the subgroup of G fixing an element of X^r is always a member of $\bar{\mathfrak{H}}$; thus, Lemma 2.2.1 demonstrates that $Z[X^r]$ is an $\mathcal{E}_0(\bar{\mathfrak{H}}) = \mathcal{E}_0(\mathfrak{H}) = \mathcal{E}(\mathfrak{H}_0)$-projective. Since the "standard complex of the permutation representation (G, X)" of [9], p. 135 has a contracting homotopy for each $H \in \mathfrak{H}$ ([9], Proposition 1.1), this complex is in $\mathcal{E}_0(\mathfrak{H}) \cap \mathcal{E}^0(\mathfrak{H})$ and the objects of the complex are $\mathcal{E}_0(\mathfrak{H})$-projectives. Therefore, the set of functors $\{H^n(X; G, \ast)|\ n \in Z\}$ as defined in [9], p. 137 is an exact doubly infinite $P(\mathfrak{H}) = P(\mathfrak{H}_0) = P(\bar{\mathfrak{H}})$-connected sequence of functors. Moreover, the results of [9], Chapter I, § 4 show that the functors $H^0(X; G, \ast)$ and $U_{\mathfrak{H}_0} = U_{\mathfrak{H}} = U_{\bar{\mathfrak{H}}}$ are naturally equivalent and Lemma 1.3.3 shows that the $\mathcal{E}_0(\mathfrak{H})$-projectives and the $\mathcal{E}^0(\mathfrak{H})$-injectives are the same; hence, the functors $H^n(X; G, \ast)$ vanish on these objects. (This, by the way, implies [9], Proposition 11.3.) But $\mathcal{E}_0(\mathfrak{H})$ has enough projectives and $\mathcal{E}^0(\mathfrak{H})$ has enough injectives and so the $P(\mathfrak{H})$-connected pair of functors $\big(H^n(X; G, \ast), H^{n+1}(X; G, \ast)\big)$ is both left $P(\mathfrak{H})$-couniversal and right $P(\mathfrak{H})$-universal by [7], Chapter XII, Theorems 7.2 and 7.6.

LEMMA 2.2.2. *Let $0 \to A \xrightarrow{u} B \xrightarrow{v} C \to 0$ be a short exact sequence of $Q(\bar{\mathfrak{H}})$. Then there exists a short exact sequence $0 \to A \xrightarrow{\bar{u}} \bar{B} \xrightarrow{\bar{v}} \bar{C} \to 0$ in $P(\mathfrak{H})$, which is the bottom row in a commutative diagram*

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "u"] \arrow[d, "1"] & B \arrow[r, "v"] \arrow[d, "\alpha"] & C \arrow[r] \arrow[d, "\beta"] & 0 \\
0 \arrow[r] & A \arrow[r, "\bar{u}"] & \bar{B} \arrow[r, "\bar{v}"] & \bar{C} \arrow[r] & 0
\end{tikzcd}$$

such that $H^n(X; G, \bar{u}) = 0$ for all $n \in Z$.

Proof. Let $A \xrightarrow{u_1} B_1$ be a $P(\mathfrak{H})$-monic with B_1 a $P(\mathfrak{H})$-injective and let

$$\begin{tikzcd}
A \arrow[r, "u_1"] \arrow[d, "u"] & B_1 \arrow[d, "w_1"] \\
B \arrow[r, "w"] & \bar{B}
\end{tikzcd}$$

be the pushout of the diagram

$$\begin{tikzcd}
A \arrow[r, "u_1"] \arrow[d, "u"] & B_1 \\
B &
\end{tikzcd}$$

Then it is easy to see that $\bar{u} = w_1 u_1\colon A \to \bar{B}$ is a $P(\mathfrak{H})$-monic and letting $\bar{v} = \operatorname{coker} \bar{u}$ we get the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "u"] \arrow[d, "1"] & B \arrow[r, "v"] \arrow[d, "w"] & C \arrow[r] \arrow[d, "\beta"] & 0 \\
0 \arrow[r] & A \arrow[r, "\bar{u}"] & \bar{B} \arrow[r, "\bar{v}"] & \bar{C} \arrow[r] & 0
\end{tikzcd}$$

which has the desired properties—proving the lemma.

THEOREM 2.2.2. *A $\big(Q(\bar{\mathfrak{H}}), \bar{\mathfrak{H}}\big)$-cohomology theory exists.*

Proof. The "standard complex" lies in $\mathcal{E}_0(\bar{\mathfrak{H}}) \subseteq \mathfrak{F}(\bar{\mathfrak{H}})$ and has objects which are $\mathfrak{F}(\bar{\mathfrak{H}})$-projectives. Hence the functors $\{H^n(X; G, \ast)|\ n \in Z\}$ form and exact $Q(\bar{\mathfrak{H}})$-connected sequence of functors.

As is well known, if Y denotes an h.f. class of epimorphisms of ${}_G\mathfrak{C}$ in the sense of [8], p. 139, exercise 5, then the class of short exact sequences $0\to A\xrightarrow{u}B\xrightarrow{v}C\to 0$ in ${}_G\mathfrak{C}$ such that $v\in Y$ is proper in the sense of [7], Chapter XII, § 4 and conversely.

Hence by Lemma 2.2.2, we may apply the generalization to proper classes of short exact sequences mentioned in [8], p. 211, exercise 1 of [8], Chapter VIII, Theorem 2.2 applied to [8], p. 198, case 2 and, of course, all duals to conclude that the pair of functors $\big(H^n(X;\,G,\divideontimes);\,H^{n+1}(X;\,G,\divideontimes)\big)$ is left $Q(\overline{\mathfrak{H}})$-couniversal and right $Q(\overline{\mathfrak{H}})$-universal for all $n\in Z$. To finish the proof it only need be reiterated that the functors $U_{\overline{\mathfrak{H}}}$ and $H^0(X;\,G,\divideontimes)$ are naturally equivalent.

Since $U_{\mathfrak{H}}=U_{\overline{\mathfrak{H}}}=U_{\mathfrak{H}_0}$, it follows that cohomology theories for $\big(Q(\overline{\mathfrak{H}}),\mathfrak{H}\big)$; $\big(Q(\overline{\mathfrak{H}}),\overline{\mathfrak{H}}\big)$ and $\big(Q(\overline{\mathfrak{H}}),\mathfrak{H}_0\big)$ are the same.

COROLLARY 2.2.1. *If P is a proper class of short exact sequences such that $P(\mathfrak{H})\subseteq P\subseteq Q(\overline{\mathfrak{H}})$, then a $(P,\mathfrak{H})$-cohomology theory exists.*

We have seen that cohomology theories for $(P(\mathfrak{H}),\mathfrak{H})$ and $\big(Q(\overline{\mathfrak{H}}),\overline{\mathfrak{H}}\big)$ comprise the same sequences of functors; this sheds new light on § 7 of [9].

The functors $H^{-1}(X;\,G,\divideontimes)$ and $H^1(X;\,G,\divideontimes)$ are computed in special cases in sections 5 and 6 of [9] and in section 4 of [1]. Moreover, Theorem 2.1.1 gives [9], Corollary 10.2.

Suppose that G is a finite group and that (G,X) is a fixed point free permutation representation; then $f\big((G,X)\big)=\mathfrak{H}$ consists only of the identity subgroup, $Q(\overline{\mathfrak{H}})$ consists of all short exact sequences of ${}_G\mathfrak{C}$ and $P(\mathfrak{H})$ consists of all Z-split short exact sequences of ${}_G\mathfrak{C}$. Thus, we have proven that the ordinary (the $\big(Q(\overline{\mathfrak{H}}),\overline{\mathfrak{H}}\big)$)-cohomology theory and the "Z-split" (the $\big(P(\mathfrak{H}),\mathfrak{H}\big)$-) cohomology theory comprise the same sequences of functors.

If ϱ is a positive integer, then (G,X^ϱ) is also a finite permutation representation. If $\mathfrak{K}=f\big((G,X^\varrho)\big)$, then $\overline{\mathfrak{K}}=\overline{\mathfrak{H}}$ which implies that $P(\mathfrak{H})=P(\mathfrak{K})$, $Q(\overline{\mathfrak{H}})=Q(\overline{\mathfrak{K}})$ and $U_{\mathfrak{H}}=U_{\mathfrak{K}}$. Hence, both (G,X) and (G,X^ϱ) give rise to exactly the same cohomology theories of Theorems 2.2.1 and 2.2.2.

Suppose that the finite permutation representation (G,X) is such that $f\big((G,X)\big)=\mathfrak{H}$ contains a subgroup H such that if $\mathfrak{K}$ consists only of H then $\mathfrak{H}\cong\mathfrak{K}$. If (G,Y) denotes the permutation representation of G on the left H-cosets, then $f\big((G,Y)\big)\cong\mathfrak{K}$, $P(\mathfrak{H})=P(\mathfrak{K})$, $U_{\mathfrak{H}}=U_{\mathfrak{K}}$ and thus for purposes of the $\big(P(\mathfrak{H}),\mathfrak{H}\big)$-cohomology theory, we may use the transitive permutation representation (G,Y) instead of (G,X).

This can be generalized to: if we can delete transitive constituents of (G,X) to get a permutation representation (G,Y) such that $f\big((G,X)\big)\cong f\big((G,Y)\big)$, then for purposes of the $\big(P(\mathfrak{H}),\mathfrak{H}\big)$-cohomology theory, we may take the permutation representation (G,Y) instead of (G,X).

For example, this will be the case if in (G,X) we delete transitive constituents which are fixed point free.

§ 3. Passage to a quotient group.

LEMMA 2.3.1. *Let H be a subgroup of the group G and let M be a subgroup of H which is normal in G. If $B\in{}_H\mathfrak{C}$, then $(Z[G]\otimes_H B)^M\cong Z[G/M]\otimes_{H/M}(B^M)$ as left G/M modules.*

Proof. Let $G=\bigcup_{i\in I}x_iH$ be the left coset decomposition of H in G where $\{x_i|\ i\in I\}$ is a left coset representative choice. As abelian groups, $Z[G]\otimes_H B\cong\bigoplus_{i\in I}(x_iZ[H]\otimes_H B)\cong\bigoplus_{i\in I}B_i$ where $B_i=B$. The induced G-action on $\bigoplus_{i\in I}B_i$ is determined by: if $g\in G$ and $gx_i=x_{i(g)}h_i$ where $h_i\in H$ for $i\in I$, then for $b\in B=B_i$, $gb=h_ib\in B_{i(g)}=B$. But if $n\in M$, then $nx_i=x_in'$ for some $n'\in M\subseteq H$ and so, $\big(\bigoplus_{i\in I}B_i\big)^M=\bigoplus_{i\in I}(B^M)_i$.

Moreover, $\{x_iM|\ i\in I\}$ is a left coset representative choice for H/M in G/M and hence $Z[G/M]\otimes_{H/M}(B^M)\cong\bigoplus_{i\in I}(B^M)_i$ as abelian groups. Finally, the induced G/M action coincides with that above.

We could have avoided this lemma by referring to [9], Proposition 10.1 which is implied by this lemma.

The kernel of a permutation representation (G,X) is defined to be: $\bigcap_{H\in\mathfrak{H}}H$ where $\mathfrak{H}=f\big((G,X)\big)$.

THEOREM 2.3.1. *Let (G,X) denote a finite permutation representation and let M denote a normal subgroup of G which is contained in the kernel of (G,X). If the functor $V\colon{}_G\mathfrak{L}\to{}_{G/M}\mathfrak{L}$ is given by $V(A)=A^M$ for $A\in{}_G\mathfrak{L}$, then*

$$\big\{H^n\big(X;\,G/M,V(\divideontimes)\big)\big|\ n\in Z\big\}\ \text{is a}\ \big(P(\mathfrak{H}),\mathfrak{H}\big)\text{-cohomology theory}$$

where $\mathfrak{H}=f\big((G,X)\big)$. Hence, for any $n\in Z$ and any $A\in{}_G\mathfrak{L}$, $H^n(X;\,G,A)\cong H^n(X;\,G/M,A^M)$ in Ab.

Proof. Let $f\big((G/M,X)\big)=\{H/M|\ H\in\mathfrak{H}\}$ be denoted by $\mathfrak{H}/M$. Then, for any $E\in P(\mathfrak{H})$, we have $V(E)\in P(\mathfrak{H}/M)$. Lemma 2.3.1 shows that V sends $\mathcal{E}_0(\mathfrak{H})$-projectives (and $\mathcal{E}^0(\mathfrak{H})$-injectives) into $\mathcal{E}_0(\mathfrak{H}/M)$-projectives (and $\mathcal{E}^0(\mathfrak{H}/M)$-injectives). Also, it is easy to see that $U_{\mathfrak{H}/M}\circ V=U_{\mathfrak{H}}$ and hence $\big\{H^n\big(X;\,G/M,V(\divideontimes)\big)\big|\ n\in Z\big\}$ is a $\big(P(\mathfrak{H}),\mathfrak{H}\big)$ cohomology theory.

Thus, we have another proof of [9], Proposition 3.1.

In the above, if $E\in Q(\overline{\mathfrak{H}})$ implies $V(E)\in Q(\overline{\mathfrak{H}/M})$ (e.g. if $M\in\mathfrak{H}$ or equivalently if M is the kernel of (G,X)), then $\big\{H^n\big(X;\,G/M,V(\divideontimes)\big)\big|\ n\in Z\big\}$ is also a $\big(Q(\overline{\mathfrak{H}}),\overline{\mathfrak{H}}\big)$-cohomology theory.

Since the lattice of subgroups of a finite cyclic group of prime power order is linearly ordered, any permutation representation (G, X) of such a group is such that $f((G, X)) = \mathfrak{H}$ contains a unique maximal subgroup which is, of course, normal.

COROLLARY 2.3.1. *If (G, X) is a finite permutation representation of the finite cyclic group G of prime power order and if M is the maximal subgroup of $\mathfrak{H} = f((G, X))$, then, for any $A \in {}_G\mathfrak{C}$:*

$$H^n(X; G, A) \cong A^G/N(A^M) \quad \text{for } n \text{ even}$$

and

$$H^n(X; G, A) \cong {}_N(A^M)/I(A^M) \quad \text{for } n \text{ odd}$$

where $N = S_{G|1}$ denotes the norm of G/M, I denotes the augmentation ideal of $Z[G/M]$, and ${}_N(A^M) = \mathrm{Kernel}(N\colon A^M \to A^M)$.

Proof. Let $\mathfrak{K}$ denote the set consisting of just M and let (G, Y) be the permutation representation of G on the M-cosets. Then, $f((G, Y)) = \mathfrak{K} \cong \mathfrak{H} = f((G, X))$. Hence, for any $A \in {}_G\mathfrak{C}$, $H^n(X; G, A) \cong H^n(Y; G, A) \cong H^n(Y; G/M, A^M)$. But $(G/M, Y)$ is a fixed point free permutation representation and so $H^n(Y, G/M, A^M)$ is the ordinary cohomology group $H^n(G/M, A^M)$.

§ 4. Relations with subgroups. Throughout this section, K will denote an arbitrary subgroup of G.

Let $\mathfrak{H} = \{H\}$ be an arbitrary collection of subgroups of G and set $\mathfrak{L} = \{H \cap K|\ H \in \mathfrak{H}\}$. This is suggested by the fact that if (G, X) is a permutation representation such that $f((G, X)) = \mathfrak{H}$, then (K, X) is a permutation representation of K such that $f((K, X)) = \mathfrak{L}$. If $\mathfrak{H}$ is a finite collection of subgroups of finite index in G, then the same can be said for $\mathfrak{L}$.

For arbitrary $\mathfrak{H}$, the "forgetful" functor $T_K\colon {}_G\mathfrak{C} \to {}_K\mathfrak{C}$ is such that if $E \in P(\mathfrak{H})$, then $T_K(E) \in P(\mathfrak{L})$ for ${}_K\mathfrak{C}$ where $\mathfrak{L} = \{K \cap H|\ H \in \mathfrak{H}\}$.

LEMMA 2.4.1. *If H, K are arbitrary subgroups of G and if $B \in {}_H\mathfrak{C}$, then $T_K(Z[G]\otimes_H B)$ is isomorphic in ${}_K\mathfrak{C}$ to a direct sum of K-modules of the form: $Z[K]\otimes_J D$ where $J = x^{-1}Hx \cap K$ for some $x \in G$ and for some $D \in {}_J\mathfrak{C}$.*

Proof. Let $G = \bigcup_{i\in I} Kx_iH$ be the (K, H)-double coset decomposition of G where $\{x_i|\ i \in I\}$ is a double coset representative choice. Clearly, $Z[G] \cong \bigoplus_{i\in I}(Z[K]x_iz[H])$ as a left K-module and as a right H-module. Hence, $Z[G]\otimes_H B \cong \bigoplus_{i\in I}(Z[K]x_iZ[H]\otimes_H B)$ as a left K-module. Let $\{k_{ij}x_i|\ k_{ij} \in K, j \in m(i)\}$ be a coset representative choice for the left H-cosets in Kx_iH for $i \in I$. As is well known, $\{k_{ij}|\ j \in m(i)\}$ forms a complete set of left coset representatives for $J = K \cap (x_iHx_i^{-1})$ in K. Let $D = B$ be the left $J = K \cap x_iHx_i^{-1}$-module with action defined by: $(x_ihx_i^{-1})b = hb$ for all $b \in B$ and $x_ihx_i^{-1} \in J$ and let $\alpha\colon (Z[K]x_i\,Z[H]\otimes_H B) \to Z[K]\otimes_J D$ denote the abelian group homomorphism defined by $\alpha(kx_ih\otimes_H b) = k\otimes_J hb$ for all $k \in K$, $h \in H$ and $b \in B$. Finally, let $\beta\colon Z[K]\otimes_J D \to (Z[K]x_iZ[H])\otimes_H B$ denote the abelian group homomorphism defined by: $\beta(k\otimes_J b) = kx_i\otimes_H b$ for all $k \in K$, and $b \in B$. It is straightforward to prove that both α and β are well defined K-homomorphisms which are inverses of each other.

For the rest of this section, we assume that there is a finite permutation representation (G, X) such that $f((G, X)) = \mathfrak{H}$. Hence, (K, X) is a finite permutation representation of K such that $f((K, X)) = \{H \cap K|\ H \in \mathfrak{H}\} = \mathfrak{L}$. Moreover, $\{H^n(X; K, T_K(*))|\ n \in Z\}$ is an exact $P(\mathfrak{H})$-connected sequence of functors each of which (by Lemma 2.4.1) vanishes on the $\mathcal{E}^0(\mathfrak{H})$-injectives and $\mathcal{E}_0(\mathfrak{H})$-projectives. Thus, for each $n \in Z$, the ordered pair of functors $(H^n(X; K, T_K(*)),\ H^{n+1}(X; K, T_K(*)))$ is both left $P(\mathfrak{H})$-couniversal and right $P(\mathfrak{H})$-universal.

When (G, X) is fixed point free, so is (K, X) and then $H^n(X; K, T_K(*))$ is just the ordinary cohomology functor $H^n(K, T_K(*))$.

LEMMA 2.4.2. *If $x \in G$ and $A \in {}_G\mathfrak{C}$, then the mapping $c_x\colon A^K \to A^{(xKx^{-1})}$ defined by: $c_x(a) = xa$ for $a \in A^K$ is an abelian group isomorphism and is such that c_x is the identity if $x \in K$. Moreover, $c_x(S_{K|K\cap H}(A^{K\cap H})) = S_{(xKx^{-1})|(xKx^{-1})\cap(xHx^{-1})}(A^{(xKx^{-1})\cap(xHx^{-1})})$.*

Proof. Straightforward.

Thus, c_x induces a natural equivalence of functors $c_x\colon U_{\mathfrak{L}} \circ T_K \to U_{\mathfrak{M}} \circ T_{xKx^{-1}}$ where $\mathfrak{M} = \{xKx^{-1} \cap H|\ H \in \mathfrak{H}\} = f((xKx^{-1}, X))$. By the usual universality argument, c_x induces a unique isomorphism: $\{c_x^n|\ n \in Z\}$: $\{H^n(X, K, T_K(*))|\ n \in Z\} \to \{H^n(X; xKx^{-1}, T_{xKx^{-1}}(*))|\ n \in Z\}$ of exact $P(\mathfrak{H})$-connected sequences of functors. If every $E \in Q(\overline{\mathfrak{H}})$ is such that $T_K(E) \in Q(\overline{\mathfrak{L}})$ and $T_{xKx^{-1}}(E) \in Q(\overline{\mathfrak{M}})$, then this isomoprhism is also an isomorphism of $Q(\overline{\mathfrak{H}})$-connected sequences of functors.

If (G, X) is fixed point free, then $\{c_x^n|\ n \in Z\}$ coincides with the homomorphism c_x defined in [3], Chapter XII, § 8.

In order to define the restriction map we need:

LEMMA 2.4.3. *Let H be any subgroup of finite index in G and let K be any subgroup. Then $H \cap K$ is of finite index in K and if $A \in {}_G\mathfrak{C}$ then*

$$S_{G|H}(A^H) \subseteq \sum_{x\in G} S_{K|K\cap(xHx^{-1})}(A^{K\cap(xHx^{-1})}).$$

Proof. Let $G = \bigcup_{i=1}^{n} Kx_iH$ and let $\{k_{ij}x_iH|\ 1 \leqslant j \leqslant m(i)\}$ be as in

Lemma 2.4.1. If $a \in A^H$, then $S_{G|H}(a) = \sum_{i=1}^{n}\sum_{j=1}^{n} k_{ij}x_i a$. But $x_i a \in A^{x_i H x_i^{-1}}$ and $\{k_{ij}|\ 1 \leqslant j \leqslant m(i)\}$ is a set of left coset representatives for $K \cap (x_i H x_i^{-1})$ in K. Hence,

$$S_{G|H}(a) = \sum_{i=1}^{n} S_{K|K\cap x_i H x_i^{-1}}(x_i a) \in \sum_{x\in G} S_{K|K\cap(xHx^{-1})}(A^{K\cap(xHx^{-1})}).$$

Suppose now that L is an arbitrary subgroup of K, that $f\big((K, X)\big) = \mathfrak{L} = \{K \cap H|\ H \in \mathfrak{H}\}$ and that $f\big((L, X)\big) = \mathfrak{M} = \{L \cap H|\ H \in \mathfrak{H}\}$. Then for any $A \in {}_G\mathfrak{C}$, the inclusion map: $A^K \to A^L$ induces a natural transformation of the functors $U_{\mathfrak{L}} \circ T_K \to U_{\mathfrak{M}} \circ T_L$ which by universality induces a unique morphism: $\{\mathrm{Res}^n(L, K)|\ n \in Z\}$: $\{H^n(X; K, T_K(*))|\ n \in Z\} \to \{H^n(X; L, T_L(*))|\ n \in Z\}$ of exact $P(\mathfrak{H})$-connected sequences of functors which is called the restriction mapping.

Clearly, this restriction mapping coincides with that of [9], Chapter I, section 8.

Now we further assume that L is of finite index in K. Then for every $A \in {}_G\mathfrak{C}$, the trace $S_{K|L}$: $A^L \to A^K$ induces a natural transformation $U_{\mathfrak{M}} \circ T_L \to U_{\mathfrak{L}} \circ T_K$, since $S_{K|L}\big(S_{L|L\cap H}(A^{L\cap H})\big) \subseteq S_{K|L\cap H}(A^{L\cap H}) \subseteq S_{K|K\cap H}(A^{K\cap H})$ by Lemma 2.1.1. This natural transformation induces a unique morphism:

$\{\mathrm{Cor}^n(K, L)|\ n \in Z\}$: $\{H^n(X; L, T_L(*))|\ n \in Z\} \to \{H^n(X; K, T_K(*))|\ n \in Z\}$

of $P(\mathfrak{H})$-connected sequences of functors which is called the corestriction or transfer mapping.

Clearly, this corestriction mapping coincides with that of [9], Chapter I, section 9.

If (G, X) is fixed point free, both the restriction and corestriction mappings defined here coincide with the ordinary cohomology theory restriction and corestriction maps.

When $T_K(E) \in Q(\bar{\mathfrak{L}})$ and $T_L(E) \in Q(\bar{\mathfrak{M}})$ for every $E \in Q(\bar{\mathfrak{H}})$, then the restriction and corestriction mappings are also morphisms of $Q(\bar{\mathfrak{H}})$-connected sequences of functors.

Via a routine check in dimension zero one can prove:

Proposition 2.4.1. $\{c_x^n|\ n \in Z\} \circ \{c_y^n|\ n \in Z\} = \{c_{xy}|\ n \in Z\}$ *for all* $x, y \in G$.

Proposition 2.4.2. *If* $x \in K$, *then* $\{c_x^n|\ n \in Z\}$: $\{H^n(X; K, T_K(*))|\ n \in Z\} \to \{H^n(X; K, T_K(*))|\ n \in Z\}$ *is the identity and hence if* $K \lhd G$, *then each* $H^n(X; K, T_K(A))$ *becomes a* G/K*-module for any* $A \in {}_G\mathfrak{C}$.

Proposition 2.4.3. *If* $J \subseteq L \subseteq K$ *is an ascending sequence of subgroups of* G, *then* $\{\mathrm{Res}^n(J, L)|\ n \in Z\} \circ \{\mathrm{Res}^n(L, K)|\ n \in Z\} = \{\mathrm{Res}^n(J, K)|\ n \in Z\}$. *If further*, J *is of finite index in* K, *then*

$$\{\mathrm{Cor}^n(K, L)|\ n \in Z\} \circ \{\mathrm{Cor}^n(L, J)|\ n \in Z\} = \{\mathrm{Cor}^n(K, J)|\ n \in Z\}.$$

Proposition 2.4.4. *If* L *is a subgroup of* K, *then for any* $x \in G$:

$$\{c_x^n|\ n \in Z\} \circ \{\mathrm{Res}^n(L, K)|\ n \in Z\} = \{\mathrm{Res}^n(xLx^{-1}, xKx^{-1})|\ n \in Z\} \circ \{c_x^n|\ n \in Z\}.$$

If further L *is of finite index in* K, *then for any* $x \in G$:

$$\{c_x^n|\ n \in Z\} \circ \{\mathrm{Cor}^n(K, L)|\ n \in Z\} = \{\mathrm{Cor}^n(xKx^{-1}, xLx^{-1})|\ n \in Z\} \circ \{c_x^n|\ n \in Z\}.$$

Using both a check in dimension zero and a dimension shift, we obtain (see the proof of [6], Chapter II, Proposition 13):

Proposition 2.4.5. *If* L *is a subgroup of finite index in* K, *if* $A, B \in {}_G\mathfrak{C}$ *and if* u: $T_L(A) \to T_L(B)$ *is an* L*-homomorphism, let* u^n: $H^n(X; L, T_L(A)) \to H^n(X; L, T_L(B))$ *and let*

$$\big(S_{K|L}(u)\big)^n\colon H^n(X; K, T_K(A)) \to H^n(X; K, T_K(B))$$

denote the maps induced by u *and* $S_{K|L}(u)$ *respectively. Under these hypotheses:*

$$\{\mathrm{Cor}^n(K, L)|\ n \in Z\} \circ \{u^n|\ n \in Z\} \circ \{\mathrm{Res}^n(L, K)|\ n \in Z\} = \{S_{K|L}(u)^n|\ n \in Z\}.$$

Hence, taking u *to be the identity map, for any* $n \in Z$ *and any* $a \in H^n(X; K, T_K(A))$ *we have:* $\mathrm{Cor}^n(K, L) \circ \mathrm{Res}^n(L, K)\, a = |K:L|\, a$ *where* $|K:L|$ *denotes the index of* L *in* K.

This is [7], Lemma 10.1.

The result [3], Chapter XII, Proposition 9.1 may also be generalized by a check in dimension zero:

Proposition 2.4.6. *For any subgroup* J *of finite index in* K *and for an arbitrary subgroup* L *of* K, *if* $K = \bigcup_{i=1}^{t} Lx_iJ$ *is the double coset decomposition (where* $\{x_i|\ 1 \leqslant i \leqslant t\}$ *is a double coset representative choice), then*

$$\{\mathrm{Res}^n(L, K)|\ n \in Z\} \circ \{\mathrm{Cor}^n(K, J)|\ n \in Z\}$$
$$= \sum_{i=1}^{t} \{\mathrm{Cor}\big(L, L \cap (x_iJx_i^{-1})\big)|\ n \in Z\} \circ$$
$$\circ \{\mathrm{Res}\big(L \cap (x_iJx_i^{-1}), (x_iJx_i^{-1})\big)|\ n \in Z\} \circ \{c_{x_i}^n|\ n \in Z\}.$$

Hence, if also $J \lhd K$ *and* $K = \bigcup_{i=1}^{t} x_iJ$ *and if* $A \in {}_G\mathfrak{C}$, *then for any* $n \in Z$ *and any* $a \in H^n(X; J, T_J(A))$ *we have*

$$\mathrm{Res}^n(J, K) \circ \mathrm{Cor}^n(K, J)\, a = \sum_{i=1}^{t} c_{x_i} a = \sum_{i=1}^{t} S_{K/J|1}(a).$$

We can also generalize the remaining results of [3], Chapter XII, sections 9 and 10.

DEFINITION 2.4.1. If $A \in {}_G\mathfrak{C}$ and L is a subgroup of K, then an element $\alpha \in H^n(X; L, T_L(A))$ will be called *K-stable* if for each $x \in K$:

$$\operatorname{Res}^n(L \cap (xLx^{-1}), L)\,\alpha = \operatorname{Res}^n(L \cap (xLx^{-1}), (xLx^{-1})) \circ c_x^n\,\alpha$$

(or equivalently, if $\operatorname{Res}^n(L \cap (xLx^{-1}), L)\,\alpha = c_x^n \circ \operatorname{Res}^n((x^{-1}Lx) \cap L, L)\,\alpha$.

If $L \lhd K$, this reduces to $\alpha = c_x\alpha$. Thus, when $L \lhd K$, the K-stable elements of $H_n(X; L, T_L(A))$ are precisely those invariant under K/L.

The following two propositions can be proven by using the arguments of [3], Chapter XII, Propositions 9.3 and 9.4:

PROPOSITION 2.4.7. *If α is in the image of $\operatorname{Res}^n(L, K)$, then α is K-stable.*

COROLLARY 2.4.1. *If $L \lhd K$ and $A \in {}_G\mathfrak{C}$, then image* $\operatorname{Res}^n(L, K) \leqslant \left(H^n(X; L, T_L(A))\right)^{K/L}$.

PROPOSITION 2.4.8. *If L is of finite index in K, if $A \in {}_G\mathfrak{C}$ and if $\alpha \in H^n(X; L, T_L(A))$ is K-stable then,*

$$\operatorname{Res}^n(L, K) \circ \operatorname{Cor}^n(K, L)\,\alpha = |K:L|\,\alpha\,.$$

We also have:

PROPOSITION 2.4.9. *If $L \lhd K$ and is of finite index in K and if $A \in {}_G\mathfrak{C}$, then $I_K H^n(X; L, T_L(A)) \subseteq \operatorname{Kernel}\operatorname{Cor}^n(K, L)$ where I_K is the augmentation ideal of $Z[K]$.*

Proof. Let $x \in K$ and $\alpha \in H^n(X; L, T_L(A))$, then

$$\begin{aligned}\operatorname{Cor}^n(K, L)(c_x^n - c_1^n)\,\alpha &= \operatorname{Cor}^n(K, L) \circ c_x^n(\alpha) - \operatorname{Cor}^n(K, L)(\alpha)\\ &= c_x^n \circ \operatorname{Cor}^n(K, L)(\alpha) - \operatorname{Cor}^n(K, L)(\alpha)\\ &= \operatorname{Cor}^n(K, L)(\alpha) - \operatorname{Cor}^n(K, L)(\alpha) = 0\end{aligned}$$

since c_x^n is the identity on $H^n(X; K, T_K(A))$.

Now assume that K is a finite subgroup of G and that π is any set of prime integers. If $A \in {}_G\mathfrak{C}$, then (Theorem 2.1.1) every element of $H^n(X; K, T_K(A))$ is of finite order dividing $|K:1|$ and hence the π-primary component $H^n(X; K, T_K(A))_\pi$ of $H^n(X, K, T_K(A))$ is defined.

The proof of [3], Chapter XII, Theorem 10.1 can be easily adapted to prove:

THEOREM 2.4.1. *If K is a finite subgroup of G, if $A \in {}_G\mathfrak{C}$ and if L is a Hall π-subgroup of K where π is a set of prime integers, then, for each $n \in Z$:*

$$\operatorname{Cor}^n(K, L)\colon H^n(X; L, T_L(A)) \to H^n(X; K, T_K(A))_\pi$$

is an epimorphism and

$$\operatorname{Res}^n(L, K)\colon H^n(X; K, T_K(A))_\pi \to H^n(X; L, T_K(A))$$

is a monomorphism whose image consists of the K-stable elements of $H^n(X; L, T_L(A))$. Further, we have a direct sum decomposition

$$H^n(X; L, T_L(A)) = \operatorname{Image}\operatorname{Res}^n(L; K) \oplus \operatorname{Kernel}\operatorname{Cor}^n(K, L)\,.$$

If also, $L \lhd K$ then, for any $n \in Z$:

$$\operatorname{Image} N = \left(H^n(X; L, T_L(A))\right)^{K/L} = \operatorname{Image}\operatorname{Res}^n(L, K) \cong H^n(X; K, T_K(A))_\pi$$

and $\operatorname{Kernel} N = I_K\left(H^n(X; L, T_L(A))\right) = \operatorname{Kernel}\operatorname{Cor}^n(L, K)$ *where*

$$N = S_{K/L|1} = \sum_{x \in K/L} c_x\colon H^n(X; L, T_L(A)) \to H^n(X; L, T_L(A))\,.$$

COROLLARY 2.4.2. *If K is a finite subgroup of G, if π denotes the prime integers dividing $|K:1|$ and if for each prime $p \in \pi$ the p-Sylow subgroup of K is denoted by K_p, then for each $A \in {}_G\mathfrak{C}$ and for each $n \in Z$, there is a monomorphism*

$$H^n(X; K, T_K(A)) \to \bigoplus_{p \in \pi} H^n(X; K_p, T_{K_p}(A))\,.$$

If, moreover, K is nilpotent, then

$$H^n(X; K, T_K(A)) \cong \bigoplus_{p \in \pi} \left(H^n(X; K_p, T_{K_p}(A))^K\right).$$

III. Relative deriveds and spectral sequences

§ 1. Relative deriveds. Throughout this chapter, we shall adhere to the convention that (G, X) denotes a finite permutation representation and $\mathfrak{H} = f((G, X))$.

The complex E on p. 137 of [9], which (except for the Z-term) comprises the "upper half" of the "standard resolution", is such that $T_H(E)$ has a contracting homotopy in ${}_H\mathfrak{C}$ for each $H \in \mathfrak{H}$. Hence, E is an $\mathcal{E}_0(\mathfrak{H})$-projective resolution and an $\mathfrak{F}(\mathfrak{H})$-projective resolution of the trivial G-module Z. Thus, for any $A \in {}_G\mathfrak{C}$ and any integer $n \geqslant 1$,

$$\left(R^n_{\mathcal{E}_0(\mathfrak{H})}\operatorname{Hom}_G(\ast, A)\right)(Z) = \left(R^n_{\mathfrak{F}(\mathfrak{H})}\operatorname{Hom}_G(\ast, A)\right)(Z) = H^n(X; G, A)$$

and (since Hom_G is left exact)

$$\left(R^0_{\mathcal{E}_0(\mathfrak{H})}\operatorname{Hom}_G(\ast, A)\right)(Z) = \left(R^0_{\mathfrak{F}(\mathfrak{H})}\operatorname{Hom}_G(\ast, A)\right)(Z) = \operatorname{Hom}_G(Z, A) \cong A^G.$$

Thus, the sequence of functors $\{\operatorname{Hom}_G(Z, \ast);\ H^n(X; G, \ast) \text{ for } n \geqslant 1\}$ has interpretations as $P(\mathfrak{H})$ and $Q(\mathfrak{H})$-right derived functors (see [7], Chapter XII, § 9) and consequently for each $A \in {}_G\mathfrak{C}$ and $n \geqslant 1$, $H^n(X; G, A) = \mathrm{Ext}^n_{P(\mathfrak{H})}(Z, A) = \mathrm{Ext}^n_{Q(\overline{\mathfrak{H}})}(Z, A)$ has two interpretations in terms of composites of short exact sequences in $P(\mathfrak{H})$ and in $Q(\overline{\mathfrak{H}})$ (for details see [7], Chapter XII, § 4).

Since $\mathcal{E}_0(\mathfrak{H})$ and $\mathcal{E}^0(\mathfrak{H})$ are complementary classes of sequences, for any $A \in {}_G\mathfrak{C}$, the sequences of groups $\{\mathrm{Hom}_G(Z, A);\ H^n(X; G, A)$ for $n \geqslant 1\}$ can be computed by utilizing an $\mathcal{E}^0(\mathfrak{H})$-injective resolution of A.

It is clear that the results of Chapter II, § 3 have analogs for the positive sequence of functors $\{\mathrm{Hom}_G(Z, \ast);\ H^n(X; G, A)$ for $n \geqslant 1\}$ by means of universality. There are also analogues for most of the results of Chapter II, § 4.

It should be noted that the complex F on p. 137 of [9], which (except for the Z-term) comprises the "lower half" of the "standard resolution", is an $\mathcal{E}^0(\mathfrak{H})$-injective resolution of Z. Hence for any $A \in {}_G\mathfrak{C}$,

$$\big(L^{n-1}_{\mathcal{E}^0(\mathfrak{H})}\mathrm{Hom}_G(\ast, A)\big)(Z) = H^{-n}(X; G, A) \quad \text{for all } n \geqslant 2$$

and $\big(L^0_{\mathcal{E}^0(\mathfrak{H})}\mathrm{Hom}_G(\ast, A)\big)(Z) = Z^{-1}(X; G, A)$ which is described in [9], Proposition 5.1. The couniversality property of relative left derived functors can then similarly be utilized to develop properties of the exact $P(\mathfrak{H})$-connected sequence of functors $\{Z^{-1}(X; G, \ast);\ H^n(X; G, \ast)$ for $n \leqslant -2\}$.

§ 2. Spectral sequences. The results of [4], Chapter IV, § 5 may be applied to obtain various spectral sequences involving the sequence of functors

$$\{\mathrm{Hom}_G(Z, \ast);\ H^n(X; G, \ast) \text{ for } n \geqslant 1\}.$$

Thus, let $\mathfrak{F}$ be an arbitrary projective class in ${}_G\mathfrak{C}$ such that $\mathcal{E}_0(\mathfrak{H}) \subseteq \mathfrak{F}$. Then if $A \in {}_G\mathfrak{C}$ and the spectral sequence of [2], p. 171 which is rederived in [4], Chapter IV, § 5 is applied to the complex E using the contravariant functor $\mathrm{Hom}_G(\ast, A)$ we obtain:

THEOREM 3.1. $R^p_{\mathcal{E}_0(\mathfrak{H})}\big(R^q_{\mathfrak{F}}\mathrm{Hom}_G(\ast, A)\big)(Z) \Rightarrow \big(R^n_{\mathfrak{F}}\mathrm{Hom}_G(\ast, A)\big)(Z)$, *whenever $\mathfrak{F}$ is a projective class in ${}_G\mathfrak{C}$ such that $\mathfrak{F} \supseteq \mathcal{E}_0(\mathfrak{H})$.*

If $\mathfrak{F}$ is taken to be the projective class of all exact sequences of ${}_G\mathfrak{C}$, then this theorem applies and we obtain the spectral sequence of [9], Theorem 12.1. Hence we have generalized [9], Theorem 12.1.

When the permutation representation (G, Y) is obtained from (G, X) by deleting any set of transitive constituents, then

$$\mathfrak{H} = f\big((G, X)\big) \geqslant f\big((G, Y)\big) = \mathfrak{K}, \qquad \mathcal{E}_0(\mathfrak{H}) \subseteq \mathcal{E}_0(\mathfrak{K})$$

and again Theorem 3.1 applies.

Finally we sketch new derivations of the spectral sequences of [9], Chapter 3. As defined in [9], a morphism θ: $(G, X) \to (L, Y)$ of the permutation representation (G, X) of G into the permutation representation (L, Y) of L consists of a pair $\theta = (\varphi, \beta)$ where φ: $G \to L$ is a group homomorphism and β: $X \to Y$ is a function such that $\beta(\sigma x) = \varphi(\sigma)\beta(x)$ for all $\sigma \in G$ and all $x \in X$. Assume that φ is an epimorphism, let $N = \mathrm{Kernel}(\varphi)$ and let C^+ denote the positive half of the standard complex for (G, X) as given at the bottom of [9] page 155.

If (L, Y) is fixed point free, then we are in the ordinary cohomology situation and we may apply the spectral sequence of hyperhomology to the complex $\mathrm{Hom}_N(C^+, A)$ in ${}_L\mathfrak{C}$ using the functor $\mathrm{Hom}_L(Z, \ast)$. Then

$$
\begin{aligned}
{}_{\mathrm{I}}E_2^{p,q} &= H^p\big(\mathrm{Ext}^q(Z, \mathrm{Hom}_N(C^+, A))\big) = H^p\big(H^q(L, \mathrm{Hom}_N(C^+, A))\big)\\
&= \begin{cases} 0 & \text{if } q > 0 \text{ by Lemma 2.3.1} \\ H^p\big(\mathrm{Hom}_G(C^+, A) = H^p(X; G, A)\big) \end{cases}
\end{aligned}
$$

and ${}_{\mathrm{II}}E_2^{p,q} = \mathrm{Ext}^p\big(Z, H^q(X; N, A)\big) = H^p\big(L, H^q(X; N, A)\big)$ which is [9], Theorem 19.1.

One can also get [9], Theorem 21.1 using the Serre–Hochschild spectral sequence and the spectral sequence of [2], p. 171. To see this, let $\mathfrak{F}$ denote the "absolute exact" injective class in ${}_L\mathfrak{C}$ then $\mathcal{E}^0\big(f(L, Y)\big) \subseteq \mathfrak{F}$. If $B \in {}_L\mathfrak{C}$, then the spectral sequence of [2], p. 171 gives:

$$(\ast) \qquad H^p[0 \to \mathrm{Ext}^q_L(D_0, B) \to \mathrm{Ext}^q_L(D_1, B) \to \ldots] \Rightarrow H^n(L, B)$$

where D^+: $\ldots \to D_1 \to D_0 \to 0$ is the positive half of the standard complex for (L, Y). If $A \in {}_G\mathfrak{C}$, if (G, X) is fixed point free and if N is such that $H^j(N, A) = 0$ for all $j > 0$ then the Serre–Hochschild spectral sequence $H^p\big(L, H^q(N, A)\big) \Rightarrow H^n(G, X)$ collapses to give: $H^n(L, A^N) \simeq H^n(G, A)$. Replacing B by A^N in $(\ast)$ gives [9], Theorem 21.1.

References

[1] I. F. Adamson, *Cohomology theory for non-normal subgroups and non-normal fields*, Proc. Glasgow Math. Assoc. 2 (1954), pp. 66-76.

[2] M. R. C. Butler and G. Horrocks, *Classes of extensions and resolutions*, Philos. Trans. Roy. Soc. London, Ser. A, 254 (1961), pp. 155-222.

[3] H. Cartan and S. Eilenberg, *Homological algebra*, Princeton, 1956.

[4] S. Eilenberg and J. C. Moore, *Foundations of relative homological algebra*, Mem. Amer. Math. Soc. 55 (1965).

[5] G. Hochschild, *Relative homological algebra*, Trans. Amer. Math. Soc. 82 (1956), pp. 246-269.

[6] S. Lang, *Rapport sur la cohomologie des groupes*, New York 1966.

[7] S. MacLane, *Homology*, New York 1963.

[8] B. Mitchell, *Theory of categories*, New York 1965.

[9] E. Snapper, *Cohomology of permutation representations I. Spectral sequences*, J. Math. Mech. 13 (1964), pp. 133-161.

[10] E. Snapper, *Cohomology of permutation representations II. Cup product*, J. Math. Mech., 13 (1964), pp. 1047-1064.

[11] — *Duality in the cohomology ring of transitive permutation representations*, J. Math. Mech., 14 (1965), pp. 323-336.

[12] — *Inflation and deflation for all dimensions*, Pacific J. Math. 15 (1965), pp. 1061-1081.

[13] — *Spectral sequences and Frobenius groups*, Trans. Amer. Math. Soc. 114 (1965), pp. 133-146.

THE UNIVERSITY OF ILLINOIS
at Chicago Circle

Reçu par la Rédaction le 6. 1. 1968

Some theorems on the embeddability of ANR-spaces into Euclidean spaces

by

H. Patkowska (Warszawa)

1. Introduction. In 1930 C. Kuratowski (see [7]) has characterized the local dendrites (i.e. connected, 1-dimensional ANR-sets) which are embeddable into the plane E^2 as those which do not contain homeomorphic images of the two graphs K_1 and K_2. K_1 and K_2 will be called the *graphs of Kuratowski*. K_1 is the 1-skelton of a 3-simplex in which the midpoints of a pair of non-adjacent edges are joined by a segment; K_2 is the 1-skelton of a 4-simplex. C. Kuratowski has also described two locally connected curves K_3 and K_4 which are not ANR-sets and he has conjectured the characterization of locally connected continua which are embeddable into the sphere S^2 as those which do not contain homeomorphic images of the four curves K_i, $i = 1, 2, 3, 4$. This was proved in 1937 by S. Claytor (see [3] and [4]). As a corollary, Claytor obtained the following result ([3], p. 632), which will be useful for us: Each cyclic locally connected continuum which does not contain homeomorphic images of the graphs of Kuratowski is embeddable into S^2. Recall that a connected space is cyclic (in the sense of Whyburn) if it is separated by no point.

In 1966 S. Mardešić and J. Segal (see [9] and [10]) showed that the connected polyhedra which are embeddable into S^2 can be characterized as those which do not contain homeomorphic images of three polyhedra, namely K_1, K_2 and $\perp$, where $\perp$ is the one-point union of a 2-simplex and of a segment relative to an interior point of the 2-simplex and an end-point of the segment. They raised the question if this characterization can be extended to the connected ANR-sets (they are always assumed to be compact) which are embeddable into S^2. We shall show in this paper that this is in fact true. Namely, we shall derive this from Claytor's result mentioned above and from the positive answer to the following question for $n = 2$: If X is a connected ANR containing no n-umbrella and if the cyclic elements of X are embeddable into E^n, is X also embeddable into E^n? By an *n-umbrella* we shall mean here a one-point union of a (topological) n-ball and of an arc relative to an interior point of the n-ball and an end-point of the arc. For the definition of cyclic elements see section 3.