Boolean powers

by

P. Ribenboim (Kingston, Ontario)

This paper is devoted to a systematic presentation of the construction of Boolean powers, which have been used by several authors ([5], [1], [3], [6]) in various problems. It was our first task to give a rather general definition for the Boolean power and to show how several kinds of structures are inherited by the Boolean power.

Our only application has been to establish a natural isomorphism between the category of Boolean algebras and a certain category of lattice ordered abelian groups.

It is quite apparent that the study began in the present paper should be pursued in several different directions.

1. Let B be a Boolean algebra, and F a filter of B ($F \subseteq B$).

Let G be a set with a distinguished point 0. Let $\mathfrak{J}$ be an ideal of subsets of G.

We shall make the following hypothesis:

(H) *If $S \in \mathfrak{J}$, if $(x_s)_{s \in S}$ is any family of elements of B, indexed by S, then $\bigvee_{s \in S} x_s$ exists in B.*

For example, if B is an α-complete Boolean algebra, where $\#(G) \leqslant \alpha$, then for any ideal $\mathfrak{J}$ this hypothesis is satisfied.

Similarly, the hypothesis holds also when $\mathfrak{J} = \mathfrak{J}_0$ is the ideal of all finite subsets of G.

Definition 1. Let $X = \bigtimes_{B,F}(G, \mathfrak{J})$ be the set of all elements $x = (x_g)_{g \in G} \in B^G$ such that:

(1) $x_0 \in F$,

(2) if $g_1, g_2 \in G$, $g_1 \neq g_2$, then $x_{g_1} \wedge x_{g_2} = 0$,

(3) $\operatorname{supp}(x) = \{g \in G \mid x_g \neq 0\} \in \mathfrak{J}$,

(4) $\bigvee_{g \in G} x_g = 1$ (last element of B).

X is called the *Boolean power of G over B* (relative to $\mathfrak{J}, F$) [1].

More generally, let us assume that B is a Boolean lattice, that is, a distributive lattice with first element 0, such that for every $x \in B$ the principal ideal $\mathrm{Id}(x) = \{b \in B \mid 0 \leqslant b \leqslant x\}$ is a Boolean algebra.

Clearly, $B = \bigcup_{s \in F} \mathrm{Id}(s)$. Let $F_s = \{u \wedge s \mid u \in F\}$, so it is a filter of the Boolean algebra $\mathrm{Id}(s)$. If $s \leqslant t$ then $\mathrm{Id}(s) \subseteq \mathrm{Id}(t)$ and $F_s = \{u \wedge s \mid u \in F_t\}$. If $X_t = \bigtimes_{\mathrm{Id}(t), F_t} (G, \mathfrak{J})$, $X_s = \bigtimes_{\mathrm{Id}(s), F_s} (G, \mathfrak{J})$, there exists a natural mapping $\pi_s^t\colon X_t \to X_s$, namely $\pi_s^t(x) = y$ where $y_g = x_g \wedge s$ (for every $g \in G$). There exists also a natural mapping $\iota_s^t\colon X_s \to X_t$, namely $\iota_s^t(y) = x$ where $x_0 = y_0 \vee s^*$ (with $s \vee s^* = t$, $s \wedge s^* = 0$), $x_g = y_g$ for $g \neq 0$; so $x_0 \in F_t$ and $x \in X_t$. We have $\pi_s^t \circ \iota_s^t$ equal to the identity mapping of X_s, hence π_s^t is surjective, while ι_s^t is injective. Moreover, if $s \leqslant t \leqslant u$ are elements in F, then $\pi_s^u = \pi_s^t \circ \pi_t^u$, π_s^s is the identity, and also $\iota_s^u = \iota_t^u \circ \iota_s^t$, ι_s^s is the identity.

We may consider the inverse limit $X = \varprojlim X_s$. Let $\pi_s\colon X \to X_s$ be the canonical mapping, hence π_s is also surjective; let $\iota_s\colon X_s \to X$ be the mapping defined by the family of mappings ι_s^t (for $s \leqslant t$), then $\pi_s \circ \iota_s$ is the identity mapping, so ι_s is also injective.

DEFINITION 2. With above notations, we say that X is the *Boolean power of G over B* (relative to $\mathfrak{J}, F$) and we write again $X = \bigtimes_{B,F} (G, \mathfrak{J})$.

Before proceeding, we want to illustrate this concept with a few examples.

EXAMPLE 1. Let I be a set, let $B = \mathfrak{B}(I) = F$ (Boolean algebra of subsets of I) let G be any set with a distinguished point $0 \in G$, $\mathfrak{J} = \mathfrak{B}(G)$ (set of subsets of G). Then there is a natural bijection between X and G^I. In fact, given $x = (x_g)_{g \in G} \in X$ and given $i \in I$ there exists one and only one element $g \in G$ such that $i \in x_g$ (by properties (2), (4)). We define $\xi\colon I \to G$ by letting $\xi(i) = g$ when $i \in x_g$. The mapping $X \to G^I$ defined by $x \to \xi$, is a bijection.

In fact, if $x \neq y$, there exists $g \in G$ such that $x_g \neq y_g$, so there exists $i \in I$ such that $i \in x_g$, $i \notin y_g$ (or vice-versa), hence $\xi(i) = g$, $\eta(i) \neq g$. On the other hand, given $\xi\colon I \to G$, let $x_g = \{i \in I \mid \xi(i) = g\}$, then $x = (x_g)_{g \in G} \in X$ and ξ is clearly the image of x.

EXAMPLE 2. Let I be a set, $B = \mathfrak{B}(I)$, let F be the filter of cofinite subsets of I, let G be a set with a distinguished point 0, $\mathfrak{J} = \mathfrak{B}(G)$. Then there is a natural bijection between X and the "direct sum" of $\#(I)$ copies of G, that is the set of mappings $\xi\colon I \to G$ such that $\mathrm{supp}(\xi)$ is finite.

[1] We remark that, for this definition, we need only the fact that B is a distributive lattice, with first and last element. However, if $x \in X$ then $x_{g_0} \in L$ has complement $\bigvee_{g \neq g_0} x_g$ in B, so there is no loss of generality in replacing B by the largest Boolean algebra contained in B.

EXAMPLE 3. If $B = F = \mathfrak{B}(I)$, G as before, if $\mathfrak{J}$ is the ideal of finite subsets of G, then there is a natural bijection between X and the set of mappings $\xi\colon I \to G$ such that $\xi(I)$ is finite.

EXAMPLE 4. If $B = F = \mathfrak{B}(I)$, if $G = \boldsymbol{R}$ (real numbers) if $\mathfrak{J}$ is the ideal generated by all intervals $(-a, +a)$ then there is a natural bijection from X to the set of bounded mappings from I to $\boldsymbol{R}$.

EXAMPLE 5. If $B = \mathfrak{B}(\boldsymbol{R})$, if F is the filter generated by the complements of the intervals $(-n, +n)$ (for every integer n), if $G = \boldsymbol{R}$ and $\mathfrak{J}$ as in example 4, then there is a natural bijection from X to the set of all bounded mappings from $\boldsymbol{R}$ to $\boldsymbol{R}$, vanishing outside some closed interval.

Remark. It is possible to consider a more general construction than the Boolean power, in the following situation. Let $(G_i)_{i \in I}$ be a family of sets. Let B be a Boolean algebra and $X = \bigtimes_{B,B} (I, \mathfrak{I})$, where $\mathfrak{I}$ is an ideal of subsets of I, and the hypothesis (H) is satisfied for $B, \mathfrak{I}$.

For every $x \in X$, we define the set Y_x in the following manner. If G is the disjoint union of the sets G_i (for $i \in I$), we let Y_x be the subset of $\bigtimes_{B,B} (G, \mathfrak{J})$ (where $\mathfrak{J}$ is an ideal subsets of G, and hypothesis (H) is satisfied for $B, \mathfrak{J}$) consisting of all elements $(y_g)_{g \in G}$ such that:

(1) if $g, h \in G$, $g \neq h$ then $y_g \wedge y_h = 0$,

(2) for every $i \in I$, $\mathrm{supp}_i(y) = \{g \in G_i \mid y_g \neq 0\} \in \mathfrak{J}$,

(3) for every $i \in I$, $x_i = \bigvee_{g \in G_i} y_g$.

For example, let $B = \mathfrak{B}(I)$, $\mathfrak{I} = \mathfrak{B}(I)$, let $x = (x_i)_{i \in I}$ be such that $x_i = \{i\}$ for every $i \in I$. If $\mathfrak{J} = \mathfrak{B}(G)$ there exists a bijection from Y_x onto $\prod_{i \in I} G_i$.

We shall not try to explore the properties satisfied by this more general construction.

In order to compare Boolean powers defined for different pairs (B_1, F_1), (B_2, F_2), where B_1, B_2 are Boolean lattices, and F_1, F_2 are filters, we define the following concepts.

A morphism $\alpha\colon (B_1, F_1) \to (B_2, F_2)$ is a mapping $\alpha\colon B_1 \to B_2$ such that $\alpha(F_1) \subseteq F_2$, α preserves the Boolean operations and $\alpha(B_1)$ is cofinal in B_2 (in particular, if B_1 is a Boolean algebra, then B_2 is a Boolean algebra, $\alpha(1) = 1$).

If B_1 is a Boolean algebra, $\alpha\colon (B_1, F_1) \to (B_2, F_2)$ a morphism, if G is a set, and $\mathfrak{J}$ an ideal of subsets of G, if $X_1 = \bigtimes_{B_1, F_1} (G, \mathfrak{J})$, $X_2 = \bigtimes_{B_2, F_2} (G, \mathfrak{J})$, then α induces a mapping $\alpha_*\colon X_1 \to X_2$, which is so defined: if $x = (x_g)_{g \in G} \in X_1$ then $\alpha_*(x) = y$, where $y_g = \alpha(x_g)$ for every $g \in G$.

More generally, if B_1, B_2 are Boolean lattices, for every $s \in B_1$, the restriction of α to $\mathrm{Id}(s)$ is a morphism $\alpha_s\colon \big(\mathrm{Id}(s), (F_1)_s\big) \to \big(\mathrm{Id}(\alpha(s)), (F_2)_{\alpha(s)}\big)$, hence it induces $(\alpha_s)_*\colon X_{1,s}\to X_{2,\alpha(s)}$ (where these Boolean powers correspond to the above pairs). Since $X_1 = \varprojlim X_{1,s}$, $X_2 = \varprojlim X_{2,\alpha(s)}$ (because $\alpha(B_1)$ is cofinal in B_2), then there is a natural mapping $\alpha_*\colon X_1\to X_2$, which is defined as the inverse limit of the mappings α_s.

If $\alpha\colon (B_1, F_1)\to(B_2, F_2)$, $\beta\colon (B_2, F_2)\to(B_3, F_3)$ are morphisms, then $\beta\circ\alpha$ is also a morphism, and $(\beta\circ\alpha)_* = \beta_*\circ\alpha_*$. If α is injective, then α_* is also injective. If there exists a morphism $\beta\colon (B_2, F_2)\to(B_1, F_1)$, such that $\alpha\circ\beta$ is the identity, then α_* splits (that is $\alpha_*\circ\beta_*$ is the identity of X_2).

However, it is not true in general that if α is surjective then so is α_* too.

We shall also deal with pairs $(G, \mathfrak{J})$, where $\mathfrak{J}$ is an ideal of subsets of G. A morphism $\beta\colon (G_1, \mathfrak{J}_1)\to(G_2, \mathfrak{J}_2)$ is a mapping $\beta\colon G_1\to G_2$ such that $\beta(0) = 0$, $\beta(\mathfrak{J}_1)\subseteq\mathfrak{J}_2$. In similar way, we define ${}_*\beta\colon X_1\to X_2$, by letting ${}_*\beta(x) = y$, where $y_g = \bigvee_{\beta(h)=g} x_h$ for every $g\in G_2$; in particular, if $g\notin\beta(G_1)$ then $y_g = 0$ (we note also that since $\operatorname{supp}(x)\in\mathfrak{J}_1$, the supremum exists).

Moreover, if $\gamma\colon (G_2, \mathfrak{J}_2)\to(G_3, \mathfrak{J}_3)$ is another morphism, then $\gamma\circ\beta$ is also a morphism and ${}_*(\gamma\circ\beta) = {}_*\gamma\circ{}_*\beta$.

Let us consider the following special case: $B_1 = F_1 = \mathfrak{B}(I_1)$, $B = F_2 = \mathfrak{B}(I_2)$ where I_1, I_2 are sets, let α be a complete homomorphism of B_1 to B_2.

Then α induces a mapping $\alpha^*\colon I_2\to I_1$, as follows: given $i\in I_2$, the set $\{b\in B_1|\ \alpha(b)\ni i\}$ is an ultrafilter of B_1; if $\bigcap_{\alpha(b)\ni i} b = \emptyset$ then $\emptyset = \alpha(\emptyset) = \alpha(\bigcap_{\alpha(b)\ni i} b) = \bigcap_{\alpha(b)\ni i}\alpha(b)\ni i$, impossible; hence the above intersection is a set consisting of only one point which is defined to be $\alpha^*(i)$.

Let $\tilde{\alpha}^*\colon G^{I_1}\to G^{I_2}$ be the mapping such that $\tilde{\alpha}^*(\varphi) = \varphi\circ\alpha^*$ for every $\varphi\in G^{I_1}$.

If $\theta_1\colon X_1\to G^{I_1}$, $\theta_2\colon X_2\to G^{I_2}$ are the natural bijections indicated in example 1, then $\theta_2\circ\alpha_* = \tilde{\alpha}^*\circ\theta_1$, as it is easy to verify.

Similarly, let $B = F = \mathfrak{B}(I)$, let β be a morphism from $(G_1, \mathfrak{J}_1)$ to $(G_2, \mathfrak{J}_2)$ where $\mathfrak{J}_1 = \mathfrak{B}(G_1)$, $\mathfrak{J}_2 = \mathfrak{B}(G_2)$.

Then β induces a mapping $\tilde{\beta}\colon G_1^I\to G_2^I$, namely $\tilde{\beta}(\varphi) = \beta\circ\varphi$ for every $\varphi\in G_1^I$.

If $\theta_1\colon X_1\to G_1^I$, $\theta_2\colon X_2\to G_2^I$ are the natural bijections of example 1, then $\tilde{\beta}\circ\theta_1 = \theta_2\circ{}_*\beta$.

2. Following the idea in example 1, we shall now indicate a representation of the Boolean power X.

Let B be a Boolean lattice; we recall Stone's representation theorem. Let $\mathfrak{U}^*$ be the set of ultrafilters of B; for every filter H of B, let $\varrho(H) = \{U\in\mathfrak{U}^*|\ H\subseteq U\}$, in particular, if H is the principal filter of $x\in B$, namely $H = \mathrm{Fi}(x) = \{y\in B|\ y\geqslant x\}$, we write $\varrho(x) = \varrho(H)$.

We define a topology on $\mathfrak{U}^*$ by stating that $\{\varrho(H)|\ H$ is a filter of $B\}$ is the collection of closed sets of $\mathfrak{U}^*$. Equivalently, for every $U\in\mathfrak{U}^*$, a fundamental system of neighborhoods is given by $\{N_x(U)|\ x\in U\}$, where $N_x(U) = \{U'\in\mathfrak{U}^*|\ x\in U'\}$. With this topology $\mathfrak{U}^*$ becomes a Hausdorff, locally compact and totally disconnected space. For every $x\in B$, the set $\varrho(x)$ is an open and compact subset of $\mathfrak{U}^*$. The mapping $\varrho\colon B\to\mathcal{OC}(\mathfrak{U}^*)$ (set of open and compact subsets of $\mathfrak{U}^*$) is an isomorphism of Boolean lattices:

$$\varrho(x\vee y) = \varrho(x)\cap\varrho(y)\,,\qquad \varrho(x\wedge y) = \varrho(x)\cap\varrho(y)\,,$$

hence if

$$y\vee y' = x\,,\ y\wedge y' = 0$$

then

$$\varrho(y)\cup\varrho(y') = \varrho(x)\,,\qquad \varrho(y)\cap\varrho(y') = \emptyset;$$

moreover θ is an injection and maps B onto $\mathcal{OC}(\mathfrak{U}^*)$.

Conversely, if $\mathcal{S}$ is a Hausdorff, locally compact, totally disconnected space, then $\mathcal{OC}(\mathcal{S}) = B$ is a Boolean lattice, there is a homeomorphism between the topological space $\mathfrak{U}^*$ of ultrafilters of B and the given space $\mathcal{S}$, and $\varrho(B) = \mathcal{OC}(\mathcal{S})$.

Actually, in the above theorem it is enough to consider a subset $\mathfrak{U}$ of $\mathfrak{U}^*$, which separates elements of B: if $x, y\in B$, $x\neq y$, there exists $U\in\mathfrak{U}$ such that either $x\in U$, $y\notin U$ or $y\in U$, $x\notin U$.

Thus, if $B = \mathfrak{B}(I)$, for a set I, then $\mathfrak{U}$ may be taken to be the set of principal ultrafilters of B, which is in one-to-one correspondence with I.

We shall make use of the following easy result:

(a) *If F, F' are filters of the Boolean algebra B, if $F\vee F' = B$, $F\cap F' = \{1\}$, then there exists $x\in B$ such that $F = \mathrm{Fi}(x)$, $F' = \mathrm{Fi}(x')$.*

The proof is straightforward (see Hermes [4]).

We may prove:

THEOREM 1. *Let B be a Boolean algebra (with last element 1). There exists a natural bijection θ from $X = \bigtimes_{B,F}(G, \mathfrak{J}_0)$ onto the set $\mathcal{F}$ of all functions $\xi\colon\mathfrak{U}\to G$ such that:*

(1) *$\xi(U)$ is finite,*

(2) *ξ is continuous,*

(3) $\bigcap_{\xi(U)=0} U\subseteq F$.

Proof. Let $x = (x_g)_{g\in G}\in X$, we shall define $\xi = \theta(x)\colon\mathfrak{U}\to G$. From $1 = \bigvee_{g\in\operatorname{supp}(x)} x_g$, $\operatorname{supp}(x)\in\mathfrak{J}_0$ (ideal of finite subsets of G), it follows by Stone's representation theorem, that $\mathfrak{U} = \varrho(1) = \bigcup_{g\in\operatorname{supp}(x)}\varrho(x_g)$. Moreover, if $g_1\neq g_2$

then $\emptyset = \varrho(0) = \varrho(x_{g_1} \wedge x_{g_2}) = \varrho(x_{g_1}) \cap \varrho(x_{g_2})$. Thus, for every $U \in \mathfrak{U}$ there exists exactly one element $g \in G$ such that $U \in \varrho(x_g)$, that is $x_g \in U$. We denote this element by g_U and define $\xi(U) = g_U$ for every $U \in \mathfrak{U}$.

Since $\mathrm{supp}(x) \in \mathfrak{F}_0$ then ξ assumes only finitely many distinct values.

Also, ξ is continuous, because given any $U \in \mathfrak{U}$, let $U' \in N_{x_g}(U)$, where $x_g \in U$, that is $\xi(U) = g$; then $x_g \in U'$, so $\xi(U') = g$.

We have $x_0 \in U$ for every $U \in \mathfrak{U}$ such that $\xi(U) = 0$. On the other hand, if $t \in \bigcap_{\xi(U)=0} U$ and $t \not\geqslant x_0$ then $\varrho(t) \not\supseteq \varrho(x_0)$, hence there exists $U \in \mathfrak{U}$ such that $U \in \varrho(x_0)$, $U \notin \varrho(t)$, that is $x_0 \in U$, $t \notin U$; however, from $x_0 \in U$ it follows that $\xi(U) = 0$, hence by hypothesis $t \in U$, which is a contradiction. Therefore $\bigcap_{\xi(U)=0} U = \mathrm{Fi}(x_0) \subseteq F$, since $x_0 \in F$, by hypothesis.

We show that θ is an injection. If $x \neq y$, there exists $g \in G$ such that $x_g \neq y_g$; since $\mathfrak{U}$ separates points, there exists $U \in \mathfrak{U}$ such that $x_g \in U$, $y_g \notin U$ (or vice-versa), hence $\xi(U) = g$, $\eta(U) \neq g$ (where $\theta(x) = \xi$, $\theta(y) = \eta$).

Now, we prove that θ is surjective. Let $\xi\colon \mathfrak{U} \to G$ be a mapping satisfying the above conditions (1), (2), (3). For every $g \in G$ let $\mathfrak{U}_g = \{U \in \mathfrak{U} \mid \xi(U) = g\}$, thus $\mathfrak{U}$ is the union of finitely many pairwise disjoint sets $\mathfrak{U}_g$, and $\mathfrak{U}_g \neq \emptyset$ if and only if $g \in \xi(\mathfrak{U})$.

Let $H_g = \bigcap_{U \in \mathfrak{U}_g} U$, when $g \in \xi(\mathfrak{U})$.

We show that $\mathfrak{U}_g = \{U \in \mathfrak{U} \mid U \supseteq H_g\}$. Clearly if $U \in \mathfrak{U}_g$ then $U \supseteq H_g$. Conversely, let $U \in \mathfrak{U}$, $H \supseteq H_g$ and assume that $\xi(U) \neq g$. By continuity of ξ, there exists $z \in U$ such that if $z \in U' \in \mathfrak{U}$ then $\xi(U') = \xi(U) \neq g$; thus $z \notin V$ for every $V \in \mathfrak{U}_g$, hence $z' \in V$ for every $V \in \mathfrak{U}_g$, so $z' \in \bigcap_{V \in \mathfrak{U}_g} V = H_g \subseteq U$, and therefore $0 = z \wedge z' \in U$, which is impossible.

For every $g \in \xi(\mathfrak{U})$, let $H'_g = \bigcap_{\xi(U) \neq g} U$. We have $H'_g = \bigcap_{\substack{h \in \xi(\mathfrak{U}) \\ h \neq g}} H_h$ (this being a finite intersection). Now, if $U \in \mathfrak{U}$ then $\xi(U) = g$ or $\xi(U) = h \neq g$, and then $U \supseteq H_g$ or $U \supseteq H_h \supseteq H'_g$, so that in any case, $U \supseteq H_g \cap H'_g$, showing that $H_g \cap H'_g = \{1\}$, because $\mathfrak{U}$ separates points. Moreover, $H_g \vee H'_g = B$, because if the filter $H_g \vee H'_g$ is different from B, then it is contained in some ultrafilter U; since $H_g \subseteq U$ then $\xi(U) = g$; on the other hand, since $H'_g \subseteq U$, and H'_g is a finite intersection, then $H_h \subseteq U$ for some $h \neq g$, so $\xi(U) = h \neq g$, which is a contradiction.

By (a), for every $g \in \xi(U)$, there exists an element $x_g \in B$ such that $H_g = \mathrm{Fi}(x_g)$. If $g \notin \xi(\mathfrak{U})$, we put $x_g = 0$.

Let $x = (x_g)_{g \in G}$ and let us prove that $x \in X$. By hypothesis $\mathrm{Fi}(x_0) = H_0 = \bigcap_{\xi(U)=0} U \subseteq F$, so $x_0 \in F$.

If $g_1, g_2 \in G$, and $x_{g_1} \wedge x_{g_2} \neq 0$, there exists $U \in \mathfrak{U}$ such that $x_{g_1} \wedge x_{g_2} \in U$, hence $H_{g_1} = \mathrm{Fi}(x_{g_1}) \subseteq U$, $H_{g_2} = \mathrm{Fi}(x_{g_2}) \subseteq U$, so $g_1 = \xi(U) = g_2$.

Since $\xi(\mathfrak{U})$ is finite, then $\mathrm{supp}(x)$ is finite. Finally, if $t \geqslant \bigvee_{g \in \mathrm{supp}(x)} x_g$ then

$$t \in \bigcap_{g \in \mathrm{supp}(x)} \Big(\bigcap_{U \in \mathfrak{U}_g} U\Big) = \bigcap_{U \in \mathfrak{U}} U = \{1\}$$

(because $\mathfrak{U}$ separates points), so $\bigvee_{g \in \mathrm{supp}(x)} x_g = 1$.

Moreover, $\theta(x) = \xi$, because if $U \in \mathfrak{U}$, we have $\theta(x)(U) = g_U$ (the only element of G such that U contains x_{g_U}), while $\xi(U) = g$ if and only if $x_g \in U$. ∎

Now we shall generalize the preceding result to the case where B is a Boolean lattice which is not assumed to have last element.

THEOREM 2. *Let B be a Boolean lattice (without last element). There exists a natural bijection θ from $X = \bigtimes_{B,F} (G, \mathfrak{F}_0)$ onto the set $\mathcal{F}$ of all functions $\xi\colon \mathfrak{U} \to G$ such that:*

(1) *ξ is finite-valued on every open and compact subset of $\mathfrak{U}$,*

(2) *ξ is continuous,*

(3) *for every $s \in B$, $\bigcap_{\xi(U)=0} U_s \subseteq F_s$ (where $U_s = \{u \wedge s \mid u \in U\}$).*

Proof. For every $s \in F$, $\mathrm{Id}(s) = \{x \in G \mid 0 \leqslant x \leqslant s\}$ is a Boolean algebra with last element s, $F_s = \{u \wedge s \mid u \in F\}$ is a filter in $\mathrm{Id}(s)$. Let $X_s = \bigtimes_{\mathrm{Id}(s), F_s} (G, \mathfrak{F}_0)$.

If $s \leqslant t$ are elements in F, let $\pi^t_s\colon X_t \to X_s$ be the mapping already considered before definition 2, so that $X = \varprojlim X_s$.

For every $s \in B$, let $\mathfrak{U}_s$ be the set of ultrafilters of the Boolean algebra $\mathrm{Id}(s)$, with its Stone topology. Then $\mathfrak{U}_s = \{U_s \mid U \in \mathfrak{U}\}$, where $U_s = \{u \wedge s \mid u \in U\}$. In fact, given $V \in \mathfrak{U}_s$ let U be the filter of B generated by V, thus $s \in U \neq B$, $U_s = U \cap \mathrm{Id}(s)$ is a filter of $\mathrm{Id}(s)$, containing V hence equal to V. If U' is any filter of B such that $U'_s = U \cap \mathrm{Id}(s)$, then $U' = U$ (if $t \in U'$ then $t \wedge s \in U'_s = U \cap \mathrm{Id}(s)$ hence $t \in U$, and conversely); hence U is an ultrafilter of B, for if U' is an ultrafilter of B, $U' \supseteq U$, then $U'_s = U' \cap \mathrm{Id}(s) = U \cap \mathrm{Id}(s) = V$, so $U' = U$. Conversely, if $U \in \mathfrak{U}$ then U_s is an ultrafilter of $\mathrm{Id}(s)$, as one sees easily.

Thus, the mapping $\lambda_s\colon \mathfrak{U}_s \to \mathfrak{U}$, $\lambda_s(V) = U$, where $U_s = V$, is well defined (as we have seen) and an injection. Moreover, the topology on $\mathfrak{U}_s$ is induced by the topology on $\mathfrak{U}$, through the natural injection λ_s. Also, if $s \leqslant t$, there is a natural injection $\lambda^t_s\colon \mathfrak{U}_s \to \mathfrak{U}_t$, defined by $\lambda^t_s(U_s) = U_t$ for $U \in \mathfrak{U}$; clearly, $\lambda_t \circ \lambda^t_s = \lambda_s$, and if $s \leqslant t \leqslant u$ then $\lambda^u_t \circ \lambda^t_s = \lambda^u_s$, λ^s_s being the identity mapping.

Let $\tilde{\lambda}^t_s\colon G^{\mathfrak{U}_t} \to G^{\mathfrak{U}_s}$ be defined by $\tilde{\lambda}^t_s(\varphi) = \varphi \circ \lambda^t_s$ for every $\varphi\colon \mathfrak{U}_t \to G$. Then, if $s \leqslant t \leqslant u$, we have $\tilde{\lambda}^u_s = \tilde{\lambda}^t_s \circ \tilde{\lambda}^u_t$, and $\tilde{\lambda}^s_s$ is the identity mapping.

Let $\mathcal{F}_s$ be the set of mappings $\varphi\colon \mathfrak{U}_s \to G$ satisfying properties (1), (2), (3) of theorem 1 (with respect to the Boolean algebra $\mathrm{Id}(s)$).

Next, we show that the following diagram is commutative (for $s \leqslant t$ in F):

$$\begin{array}{ccc} X_t & \xrightarrow{\theta_t} & G^{\mathfrak{U}_t} \\ {\scriptstyle \pi_s^t}\downarrow & & \downarrow{\scriptstyle \tilde{\lambda}_s^t} \\ X_s & \xrightarrow{\theta_s} & G^{\mathfrak{U}_s} \end{array}$$

In fact, let $x = (x_g)_{g\in G} \in X_t$, so $x_g \leqslant t$ for every $g \in G$. Then, if $\xi_s = \theta_s(\pi_s^t(x)) = \theta_s((x_g \wedge s)_{g\in G})$, we have $\xi_s(U_s) = g$ if and only if $x_g \wedge s \in U_s$ (for every $U \in \mathfrak{U}$). On the other hand, if $\xi_t = \theta_t(x)$ then $\xi_t(U_t) = g$ if and only if $x_g \in U_t$ (for $U \in \mathfrak{U}$) and therefore $[\tilde{\lambda}_s^t(\xi_t)](U_s) = \xi_t(\lambda_s^t(U_s)) = \xi_t(U_t)$. Thus, if $U \in \mathfrak{U}$, from $x_g \leqslant t$ it follows that $x_g \in U_t$ if and only if $x_g \wedge s \in U_s$, showing the commutativity of the diagram.

Since $\theta_s(X_s) = \mathcal{F}_s$, $\theta_t(X_t) = \mathcal{F}_t$, then $\tilde{\lambda}_s^t(\mathcal{F}_t) \subseteq \mathcal{F}_s$. We may therefore consider the inverse limits $\varprojlim \mathcal{F}_s$, $\varprojlim G^{\mathfrak{U}_s}$. Since $\mathcal{F}_s \subseteq G^{\mathfrak{U}_s}$ for every $s \in F$, it follows that $\varprojlim \mathcal{F}_s \subseteq \varprojlim G^{\mathfrak{U}_s}$. Moreover, $G^{\mathfrak{U}} = \varprojlim G^{\mathfrak{U}_s}$.

In fact, if $s \in B$ we define $\tilde{\lambda}_s\colon G^{\mathfrak{U}} \to G^{\mathfrak{U}_s}$ by $[\tilde{\lambda}_s(\varphi)](U_s) = \varphi(U)$ where $U \in \mathfrak{U}$, $\varphi \in G^U$, thus $\tilde{\lambda}_s(\varphi) = \varphi \circ \lambda_s$ and therefore $\tilde{\lambda}_s^t \circ \tilde{\lambda}_t = \tilde{\lambda}_s$ when $s \leqslant t$. On the other hand, if $\mathcal{H}$ is a set, if $\mu_s\colon \mathcal{H} \to G^{\mathfrak{U}_s}$ is a mapping (for every $s \in B$), such that if $s \leqslant t$ then $\mu_s = \tilde{\lambda}_s^t \circ \mu_t$, we define $\mu\colon \mathcal{H} \to G^{\mathfrak{U}}$ as follows: if $h \in \mathcal{H}$, if $U \in \mathfrak{U}$ we put $\mu(h)(U) = \mu_s(h)(U_s)$ (and this definition is independent of $s \in B$); then $\tilde{\lambda}_s \circ \mu = \mu_s$ for every $s \in B$. Therefore $G^{\mathfrak{U}} = \varprojlim G^{\mathfrak{U}_s}$.

So, every element of $\mathcal{F} = \varprojlim \mathcal{F}_s$ is a mapping from $\mathfrak{U}$ into G; since $\theta_s\colon X_s \to \mathcal{F}_s \subseteq G^{\mathfrak{U}_s}$ for every $s \in B$, by the commutativity $\theta_s \circ \pi_s^t = \tilde{\lambda}_s^t \circ \theta_t$ it follows that there exists $\theta\colon X \to \mathcal{F}$ such that $\theta_s \circ \pi_s = \tilde{\lambda}_s \circ \theta$ (for every $s \in B$), where $\pi_s\colon X \to X_s$, $\tilde{\lambda}_s\colon \mathcal{F} \to \mathcal{F}_s$ are the canonical mappings.

Moreover, since each mapping θ_s is a bijection, the same holds for θ. We describe now the properties of the mappings belonging to $\mathcal{F}$; precisely, we show that $\mathcal{F}$ is the set of mappings from $\mathfrak{U}$ to G satisfying conditions (1), (2), (3) of the statement of the theorem.

Let $\xi = \theta(x) \in \mathcal{F}$, let $\mathfrak{C}$ be an open and compact subset of $\mathfrak{U}$; by Stone's theorem, there exists $s \in B$ such that $\mathfrak{C} = \varrho(s) = \{U \in \mathfrak{U} \mid s \in U\}$.

Now $\theta(x)(\varrho(s)) = \theta(x)(\lambda_s(\mathfrak{U}_s)) = (\theta(x) \circ \lambda_s)(\mathfrak{U}_s) = [\tilde{\lambda}_s(\theta(x))](U_s) = [\theta_s(\pi_s(x))](U_s)$ and this set is finite, by theorem 1.

Also, $\xi = \theta(x)$ is continuous, because if $V \in \mathfrak{U}$, if $t \in V$, then $\theta_t(\pi_t(x))$ is continuous at V_t (by theorem 1), so there exists $s \in V_t$ such that if $s \in U_t$, then $[\theta_t(\pi_t(x))](U_t) = [\theta_t(\pi_t(x))](V_t)$, hence $\theta(x)(U) = \theta(x)(V)$.

We show that $\theta(x)$ satisfies property (3). Let $s \in B$, let $u \in B$ be such that if $\theta(x)(U) = 0$ then $u \in U_s$.

Then $u \wedge s$ is such that if $\theta_s(\pi_s(x))(U_s) = 0$ then $\theta(x)(U) = 0$ so $u \in U_s$ hence $u \wedge s \in F_s$ and $u \in F$ (since (3) is satisfied by hypothesis by $\theta_s(\pi_s(x)) \in \mathcal{F}_s$).

Conversely, let $\xi\colon \mathfrak{U} \to G$ be a mapping satisfying conditions (1), (2), (3). We shall show that, for every $s \in B$, the mapping $\tilde{\lambda}_s(\xi)\colon \mathfrak{U}_s \to G$ satisfies (1), (2), (3) of theorem 1; so $\tilde{\lambda}_s(\xi) \in \mathcal{F}_s$ and there exists a unique element $x^s \in X_s$ such that $\tilde{\lambda}_s(\xi) = \theta_s(x^s)$. Also, if $s \leqslant t$ then $\theta_s(x^s) = \tilde{\lambda}_s(\xi) = \tilde{\lambda}_s^t(\tilde{\lambda}_t(\xi)) = \tilde{\lambda}_s^t(\theta_t(x^t)) = \theta_s(\pi_s^t(x^t))$ hence $x^s = \pi_s^t(x^t)$. Thus, there exists $x \in X$ such that $\pi_s(x) = x^s$ for every $s \in B$. Then if $s \in U$, $\theta(x)(U) = \tilde{\lambda}_s(\theta(x))(U_s) = [\theta_s(\pi_s(x))](U_s) = \theta_s(x^s)(U_s) = \tilde{\lambda}_s(\xi)(U_s) = \xi(U)$, that is, $\xi \in \mathcal{F}$.

So, we have only to prove that $\lambda_s(\xi)$ satisfies conditions (1), (2), (3) of theorem 1. But these are automatically verified in virtue of the hypothesis on ξ, as one may check without any difficulty.

We may obtain another representation theorem, under broader hypothesis:

THEOREM 3. *Let G be a set $\#(G) = a$, let B be a a-complete Boolean algebra, F a filter of B, $\mathfrak{J}$ an ideal of subsets of G; let $X = \bigtimes_{B,F}(G, \mathfrak{J})$. There exists a natural injection θ from X into the set $\mathcal{F}$ of all functions ξ, defined in some open and dense subset $\mathcal{O}$ of $\mathfrak{U}$ (topological space of ultrafilters of B), with values in G, such that:*

(1) *ξ is continuous*

(2) $\bigcap_{\xi(U)=0} U \subseteq F$.

Moreover, if $\mathfrak{J} = \mathfrak{B}(G)$, then θ is a bijection (in the following sense: every $\xi \in \mathcal{F}$ is the restriction of a function $\theta(x)$, where $x \in X$).

Proof. Since this result is analogous to theorem 1, we shall only sketch the main points of the proof.

Let $x = (x_g)_{g\in G} \in X$, $\varrho(x_g) = \{U \in \mathfrak{U} \mid x_g \in U\}$, $\mathcal{O}_x = \bigcup_{g\in G} \varrho(x_g)$. Since $\mathrm{supp}(x)$ is not necessarily finite, we cannot conclude in general that $1 = \bigvee_{g\in G} x_g$ implies that $\mathfrak{U} = \varrho(1)$ is equal to $\mathcal{O}_x$. However $\mathcal{O}_x$ is an open set in $\mathfrak{U}$ (since each $\varrho(x_g)$ is open). Moreover, $\mathcal{O}_x$ is dense; in fact, given any $V \in \mathfrak{U}$ and any fundamental neighborhood $N_z(V) = \{U \in \mathfrak{U} \mid U \ni z\}$, where $z \in V$, from $z = z \wedge 1 = z \wedge (\bigvee_{g\in G} x_g) = \bigvee_{g\in G}(z \wedge x_g)$, there exists $g \in G$ such that $z \wedge x_g \neq 0$; let $U \in \mathfrak{U}$ be such that $z \wedge x_g \in U$, then $z \in U$, so $U \in \mathcal{O}_x \cap N_z(V)$.

As in theorem 1, we define $\theta(x) = \xi\colon \mathcal{O}_x \to G$ by letting $\xi(U) = g$ when $x_g \in U$ (g is unique with this property).

It follows from the above-mentioned proof, that ξ is continuous, $\bigcap_{\xi(U)=0} U \subseteq F$ and θ is an injection.

We proceed now to prove that θ is surjective, when $\mathfrak{J} = \mathcal{B}(G)$. Let $\xi\colon \mathcal{O} \to G$ be a continuous mapping, where $\mathcal{O}$ is a dense and open subset of $\mathfrak{U}$, and G has the discrete topology. For every $g \in G$, the set $\xi^{-1}(\{g\})$ is an open and closed subset of $\mathcal{O}$, that is, the intersection with $\mathcal{O}$ of an open and closed subset of $\mathfrak{U}$; therefore, there exists an element $x_g \in B$ such that $\xi^{-1}(\{g\}) = \varrho(x_g) \cap \mathcal{O}$. Also, x_g is uniquely defined, because if $y_g \in B$, $\varrho(y_g) \cap \mathcal{O} = \varrho(x_g) \cap \mathcal{O}$, and if $x_g \neq y_g$ then for example there exists $U \in \mathfrak{U}$ such that $x_g \in U$, $y_g \notin U$; thus $y_g' \in U$. Consider the neighbourhood $N_{x_g \wedge y_g'}(U)$; by density of $\mathcal{O}$, there exists $V \in N_{x_g \wedge y_g'}(U) \cap \mathcal{O}$, hence $x_g \wedge y_g' \in V$, so $x_g \in V$, $y_g \notin V$ and therefore $\varrho(y_g) \cap \mathcal{O} \neq \varrho(x_g) \cap \mathcal{O}$, a contradiction.

Let $x = (x_g)_{g \in G}$. We shall show that $x \in X$.

If $U \in \mathcal{O}$ and $\xi(U) = 0$ then $U \in \xi^{-1}(0) = \varrho(x_0) \cap \mathcal{O}$ so $x_0 \in U$. From the hypothesis, we have $x_0 \in \bigcap_{\xi(U)=0} U \subseteq F$.

If $g_1, g_2 \in G$, $g_1 \neq g_2$, then $x_{g_1} \wedge x_{g_2} = 0$. In fact, if $x_{g_1} \wedge x_{g_2} \neq 0$, there exists $U \in \mathfrak{U}$ such that $x_{g_1} \wedge x_{g_2} \in U$; by density of $\mathcal{O}$, there exists $V \in \mathcal{O} \cap N_{x_{g_1} \wedge x_{g_2}}(U)$ so $V \in \mathcal{O}$, $x_{g_1} \in V$, $x_{g_2} \in V$ hence $V \in \mathcal{O} \cap \varrho(x_{g_1})$, $V \in \mathcal{O} \cap \varrho(x_{g_2})$, thus $\xi(V) = g_1$, $\xi(V) = g_2$, and $g_1 = g_2$.

Let $b = \bigvee_{g \in G} x_g$; if $b \neq 1$, there exists $U \in \mathfrak{U}$ such that $b \notin U$, hence $b' \in U$. By the density of $\mathcal{O}$ there exists $W \in \mathcal{O} \cap N_{b'}(U)$, so $b' \in W$. On the other hand, let $g = \xi(W)$, hence $W \in \xi^{-1}(\{g\}) = \varrho(x_g) \subseteq \varrho(b)$, thus $b \in W$ and $0 = b \wedge b' \in W$, which is impossible.

This shows that $x = (x_g)_{g \in G} \in X$ (under the hypothesis that $\mathfrak{J} = \mathcal{B}(G)$). Next, we prove that ξ is the restriction of $\theta(x)$. For this, we note that $\theta(x)$ is defined on the open and dense set $\mathcal{O}_x = \bigcup_{g \in G} \varrho(x_g)$, and $\mathcal{O} \subseteq \mathcal{O}_x$ (because if $U \in \mathcal{O}$ and $\xi(U) = g$ then $U \in \varrho(x_g) \cap \mathcal{O} \subseteq \varrho(x_g) \subseteq \mathcal{O}_x$). Now, by definition, we have $\theta(x)(U) = g$ if and only if $x_g \in U$, that is $U \in \mathcal{O}_x \cap \varrho(x_g)$; hence if $U \in \mathcal{O} \cap \varrho(x_g)$ then $\xi(U) = g$. ■

3. We shall define a natural topology on $X = \mathop{\times}\limits_{B,F} (G, \mathfrak{J})$, even without assuming any (non-discrete) topology on G or B.

An element $c \in B$ is said to be *compact* if the following property is satisfied: if $(s_i)_{i \in I}$ is any family of elements of B, if $\bigvee_{i \in I} s_i$ exists and $c \leqslant \bigvee_{i \in I} s_i$ then there exists a finite subset I_0 of I such that $c \leqslant \bigvee_{i \in I_0} s_i$.

(b) *The set C of compact elements of B is an ideal.*

Proof. We have $0 \in C$. If $c_1, c_2 \in C$, if $(s_i)_{i \in I}$ is a family of elements of B, such that $\bigvee_{i \in I} s_i$ exists and $c_1 \vee c_2 \leqslant \bigvee_{i \in I} s_i$, then there exist finite sets $I_1 \subseteq I$, $I_2 \subseteq I$ such that $c_1 \leqslant \bigvee_{i \in I_1} s_i$, $c_2 \leqslant \bigvee_{i \in I_2} s_i$, hence $c_1 \vee c_2 \leqslant \bigvee_{i \in I_1 \cup I_2} s_i$.

If $d \leqslant c$, $c \in C$, if $d \leqslant \bigvee_{i \in I} s_i$, and if d^* is such that $d \vee d^* = c$, $d \wedge d^* = 0$, then $c = d \vee d^* \leqslant (\bigvee_{i \in I} s_i) \vee d^*$, hence by hypothesis there exists a finite subset I_0 of I, such that $c \leqslant (\bigvee_{i \in I_0} s_i) \vee d^*$ and therefore

$$d = c \wedge d \leqslant [(\bigvee_{i \in I_0} s_i) \vee d^*] \wedge d = (\bigvee_{i \in I_0} s_i) \wedge d \leqslant \bigvee_{i \in I_0} s_i . \ ■$$

We shall require the following result on Boolean algebras:

(c) *Let B be a Boolean algebra, with last element* 1. *Then the following statements are equivalent*:

(1) *B is finite*;

(2) *$1 \in C$ and B is $\aleph_0$-complete*;

(3) *the ascending chain condition holds for elements of B*;

(4) *the descending chain condition holds for elements of B.*

Proof. It is obvious that (1)→(2) and (3)→(4). We show that (2)→(3). Let $(a_n)_n$ be an infinite increasing chain of elements of B. Since B is $\aleph_0$-complete, there exists $b \in B$, $b = \bigvee_{n=1}^{\infty} a_n$. From $1 = b \vee b' = (\bigvee_{n=1}^{\infty} a_n) \vee b'$ and the hypothesis that 1 is compact, it follows that there exists m such that $1 = a_m \vee b'$, hence $b = b \wedge (a_m \vee b') = b \wedge a_m = a_m$, thus $a_n = a_m$ for every $n > m$.

Finally, we prove that (4)→(1). By known result (see [4]), the distributive lattice B, with descending chain condition has the following property: every element of B may be written uniquely as the supremum of finitely many irreducible elements; in particular $1 = b_1 \vee \ldots \vee b_m$. Since B is a Boolean algebra, an irreducible element is an atom; if $b \in B$ is any atom then $b_0 \leqslant 1 = b_1 \vee \ldots \vee b_m$ hence $b_0 = b_i$ (for some i); thus B has only finitely many atoms $b_1, \ldots, b_m$, which generate B, so B is finite. ■

We define a topology on X as follows. For every $x \in X$ a fundamental systems of neighborhoods of x is the collection $\{N_c(x) \mid c \in C\}$, where $N_c(x) = \{y \in X \mid \pi_c(y) = \pi_c(x)\}$ (we recall that $\pi_c\colon X \to X_c$ is the canonical mapping, and if B has last element 1, then $\pi_c(x) = \pi_c^1(x) = (x_g \wedge c)_{g \in G}$).

It is straightforward to check that this defines indeed a topology on X.

If 0 is the only compact element, then the topology is such that X is the only neighborhood of any $x \in X$.

(d) *If G has at least 2 elements then the following statements are equivalent:*
(1) *the topology on X is discrete;*
(2) (α) *for every $s \in F$ there exists $c \in C$ such that $F_s \subseteq \mathrm{Fi}(a^*)$, where $a = c \wedge s$, $a \wedge a^* = 0$, $a \vee a^* = s$,*
(β) *for every $x \in X$ there exists $s \in B$ such that $\pi_s^{-1}(\pi_s(x)) = \{x\}$.*

Proof. (1)$\to$(2). Let us assume that (α) is not satisfied, so there exists $s \in F$ such that for every $c \in C$ we have $F_s \nsubseteq \mathrm{Fi}(a^*)$. Let $d \in F$, be such that $a^* \nleqslant d \wedge s$, hence if $b = (d \wedge s) \vee a = (d \vee c) \wedge s$, we have $a \leqslant b < s$.
We define elements $x^s, y^s \in X_s$ (y^s depends on c) as follows:

$$x_0^s = s, \ x_g^s = 0, \ x_h^s = 0 \quad \text{and} \quad y_0^s = b, \ y_g^s = b^*, \ y_h^s = 0$$

(where $g \neq 0$, and $h \in G$, $h \neq g$, $h \neq 0$) and where $b \wedge b^* = 0$, $b \vee b^* = s$.

Then x^s, y^s are distinct elements in X_s, and $\pi_a^s(x^s) = \pi_a^s(y^s)$. Let $x = \iota_s(x^s)$, $y = \iota_s(y^s)$ be elements in X, so $x \neq y$ (because ι_s is injective).
Then $y \in N_c(x)$ for every $c \in C$, where y is defined as above, for the element c. In fact, $\pi_c(y) = \pi_c^{s \vee c} \circ \pi_{s \vee c} \circ \iota_{s \vee c} \circ \iota_s^{s \vee c}(y^s) = \pi_c^{s \vee c} \circ \iota_s^{s \vee c}(y^s)$ thus

$$\big(\pi_c(y)\big)_0 = [(y^s)_0 \vee s^*] \wedge c = (b \vee s^*) \wedge c = a \vee s^* = (c \wedge s) \vee s^* = c \wedge (s \vee c) = c,$$

$$\big(\pi_c(y)\big)_g = (y^s)_g \wedge c = b^* \wedge c = (b^* \wedge s) \wedge c = b^* \wedge a = 0,$$

$$\big(\pi_c(y)\big)_h = 0 \quad \text{for} \quad h \neq g, \ h \neq 0,$$

and similarly

$$\big(\pi_c(x)\big)_0 = c, \qquad \big(\pi_c(x)\big)_g = 0, \qquad \big(\pi_c(x)\big)_h = 0 \quad \text{for} \quad h \neq g, \ h \neq 0.$$

Thus, the topology of X is not discrete.
Now, if X is discrete, for every $x \in X$ there exists $c \in C$ such that $N_c(x) = \{x\}$; taking $s = c$, if $y \in X$ is such that $\pi_s(y) = \pi_s(x)$, then $y \in N_s(x) = \{x\}$ and this proves (β).
(2)$\to$(1). Given $x \in X$, by (β) there exists $s \in B$ such that $\pi_s^{-1}(\pi_s(x)) = \{x\}$. By ($\alpha$), there exists $c \in C$ such that $F_s \subseteq \mathrm{Fi}(a^*)$ where $a = c \wedge s$, $a \wedge a^* = 0$, $a \vee a^* = s$.
We show that if $y^s \in X_s$ and $\pi_a^s(y^s) = \pi_a^s(x^s)$ then $y^s = x^s$. In fact, from $x_0^s, y_0^s \in F_s \subseteq \mathrm{Fi}(a^*)$ we have $x_0^s \vee a = y_0^s \vee a = s$; by hypothesis $y_g^s \wedge a = x_g^s \wedge a$ for every $g \in G$, in particular $y_0^s \wedge a = x_0^s \wedge a$. By distributivity, we deduce that $y_0^s = x_0^s$, hence necessarily $b = \bigvee_{g \neq 0} y_g^s = \bigvee_{g \neq 0} x_g^s$ (it is the relative complement of $x_0^s = y_0^s$ in s) and $b \leqslant a$. Therefore $x \vee a = y_g^s \vee a = a$ and by distributivity, $y_g^s = x_g^s$ (for every $g \in G$).

Now, let us consider the neighborhood $N_c(x)$. If $y \in N_c(x)$, let $y^s = \pi_s(y)$, $x^s = \pi_s(x)$. From $\pi_c(y) = \pi_c(x)$ we deduce that $\pi_a^s(y^s) = \pi_a^s(\pi_s(y)) = \pi_a(y) = \pi_a(x) = \pi_a^s(\pi_s(x)) = \pi_a^s(x^s)$ hence $y^s = x^s$. Thus $y = x$. ∎

If B has last element, condition (2) (β) above is automatically satisfied (with $s = 1$), while condition (2) (α) becomes: (α_1) *there exists $c \in C$ such that $F \subseteq \mathrm{Fi}(c')$.*
If B is finite, then $1 \in C$ and $F \subseteq \mathrm{Fi}(0)$, so the topology is discrete.

(e) *If G has at least 2 elements, if B is a complete Boolean algebra, then the following statements are equivalent:*
(1) *X is a Hausdorff space;*
(2) *X is a T_0-space;*
(3) *if $\bigvee_{c \in C} c = a$ then $F \subseteq \mathrm{Fi}(a')$.*

Proof. Clearly, (1)$\to$(2). Now, if $a = \bigvee_{c \in C} c$ let us assume that $F \nsubseteq \mathrm{Fi}(a')$, so there exists $s \in F$, $s \notin \mathrm{Fi}(a')$, hence $s \vee a \neq 1$. Let $x = (x_h)_{h \in G}$, $y = (y_h)_{h \in G}$ be defined by $x_0 = 1$, $x_g = 0$, $x_h = 0$, and $y_0 = s \vee a$, $y_g = (s \vee a)'$, $y_h = 0$ (for $g \neq 0$ and all $h \in G$, $h \neq g$, $h \neq 0$). Then $x, y \in X$, since $s \vee a \in F$, and $x \neq y$.
We have $x_0 \wedge c = c$, $y_0 \wedge c = (s \vee a) \wedge c = c$, $x_g \wedge c = 0$, $y_g \wedge c = (s \vee a)' \wedge c = s' \wedge a' \wedge c = 0$, and also $x_h \wedge c = 0 = y_h \wedge c$. Thus $x \in N_c(y)$, $y \in N_c(x)$ for every $c \in C$, showing that X is not a T_0-space.
Finally, (3)$\to$(1). In fact, let $x, y \in X$, $x \neq y$. Then from $x_0, y_0 \in F \subseteq \mathrm{Fi}(a')$, we have $x_0 \vee a = y_0 \vee a = 1$. Hence $x_g \leqslant \bigvee_{g \neq 0} x_g = x_0' \leqslant a$, and also $y_g \leqslant a$, so $x_g \vee a = y_g \vee a = a$ for every $g \in G$, $g \neq 0$. Thus, $(x_g \wedge a)_g \neq (y_g \wedge a)_g$, otherwise, by distributivity, $x_g = y_g$ for every $g \in G$, against the hypothesis.
Since $a = \bigvee_{c \in C} c$ and B is a complete Boolean algebra, then $x_g \wedge a = \bigvee_{c \in C} (x_g \wedge c)$, $y_g \wedge a = \bigvee_{c \in C} (y_g \wedge c)$, hence there exists $g \in G$ and $c \in C$ such that $x_g \wedge c \neq y_g \wedge c$. It follows that $N_c(x) \cap N_c(y) = \emptyset$ and therefore X is a Hausdorff space. ∎

We shall now consider the effect of a morphism α: $(B_1, F_1) \to (B_2, F_2)$ (between pairs of Boolean lattices and filters) on the topology of $X_1 = \bigtimes_{B_1, F_1} (G, \mathfrak{F})$ and $X_2 = \bigtimes_{B_2, F_2} (G, \mathfrak{F})$.

(f) *If α: $B_1 \to B_2$ is an injective complete homomorphism from Boolean lattices, if C_1, C_2 are respectively their ideals of compact elements, then $\alpha^{-1}(C_2) \subseteq C_1$.*

Proof. Let $a \in B_1$ be such that $\alpha(a) \in C_2$. If $a \leqslant \bigvee_{i \in I} b_i$ (where $(b_i)_{i \in I}$ is a family of elements in B_1) then $\alpha(a) \leqslant \alpha(\bigvee_{i \in I} b_i) = \bigvee_{i \in I} \alpha(b_i)$, and since $\alpha(a)$

is compact, there exists a finite subset I_0 of I, such that $\alpha(a) \leqslant \bigvee_{i\in I_0} \alpha(b_i) = \alpha(\bigvee_{i\in I_0} b_i)$. Since α is monic then $a \leqslant \bigvee_{i\in I_0} b_i$, proving that $a \in C_1$. ■

(g) *With above notations let α be an injective complete homomorphism such that $\alpha(B_1) \cap C_2$ is cofinal in C_2; then the topology on X_1 is the inverse image by α_* of the topology on X_2.*

Proof. Let $x \in X_1$, $c_2 \in C_2$, hence there exists $c_1 \in B_1$ such that $c_2 \leqslant \alpha(c_1)$; by (f), $c_1 \in C_1$. Then: $N_{c_1}(x) = \alpha_*^{-1}\big(N_{c_2}(\alpha_*(x))\big)$. In fact, if $y \in N_{c_1}(x)$, then $\pi_{\alpha(c_1)}(\alpha_*(y)) = (\alpha_{c_1})_*(\pi_{c_1}(y)) = (\alpha_{c_1})_*(\pi_{c_1}(x)) = \pi_{\alpha(c_1)}(\alpha_*(x))$, so $\alpha_*(y) \in N_{c_2}(\alpha_*(x))$. The converse is analogous: if $\pi_{c_2}(\alpha_*(y)) = \pi_{c_2}(\alpha_*(x))$ then $\pi_{\alpha(c_1)}(\alpha_*(y)) = \iota_{c_2}^{\alpha(c_1)}\pi_{c_2}(\alpha_*(y)) = \iota_{c_2}^{\alpha(c_1)}\pi_{c_2}(\alpha_*(x)) = \pi_{\alpha(c_1)}(\alpha_*(x))$; since α is monic then α_{c_1} is monic and so is $(\alpha_{c_1})_*$ too, so we conclude just as above. ■

In particular, if B_1 is a subalgebra of B_2 and $B_1 \cap C_2$ is cofinal in C_2, then X_1 has the topology induced by that of X_2.

As an illustration, if we consider example 1, where $B = F = \mathfrak{B}(I)$, $\mathfrak{J} = \mathfrak{B}(G)$, then the mapping $\theta\colon X \to G^I$ is a homeomorphism from X (with its topology) to G^I (with the product topology of discrete spaces equal to G). Similar results hold in examples 2, 3, since the topology is now the one induced by the topology on X.

4. Let us assume that G is a topological space. We shall define a topology on $X = \bigtimes_{B,F}(G, \mathfrak{J})$.

First, we consider the case where B is a Boolean algebra. Given $x \in X$, and $c \in C$ (compact elements of B) then $c = c \wedge 1 = c \wedge (\bigvee_{g\in G} x_g) = \bigvee_{g\in G}(c \wedge x_g)$.

We show that only finitely many of the elements $c \wedge x_g$ (for $g \in G$) are different from 0. In fact, since c is compact, there exist elements $g_1, \ldots, g_m \in G$ such that $c \wedge x_{g_i} \neq 0$ and $c = \bigvee_{i=1}^{m}(c \wedge x_{g_i})$. Now, if $g \neq g_i$ (for all $i = 1, \ldots, m$) then $c \wedge x_g = (c \wedge x_g) \wedge (\bigvee_{i=1}^{m} c \wedge x_{g_i}) = \bigvee_{i=1}^{m}(c \wedge x_g \wedge g_i) = 0$.

For every g_i, we consider a neighborhood V_i of g_i in G. We define:

$$N_{c;V_1,\ldots,V_m}(x) = \{y \in X \mid x_{g_i} \wedge c = \bigvee_{j=1}^{n_i}(y_{h_{ij}} \wedge c) \text{ where each } h_{ij} \in V_i\}.$$

Then, the collection of sets $N_{c;V_1,\ldots,V_m}(x)$ so defined constitutes a fundamental system of neighborhoods of x for a topology on X.

(If 0 is the only compact element of B, then X is the only neighborhood of each element $x \in X$).

In fact, $x \in N_{c;V_1,\ldots,V_m}(x)$.

Let b, c be compact elements, let $b = \bigvee_{i=1}^{m}(b \wedge x_{g_i})$, $c = \bigvee_{j=1}^{n}(c \wedge x_{h_j})$, let V_i be a neighborhood of g_i, and W_j a neighborhood of h_j in G (for every $i = 1, \ldots, m$, $j = 1, \ldots, n$); then $N_{b\vee c;V_1,\ldots,V_m,W_1,\ldots,W_n}(x) \subseteq N_{b;V_1,\ldots,V_m}(x) \cap \cap N_{c;W_1,\ldots,W_n}(x)$.

Finally, if $y \in N_{c;W_1,\ldots,W_m}(x)$ and $x_{g_i} \wedge c = \bigvee_{j=1}^{n_i}(y_{h_{ij}} \wedge c)$ where $h_{ij} \in V_i$ (for every j), there exists a neighborhood W_{ij} of h_{ij} such that $W_{ij} \subseteq V_i$; hence $N_{c;W_{ij}}(y) \subseteq N_{c;V_i}(x)$.

If B is a Boolean lattice (without last element), $X = \varprojlim X_s$ $(s \in B)$ is endowed with the topology which is the inverse limit of the topologies defined on each X_s, (we note that $\pi_s^t\colon X_t \to X_s$ is a continuous mapping, for $s \leqslant t$).

In the particular case where G is discrete, the topology of X coincides with that defined in the preceding section.

We shall not make use of the topology on X, therefore we do not investigate any of its properties.

5. We assume now that G is an ordered set (by a relation $\leqslant$) with a distinguished element 0. Let $\mathfrak{J}$ be any ideal of subsets of G.

Let B be a Boolean algebra, F a filter of B.

We define a relation $\leqslant$ on $X = \bigtimes_{B,F}(G, \mathfrak{J})$, as follows: if $x, y \in X$ then $x \leqslant y$ whenever $x_g \leqslant \bigvee_{g\leqslant h} y_h$ for every $g \in G$ (we note that the above supremum has a sense, by the usual hypothesis on $\mathfrak{J}, B$).

(h) *The relation $\leqslant$ is an order relation on X.*

Proof. Clearly $x \leqslant x$. Let $x, y, z \in X$ be such that $x \leqslant y$, $y \leqslant z$. Then, for every $g, h \in G$ we have:

$$x_g \leqslant \bigvee_{g\leqslant h} y_h\,,\quad y_h \leqslant \bigvee_{h\leqslant k} z_k\,,\quad \text{then}\quad x_g \leqslant \bigvee_{g\leqslant h}(\bigvee_{h\leqslant k} z_k) = \bigvee_{g\leqslant k} z_k\,.$$

Finally, we assume that $x \leqslant y$, $y \leqslant x$ and we show that $x = y$.

If $h \in G$ then $x_h \leqslant \bigvee_{h\leqslant k} y_k$, hence $x_h = x_h \wedge (\bigvee_{h\leqslant k} y_k) = \bigvee_{h\leqslant k}(x_h \wedge y_k)$. Similarly, if $g \in G$ then $y_g \leqslant \bigvee_{g\leqslant l} x_l$, hence

$$y_g = \bigvee_{g\leqslant l}(y_g \wedge x_l) = \bigvee_{g\leqslant h}[\{\bigvee_{h\leqslant k}(x_h \wedge y_k)\} \wedge y_g] = \bigvee_{g\leqslant h}\bigvee_{h\leqslant k}(x_h \wedge y_k \wedge y_g) = x_g \wedge y_g$$

so $y_g \leqslant x_g$. Similarly, $x_g \leqslant y_g$ hence $x_g = y_g$ for every $g \in G$. ■

If G has at least two elements, if $F = B$, for every $g \neq 0$ let $\chi_g\colon B \to X$ be defined by $\chi_g(b) = (x_h)_{h\in G}$ where $x_0 = b'$, $x_g = b$, $x_h = 0$ for every $h \neq g$, $h \neq 0$.

Similarly, let $\varrho\colon G\to X$ be defined by $\varrho(g)=(x_h)_{h\in G}$ where $x_g=1$, $x_h=0$ for every $h\neq g$.

Then each χ_g (for $g\neq 0$), and ϱ are injective mappings.

Moreover, if G is ordered, then ϱ and χ_g (for $g\geqslant 0$) preserve the order.

In the case where B is a Boolean lattice, then with previous notations $X=\varprojlim X_s$, and since each set X_s is ordered and the mappings $\pi_s^t\colon X_t\to X_s$ are clearly order-preserving, then there exists an order relation on X, such that each mapping $\pi_s\colon X\to X_s$ is order-preserving.

We shall now assume that G is a lattice. Let $\mathfrak{J}$ be a *compatible* ideal of subsets of G, that is, if $J_1, J_2\in\mathfrak{J}$ then $J_1\vee J_2=\{g_1\vee g_2|\ g_1\in J_1,\ g_2\in J_2\}\in\mathfrak{J}$ and similarly $J_1\wedge J_2=\{g_1\wedge g_2|\ g_1\in J_1,\ g_2\in J_2\}\in\mathfrak{J}$.

(i) *If B is a Boolean algebra, if G is a lattice with first element 0 and if $\mathfrak{J}$ is a compatible ideal of subsets of G, then it is possible to define operations $\vee$, $\wedge$ in X, so that X becomes a lattice with first element. If G is distributive, the same holds for X. Moreover, if $0\leqslant g$ then $\chi_g\colon B\to X$ and $\varrho\colon G\to X$ are lattice-homomorphisms.*

Proof. Let $x,y\in X$. We define $x\vee y=z$, by putting $z_g=\bigvee_{h\vee k=g}(x_h\wedge y_k)$ for every $g\in G$ (we note that the above supremum exists, by the hypothesis on $\mathfrak{J}, B$).

Similarly, we define $x\wedge y=t$, by letting $t_g=\bigvee_{h\wedge k=g}(x_h\wedge y_k)$ for every $g\in G$.

We check with no difficulty (using the hypothesis that $\mathfrak{J}$ is a compatible ideal of subsets of G) that $x\vee y=z$, $x\wedge y=t$ belong to X.

The commutativity and associativity of the operations $\vee$, $\wedge$ is immediately verified. Similarly $x\vee x=x$, $x\wedge x=x$.

And the absorption laws can be equally easily proved; for example:

$$\begin{aligned}[x\vee(x\wedge y)]_g &= \bigvee_{h\vee k=g}\Big[x_h\wedge\Big\{\bigvee_{l\wedge m=k}(x_l\wedge y_m)\Big\}\Big]=\bigvee_{h\vee k=g}\ \bigvee_{l\wedge m=k}(x_h\wedge x_l\wedge y_m)\\ &=\bigvee_{h\vee k=g}\ \bigvee_{h\wedge m=k}(x_h\wedge y_m)=\bigvee_{k\leqslant g}\ \bigvee_{g\wedge m=k}(x_g\wedge y_m)\\ &=x_g\wedge\Big(\bigvee_{m\in G}y_m\Big)=x_g\,,\qquad\text{for every}\quad g\in G.\end{aligned}$$

This shows that X is a lattice. Its first element is $z=(z_g)_{g\in G}$ where $z_0=1$, $z_g=0$ for every $g\neq 0$. In fact, $z\vee x=x$, $z\wedge x=z$ for every $x\in X$.

If G is distributive, it is straightforward to verify the same property for X.

The last assertions are immediate consequences of the definitions. ■

(j) *The operations $\vee$, $\wedge$ are the supremum and infimum defined by the order relation on X which is induced by the order relation on G.*

Proof. Let $\leqslant$ be the order relation on X defined in (h) by the order relation on G: $g\leqslant h$ if and only if $g\vee h=h$ (or equivalently $g\wedge h=g$).

We shall prove that if $x,y\in X$ then $x\leqslant x\vee y$, $y\leqslant x\vee y$, and if $z\in X$, $x\leqslant z$, $y\leqslant z$ then $x\vee y\leqslant z$.

Clearly, $x\leqslant x\vee y$, because if $g\in G$ then

$$\begin{aligned}\bigvee_{g\leqslant h}(x\vee y)_h &= \bigvee_{g\leqslant h}\Big(\bigvee_{l\vee m=h}(x_l\wedge y_m)\Big)\geqslant\bigvee_{m\in G}(x_g\wedge y_m)\\ &=x_g\wedge\Big(\bigvee_{m\in G}y_m\Big)=x_g\,.\end{aligned}$$

Similarly $y\leqslant x\vee y$. Now we assume that $x\leqslant z$ and $y\leqslant z$, that is $x_g\leqslant\bigvee_{g\leqslant h}z_h$, $y_g\leqslant\bigvee_{g\leqslant h}z_h$; then

$$\begin{aligned}(x\vee y)_g &=\bigvee_{l\vee m=g}(x_l\wedge y_m)\leqslant\bigvee_{l\vee m=g}\Big[\Big(\bigvee_{l\leqslant h}z_h\Big)\wedge\Big(\bigvee_{m\leqslant k}z_k\Big)\Big]\\ &=\bigvee_{l\vee m=g}\ \bigvee_{l\leqslant h}\ \bigvee_{m\leqslant k}(z_h\wedge z_k)=\bigvee_{g\leqslant j}z_j\,,\end{aligned}$$

showing that $x\vee y\leqslant z$.

A similar proof holds for the infimum. ■

(k) *If B is a Boolean algebra, $F=B$, if G is a Boolean algebra, if $\mathfrak{J}$ is an ideal of subsets of G, compatible with $\vee$, $\wedge$ and such that if $S\in\mathfrak{J}$ then $\{g'|\ g\in S\}\in\mathfrak{J}$, then X is also a Boolean algebra.*

Proof. Let $1\in G$ be its last element, let $e=(e_g)_{g\in G}$ be defined by $e_1=1$, $e_g=0$ for every $g\neq 1$. Then e is the last element of X.

For every $x\in X$, let $y=(y_g)_{g\in G}$ be defined by $y_g=x_{g'}$ (where g' is the complement of g in G).

Then $y\in X$ and $x\vee y=e$, $x\wedge y=0$, as one may verify immediately. By (i) it follows that X is a Boolean algebra. ■

It is immediate to verify that if $B=F=\mathfrak{B}(I)$, $\mathfrak{J}=\mathfrak{B}(G)$, if θ is the natural bijection from X onto G^I, then $x\leqslant y$ if and only if $\theta(x)\leqslant\theta(y)$ (in the pointwise order of G), while $x\wedge y=z$ ($x\vee y=z$) if and only if $\theta(x)\wedge\theta(y)=\theta(z)$ ($\theta(x)\vee\theta(y)=\theta(z)$).

Thus, if G is totally ordered, in general X will not be totally ordered.

In the more general case where B is a Boolean lattice, and G a lattice (resp. distributive lattice) the usual technique allows us to define X as a lattice, (respectively distributive lattice) namely $X=\varprojlim_{s\in B}X_s$.

6. Let G be endowed with a binary operation $+$ such that $0+g=g+0=g$.

Let B be a Boolean algebra, F a filter of B. Let $\mathfrak{J}$ be a *compatible* ideal of subsets of G, that is if $J_1, J_2\in\mathfrak{J}$ then $J_1+J_2=\{g_1+g_2|\ g_1\in J_1,\ g_2\in J_2\}\in\mathfrak{J}$.

(l) *With the above hypothesis, X has an operation, still denoted $+$, such that if the operation in G is commutative (respectively associative), the same holds for the operation in X. If G is a group (resp. abelian group), then X is also a group (resp. abelian group).*

Proof. If $x, y \in X$, let $x+y = z$ be defined by $z_g = \bigvee_{h+k=g} (x_h \wedge y_k)$ for every $g \in G$.

Since $\mathfrak{J}$ is compatible then $z = (z_g)_{g \in G}$ belongs to X.

The properties of the operation $+$ in X are the same as those for $+$ in G, and the proof is a straightforward verification (actually, the same as in (i)).

The zero element of X is $z = (z_g)_{g \in G}$, where $z_g = 0$ for $g \neq 0$, $z_0 = 1$.

If G is a group, then the symmetric of z is $z' = (z'_g)_{g \in G}$ where $z'_g = z_{-g}$ for every $g \in G$; this may be readily computed. ■

(m) *If G is a ring (resp. commutative ring) the same holds for X, provided the ideal $\mathfrak{J}$ of subsets of G is also compatible with respect to the multiplication of G.*

Proof. We let $x \cdot y = z$, where $z_g = \bigvee_{h \cdot k=g} (x_h \wedge y_k)$ for every $g \in G$. By the hypothesis on $\mathfrak{J}$, we have $z \in X$.

Then the multiplication on X has the same properties as that of G.

If 1 is the unit element of G, then $e = (e_g)_{g \in G}$, $e_1 = 1$, $e_g = 0$ for $g \neq 1$, is the unit element of X.

The distributive laws may be verified by a straightforward computation. ■

(n) *Let A be a ring and G a left-A-module. Let $\mathfrak{J}$ be an ideal of subsets of G compatible with the operation $+$ on G, and such that if $S \in \mathfrak{J}$, $a \in A$ then $a \cdot S = \{a \cdot g \mid g \in S\} \in \mathfrak{J}$. Then X is also a left-A-module.*

Proof. If $x \in X$, $a \in A$, we define $a \cdot x = y$, where $y = (y_g)_{g \in G}$, $y_g = \bigvee_{a \cdot h=g} x_h$.

We leave to the reader the task of verifying that $y \in X$ and that, with this scalar multiplication, X becomes a left A-module. ■

Let us note that $\varrho\colon G \to X$ preserves, in each case, the operations: $\varrho(g_1+g_2) = \varrho(g_1)+\varrho(g_2)$, $\varrho(g_1 \cdot g_2) = \varrho(g_1) \cdot \varrho(g_2)$, $\varrho(a \cdot g) = a \cdot \varrho(g)$.

If B is a Boolean lattice, all the definitions may be easily generalized.

It is also clear that in the situation of example 1, θ preserves the operations.

Now, let G be an ordered additive group, with zero element 0, and let $G_+ = \{g \in G \mid g \geqslant 0\}$.

(o) *If B is a Boolean algebra, if $\mathfrak{J}$ is an ideal of subsets of G, compatible with the operation $+$, then X is an ordered additive group.*

Proof. We have already defined an order relation and an operation of addition on X. Our task will be to show that the addition and the order are compatible.

Let $X_+ = \{x \in X \mid x \geqslant 0\}$. Then $x \in X_+$ if and only if $x_g = 0$ for every $g \in G$ such that $g \ngeqslant 0$ (we recall that the zero element of X, now denoted also by 0, is defined as the element with components $0_0 = 1$, $0_g = 0$ for every $g \neq 0$).

Then $X_+ + X_+ \subseteq X_+$ (easy to check), and also $X_+ \cap (-X_+) = \{0\}$, $x + X_+ - x \subseteq X_+$ for every $x \in X$.

In fact, if $x, -x \in X_+$ then $x_g = 0$ for every $g \in G$, $g \ngeqslant 0$, also $x_{-g} = 0$ for every $g \ngeqslant 0$, hence $x_g = 0$ for every $g \neq 0$, and so $x_0 = 1$, hence $x = 0$.

If $y \in X_+$ then

$$(x+y-x)_g = \bigvee_{h+k=g} \bigvee_{l+j=h} [(x_l \wedge y_j) \wedge x_{-k}] = \bigvee_{h+k=g} \bigvee_{j-k=h} (x_{-k} \wedge y_j) .$$

Now, if $j \ngeqslant 0$ then $y_j = 0$, hence the only terms to be considered are those with $j \geqslant 0$, so $g = h+k = j \geqslant 0$; that is, if $g \ngeqslant 0$ then $(x+y-x)_g = 0$, proving that $x+X_+-x \subseteq X_+$. ■

By the same procedure, if G is an ordered ring, if $\mathfrak{J}$ is an ideal of subsets of G, compatible with the operations, if B is a Boolean algebra, then X is also an ordered ring.

Similar statements may be made for the case where G is an f-ring, or G is an ordered module over an ordered ring A (with appropriate hypothesis on $\mathfrak{J}$).

The above facts may be at once generalized for the situation where B is a Boolean lattice without last element.

7. Now we shall apply some of the foregoing ideas to establish a relationship between Boolean algebras and certain ordered abelian additive groups.

We begin recalling some definitions and facts from the theory of ordered abelian additive groups (see [7], [9]).

If X is a lattice ordered abelian additive group, then it is a distributive lattice.

Let $X_+ = \{x \in X \mid x \geqslant 0\}$. If $x \in X_+$ let $D(x) = \{y \in X \mid y \wedge x = 0\}$. We define $x \equiv y$ (for $x, y \in X_+$) when $D(x) = D(y)$; the equivalence class containing x is denoted by $\bar{x}$ and called the *carrier of* x. The set $\mathcal{C}(X)$ of carriers of elements $x \in X_+$, is ordered as follows: $\bar{x} \leqslant \bar{y}$ whenever $D(x) \supseteq D(y)$; then $\mathcal{C}(X)$ satisfies the following properties:

(1) $\mathcal{C}(X)$ is a distributive lattice with first element $\bar{0} = \{0\}$; $\overline{x \vee y} = \bar{x} \vee \bar{y}$, $\overline{x \wedge y} = \bar{x} \wedge \bar{y}$;

(2) $\mathcal{C}(X)$ is disjoinctive: if $\bar{x}, \bar{y} \in \mathcal{C}(X)$, $\bar{x} \nleqslant \bar{y}$, then there exists $\bar{z} \in \mathcal{C}(X)$, $\bar{0} \neq \bar{z} \leqslant \bar{x}$ such that $\bar{z} \wedge \bar{y} = \bar{0}$.

If x is an arbitrary element of X, we define its positive and negative parts as follows: $x_+ = x \vee 0$, $x_- = (-x) \wedge 0$; by definition $\bar{x} = \bar{x}_+ \vee \bar{x}_-$.

The lattice ordered abelian group X is said to be *totally decomposable* whenever the following property is satisfied: for every $\alpha \in \mathcal{C}(X)$, $x \in X_+$, there exist elements $x_\alpha, x_\alpha^* \in X_+$ such that $x = x_\alpha + x_\alpha^*$, $\bar{x}_\alpha \leqslant \bar{\alpha}$, $\bar{x}_\alpha^* \wedge \alpha = \bar{0}$. It follows that the elements x_α, x_α^* are uniquely defined by x, α.

If X is totally decomposable then $\mathcal{C}(X)$ is a Boolean lattice.

We shall also use the following result, which is easy to prove, or may be found in [7]:

(p) *Let C be any Boolean algebra. If $\alpha_1, \ldots, \alpha_r \in C$ there exist elements $\beta_1, \ldots, \beta_s \in C$ such that $\beta_i \wedge \beta_j = 0$ (for $i \neq j$) and $\alpha_i = \bigvee_{\beta_j \leqslant \alpha_i} \beta_j$ for all $i = 1, \ldots, r$.*

Let B be any Boolean algebra, let $F = B$, let $\boldsymbol{Z}$ be the ordered abelian additive group of integers, let $\mathfrak{J}_0$ be the ideal of finite subsets of $\boldsymbol{Z}$.

Let $X = \bigtimes_{B,F} (\boldsymbol{Z}, \mathfrak{J}_0)$, thus X depends only on B, hence we shall denote it by $B^\#$. By (h), $B^\#$ is an ordered abelian additive group (actually, it is also a ring, but we shall regard it as a group only).

Explicitly, if $x = (x_n)_{n \in \boldsymbol{Z}}$, $y = (y_n)_{n \in \boldsymbol{Z}}$, then $x \leqslant y$ if and only if $x_n \leqslant \bigvee_{n \leqslant m} y_m$ for every $n \in \boldsymbol{Z}$, and this is equivalent to the following condition: if $m < n$ then $y_m \wedge x_n = 0$ (because if this holds then $x_n \leqslant (\bigvee_{m<n} y_m)' = \bigvee_{n \leqslant m} y_m$; the converse is immediate).

If $z = x \vee y$ where $x, y \in X_+$, if $k = \max\{n \in \boldsymbol{Z} \mid x_n \vee y_n \neq 0\}$ then $z_0 = x_0 \wedge y_0$. z_1 is the relative complement of z_0 in $(x_0 \vee x_1) \wedge (y_0 \vee y_1)$, z_2 is the relative complement of $(x_0 \vee x_1) \wedge (y_0 \vee y_1)$ in $(x_0 \vee x_1 \vee x_2) \wedge (y_0 \vee y_1 \vee y_2)$, and so on (hence $z_{k+1} = 0$).

Similarly, if $t = x \wedge y$, then $t_k = x_k \wedge y_k$, t_{k-1} is the relative complement of $x_k \wedge y_k$ in $(x_k \vee x_{k-1}) \wedge (y_k \vee y_{k-1})$, t_{k-2} is the relative complement of $(x_k \vee x_{k-1}) \wedge (y_k \vee y_{k-1})$ in $(x_k \vee x_{k-1} \vee x_{k-2}) \wedge (y_k \vee y_{k-1} \vee y_{k-2})$, and so on; in particular, $t_0 = x_0 \vee y_0$. Thus $x \wedge y = 0$ if and only if $x_0 \vee y_0 = 1$.

The *characteristic mapping* $\chi\colon B \to B_+^\#$ is defined as follows: $\chi(b) = (x_n)_{n \in \boldsymbol{Z}}$ where $x_1 = b$, $x_0 = b'$ (thus $\chi = \chi_1$, as defined before).

(q) *χ is an injective lattice homomorphism such that $\chi(0)$ is the zero element of $B^\#$, $\chi(b)$ is idempotent for every $b \in B$ and $\chi(b) + \chi(b') = 1$ (unit of the ring $B^\#$).*

Proof. The proof consists on a series of straightforward verifications. ■

We shall now consider the carriers of the group $B^\#$. First, we note the following useful fact:

(r) *If $x, y \in B_+^\#$ then $\bar{x} = \bar{y}$ if and only if $x_0 = y_0$.*

Proof. Indeed, $\bar{x} = \bar{y}$ is equivalent to the fact that $x \wedge z = 0$ if and only if $y \wedge z = 0$ (where $z \in B_+^\#$), that is, $x_0 \vee z_0 = 1$ if and only if $y_0 \vee z_0 = 1$; this means that $x_0 = y_0$. ■

(s) *The lattice $\mathcal{C}(B^\#)$ of carriers of $B^\#$ is a Boolean algebra and $\varphi\colon B \to \mathcal{C}(B^\#)$, defined by $\varphi(b) = \overline{\chi(b)}$, is an isomorphism.*

Proof. We have $\varphi(b \vee c) = \overline{\chi(b \vee c)} = \overline{\chi(b) \vee \chi(c)} = \overline{\chi(b)} \vee \overline{\chi(c)} = \varphi(b) \vee \varphi(c)$, and similarly, $\varphi(b \wedge c) = \varphi(b) \wedge \varphi(c)$, $\varphi(0) = \overline{\chi(0)} = \bar{0}$.

φ is injective, because if $b \neq c$ then, for example $b \nleqslant c$, hence $b' \ngeqslant c'$. We have $\chi(c') \wedge \chi(c) = 0$, while $\chi(c') \wedge \chi(b) = \chi(c' \wedge b) \neq 0$ (since χ is injective). Hence $D(\chi(c)) \neq D(\chi(b))$, so $\varphi(c) = \overline{\chi(c)} \neq \overline{\chi(b)} = \varphi(b)$.

The mapping φ is also surjective. In fact, given $\alpha \in \mathcal{C}(B^\#)$, let $x \in B_+^\#$ such that $\bar{x} = \alpha$. Then $\varphi(x_0') = \overline{\chi(x_0')} = \alpha$, because $(\chi(x_0'))_0 = (x_0')' = x_0$, so by (r), we have $\overline{\chi(x_0')} = \bar{x} = \alpha$.

Thus, $\varphi(1)$ is the last element of $\mathcal{C}(B^\#)$, which is a Boolean algebra. ■

It follows that $\psi = \chi \circ \varphi^{-1}\colon \mathcal{C}(B^\#) \to B_+^\#$ is an injective lattice homomorphism such that $\overline{\psi(\alpha)} = \alpha$, for every $\alpha \in \mathcal{C}(B^\#)$, since $\overline{\psi(\alpha)} = \overline{\chi(\varphi^{-1}(\alpha))} = \varphi(\varphi^{-1}(\alpha)) = \alpha$.

Explicitly, if $\alpha \in \mathcal{C}(B^\#)$, if $x \in B_+^\#$ is such that $\bar{x} = \alpha$, then $\psi(\alpha) = \chi(\varphi^{-1}(\alpha)) = \chi(x_0')$ because $\varphi(x_0') = \overline{\chi(x_0')} = \bar{x}$ as it was shown in (r); actually, this shows also directly that $\psi(\alpha)$ independent of the choice of x such that $\bar{x} = \alpha$. Moreover:

(t) *$B^\#$ is generated (as an additive group) by $\psi(\mathcal{C}(B^\#))$.*

Proof. Let $x = (x_n)_n \in X_+$. We shall show that $x = \sum_n n \cdot \psi(\varphi(x_n)) = \sum_n n \cdot \chi(x_n)$ (this sum is finite, that is $x_n = 0$ except for a finite number of integers n, and since $x \in X_+$ then $x_n = 0$ for $n < 0$).

Indeed, for every $n > 0$, we have $n\chi(x_n) = (y_m^{(n)})_m$ where $y_0^{(n)} = x_n'$, $y_n^{(n)} = x_n$, $y_m^{(n)} = 0$ for every $m \in \boldsymbol{Z}$, $m \neq 0$, $m \neq n$.

Then,

$$y = \sum_n n \cdot \chi(x_n) = (y_m)_m \quad \text{where} \quad y_0 = \bigwedge_{n \neq 0} x_n' = (\bigvee_{n \neq 0} x_n)' = x_0,$$

$$y_1 = x_1 \wedge (\bigwedge_{m>1} x_m') = x_1 \wedge (\bigvee_{m>1} x_m)' = x_1 \wedge (x_0 \vee x_1) = x_1,$$

$$y_2 = x_1' \wedge x_2 \wedge (\bigwedge_{m>2} x_m') = x_2 \wedge (\bigvee_{m \neq 2,0} x_m)' = x_2 \wedge (x_0 \vee x_2) = x_2;$$

similarly $y_m = x_m$ for every m, showing that $x = \sum_n n\chi(x_n)$, belonging therefore to the abelian group generated by $\psi(\mathcal{C}(B^\#))$. For a general element $x \in X_+$, we write $x = x_+ - x_-$ where $x_+ = x \vee 0$, $x_- = (-x) \vee 0$; then x_+, x_- are in the above subgroup, and so is x too. ∎

(u) *B^* is a completely decomposable lattice ordered abelian group.*

Proof. Let $x \in B^*_+$, let $a \in \mathcal{C}(B^*)$. We have to show that there exist elements $x_a, x^*_a \in B^*_+$ such that $x = x_a + x^*_a$, $\overline{x}_a \leqslant a$, $\overline{x^*_a} \wedge a = \overline{0}$.

In fact, by (s) there exists $a \in B$ such that $a = \overline{\chi(a)}$. Let $x_a = (x_{an})_{n \in Z}$, where $x_{an} = 0$ for $n < 0$, $x_{a0} = a' \vee x_0$, $x_{an} = a \wedge x_n$ for $n > 0$; then $x_a \in B^*_+$. Moreover $\overline{x}_a \leqslant a$, that is $D(x_a) \supseteq D(\chi(a))$; indeed, if $y \in B^*$ then $y \wedge \chi(a) = 0$ if and only if $y_0 \vee a' = 1$; this implies $y_0 \vee (a' \vee x_0) = 1$ hence $y \wedge x_a = 0$.

Clearly $x_a \leqslant x$. Let $x^*_a = x - x_a \in B^*_+$, thus $x^*_{an} = 0$ for $n < 0$, $x^*_{a0} = a \vee x_0$, $x^*_{an} = a' \wedge x_n$ for $n > 0$. Then $x^*_a \wedge \chi(a) = 0$ (because $x^*_{a0} \vee a' = 1$), hence $\overline{x^*_a} \wedge a = \overline{0}$. ∎

Summarizing, we have shown:

THEOREM 4. *If B is a Boolean algebra, if $X = B^*$, then:*

(1) *X is a completely decomposable lattice ordered abelian group;*

(2) *$\mathcal{C}(X)$ has a last element (hence it is a Boolean algebra);*

(3) *there exists an injective lattice homomorphism ψ: $\mathcal{C}(X) \to X_+$ such that $\overline{\psi(a)} = a$ for every $a \in \mathcal{C}(X)$;*

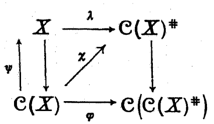
(4) *X is generated by $\psi(\mathcal{C}(x))$.*

Moreover, the Boolean algebras B, $\mathcal{C}(X)$ are isomorphic. ∎

Conversely:

THEOREM 5. *Let X be an abelian group satisfying conditions* (1)–(4) *of the above theorem. Then there exists a lattice-group-isomorphism λ: $X \to \mathcal{C}(X)^*$ such that $\overline{\lambda(x)} = \overline{\chi(\overline{x})}$ for every $x \in X$.*

Proof.



By (p) given a finite set $\{a_1, \ldots, a_r\}$ of elements of $\mathcal{C}(X)$ there exists a finite set $\{\beta_1, \ldots, \beta_s\}$ of pairwise disjoint elements of $\mathcal{C}(X)$ such that $a_i = \bigvee_{\beta_j \leqslant a_i} \beta_j$ (for every $i = 1, \ldots, r$).

Then $\psi(a_i) = \bigvee_{\beta_j \leqslant a_i} \psi(\beta_j)$ and the elements $\psi(\beta_j)$ $(j = 1, \ldots, r)$ are pairwise disjoint. Hence for every $n \in Z$, we have

$$n\psi(a_i) = n\Big(\bigvee_{\beta_j \leqslant a_i} \psi(\beta_j)\Big) = n\Big(\sum_{\beta_j \leqslant a_i} \psi(\beta_j)\Big) = \sum_{\beta_j \leqslant a_i} n\psi(\beta_j).$$

If $x \in X$, $y \in X$, by (4) we may write $x = \sum_{i=1}^{r} n_{a_i}(x)\psi(a_i)$, $y = \sum_{i=1}^{r} n_{a_i}(y)\psi(a_i)$, where $a_i \in \mathcal{C}(X)$, $n_{a_i}(x), n_{a_i}(y) \in Z$, and the elements a_i are pairwise disjoint. Then

$$x \vee y = \sum_{i=1}^{r} [n_{a_i}(x) \vee n_{a_i}(y)] \cdot \psi(a_i),$$

$$x \wedge y = \sum_{i=1}^{r} [n_{a_i}(x) \wedge n_{a_i}(y)] \cdot \psi(a_i),$$

$$x + y = \sum_{j=1}^{r} [n_{a_i}(x) + n_{a_i}(y)] \cdot \psi(a_i).$$

In fact, since the elements $\psi(a_i)$ are pairwise disjoint, the same holds for their multiples $n_{a_i}(x)\psi(a_i)$, hence

$$x = \bigvee_{i=1}^{r} n_{a_i}(x) \cdot \psi(a_i), \qquad y = \bigvee_{i=1}^{r} n_{a_i}(y) \cdot \psi(a_i)$$

and so $x \vee y$, $x \wedge y$, $x + y$ are given by the above formulae.

Now, we define the mapping λ: $X \to \mathcal{C}(X)^*$ as follows: if

$$x = \sum_{i=1}^{r} n_{a_i}(x) \cdot \psi(a_i),$$

let $\lambda(x) = (x_n)_{n \in Z}$, where if $n \neq 0$ then $x_n = \bigvee_{n_{a_i}(x) = n} a_i \in \mathcal{C}(X)$ and $x_0 = (\bigvee_{n \neq 0} x_n)' \in \mathcal{C}(X)$. Clearly $x_n \wedge x_m = 0$ when $n \neq m$ and $\bigvee_{n \in Z} x_n = 1$, so $\lambda(x) \in \mathcal{C}(X)^*$.

λ is injective, because if $\lambda(x) = 0$, that is $x_0 = 1$, $x_n = 0$ for $n \neq 0$, then $n_{a_i}(x) = 0$ for every a_i and since ψ is injective then $x = 0$.

Given any element $(\gamma_n)_{n \in Z} \in \mathcal{C}(X)^*$, let $x = \sum_{\gamma_n \neq 0} n\psi(\gamma_n)$, so $x \in X$ (since $\gamma_n = 0$ except at most for a finite number of integers); since the elements γ_n are pairwise disjoint, the same holds for the elements $\psi(\gamma_n)$, hence $\lambda(x) = (x_n)_{n \in Z}$ is such that if $n \neq 0$ then $x_n = \bigvee_{m=n} \gamma_m = \gamma_n$, and $x_0 = (\bigvee_{n \neq 0} x_n)' = (\bigvee_{n \neq 0} \gamma_n)' = \gamma_0$.

Finally $\lambda(x \vee y) = \lambda(x) \vee \lambda(y)$, $\lambda(x \wedge y) = \lambda(x) \wedge \lambda(y)$, $\lambda(x+y) = \lambda(x) + \lambda(y)$. For example, if $z = x \vee y$ then $z_n = \bigvee_{n_{a_i}(x) \vee n_{a_i}(y) = n} a_i$, while

$$\big(\lambda(x) \vee \lambda(y)\big)_n = \bigvee_{h \vee k = n} (x_h \wedge y_k) = \bigvee_{h \vee k = n} \Big[\Big(\bigvee_{n_{a_i}(x) = h} a_i\Big) \wedge \Big(\bigvee_{n_{a_j}(y) = k} a_j\Big)\Big]$$
$$= \bigvee_{h \vee k = n} \Big(\bigvee_{n_{a_i}(x) = h, n_{a_i}(y) = k} a_i\Big) = \bigvee_{n_{a_i}(x) + n_{a_i}(y) = n} a_i = z_n$$

(noting that if $a_i \neq a_j$ then $a_i \wedge a_j = 0$).

To conclude the proof, we need only to show that $\overline{\lambda(x)} = \overline{\chi(\bar{x})}$, that is $x_0 = (\chi(\bar{x}))_0$; but if $x = \sum_{i=1}^{r} n_{a_i}\psi(a_i)$ then $\bar{x} = \bigvee_{i=1}^{r} \overline{\psi(a_i)} = \bigvee_{i=1}^{r} a_i = x_0'$, hence $(\chi(\bar{x}))_0 = x_0'' = x_0$ showing the stated commutativity of mappings. ■

Now we shall use the language of the theory of categories to express the above results more precisely.

Let $\mathfrak{B}$ be the category of Boolean algebras, with lattice-homomorphisms preserving the first and last element and whose images are sublattices.

Let $\mathfrak{G}$ be the category whose objects are the abelian groups satisfying conditions (1)—(4) of Theorem 4. The morphisms $\eta\colon X \to Y$ in the category are the lattice-group-homomorphisms with the following properties:

(1) if $\bar{x}_1 = \bar{x}_2$ then $\overline{\eta(x_1)} = \overline{\eta(x_2)}$, so η preserves the carriers and induces a lattice-homomorphism $\bar{\eta}\colon \mathcal{C}(X) \to \mathcal{C}(Y)$, by defining $\bar{\eta}(\bar{x}) = \overline{\eta(x)}$.

(2) if ε is the last carrier of X then $\bar{\eta}(\varepsilon)$ is the last carrier of Y.

(3) $\psi_Y \circ \bar{\eta} = \eta \circ \psi_X$.

It is easy to check that $\mathfrak{G}$ is indeed a category.

(v) *We define a covariant functor $\mathfrak{B} \rightsquigarrow \mathfrak{G}$ by associating with every $B \in \mathfrak{B}$ the group $B^* \in \mathfrak{G}$, and with every morphism $\mu\colon B_1 \to B_2$ the mapping $\mu^*\colon B_1^* \to B_2^*$, defined by $\mu^*((x_n)_n) = (\mu(x_n)_n)$.*

Proof. Clearly $\mu^*((x_n)_n) \in B_2^*$. It is also immediate to verify μ^* is a lattice-group-homomorphism.

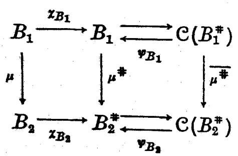
If $\bar{x}_1 = \bar{x}_2$ then $(x_1)_0 = (x_2)_0$ hence $(\mu^*(x_1))_0 = \mu((x_1)_0) = \mu((x_2)_0) = (\mu^*(x_2))_0$ hence $\overline{\mu^*(x_1)} = \overline{\mu^*(x_2)}$.

If ε is the last carrier of B_1^*, then $\varepsilon = \overline{\chi(1)}$ hence $\overline{\mu^*}(\varepsilon) = \overline{\mu^*}\overline{(\chi(1))} = \overline{\mu^*(\chi(1))} = \overline{\chi(1)}$ which is the last carrier of B_2^*.

Now, we show that $\psi_{B_2} \circ \overline{\mu^*} = \mu^* \circ \psi_{B_1}$.

Let $a_1 \in \mathcal{C}(B_1^*)$, $a_1 = \bar{x}$, where $x \in B_1^*$. Then

$$\psi_{B_2}(\overline{\mu^*}(a_1)) = \psi_{B_2}(\overline{\mu^*(x)}) = \psi_{B_2}(\overline{(\mu(x_n))_n}) = \chi_{B_2}(\mu(x_0)') = (\ldots, \mu(x_0), \mu(x_0)', 0, \ldots).$$



On the other hand, $\mu^*(\psi_{B_1}(a_1)) = \mu^*(\psi_{B_1}(\bar{x})) = \mu^*(\chi_{B_1}(x_0')) = \mu^*(\ldots, x_0, x_0', 0, \ldots) = (\ldots, \mu(x_0), \mu(x_0'), 0, \ldots)$. This shows the commutativity of the mappings, proving that μ^* is a morphism in the category $\mathfrak{G}$.

Since $(\nu \circ \mu)^* = \nu^* \circ \mu^*$, where ν, μ are morphisms in $\mathfrak{B}$, then we have a covariant functor $\mathfrak{B} \to \mathfrak{G}$. ■

(w) *For every Boolean algebra B the isomorphism $\varphi\colon B \to \mathcal{C}(B^*)$, defined in* (s), *is natural.*

Proof. Let B_1, B_2 be Boolean algebras, let $\mu\colon B_1 \to B_2$ be a morphism. We have to show that $\overline{\mu^*} \circ \varphi_{B_1} = \varphi_{B_2} \circ \mu$, which is clear from the definitions:

$$\overline{\mu^*}(\varphi_{B_1}(b)) = \overline{\mu^*}\overline{(\chi(b))} = \overline{\mu^*(\chi(b))} = \overline{(\ldots, \mu(b'), \mu(b), 0, \ldots)} = \overline{\chi(\mu(b))} = \varphi_{B_2}(\mu(b)) \,.\ ■$$

(x) *For every abelian group $X \in \mathfrak{G}$ the isomorphism $\lambda\colon X \to \mathcal{C}(X)^*$, defined in theorem* 5, *is natural.*

Proof. Let $X_1, X_2 \in \mathfrak{G}$, let $\eta\colon X_1 \to X_2$ be a morphism. We have to show that $\bar{\eta}^* \circ \lambda_{X_1} = \lambda_{X_2} \circ \eta$.

If $x \in X_1$, by hypothesis and (p), we may write $x = \sum_a n_a \psi_{X_1}(a)$, where the elements a such that $n_a \neq 0$ are pairwise disjoint. Then

$$\eta(x) = \sum_a n_a \eta(\psi_{X_1}(a)) = \sum_a n_a \psi_{X_2}(\bar{\eta}(a)) \,.$$

By definition $\lambda_{X_2}(\eta(x)) = y \in \mathcal{C}(X_2)^*$ is such that if $n \neq 0$ then $y_n = \bigvee_{n_a = n} \bar{\eta}(a) = \bar{\eta}(\bigvee_{n_a = n} a)$. On the other hand, if $\lambda_{X_1}(x) = t$, then for $n \neq 0$ we have $t_n = \bigvee_{n_a \neq 0} a$ hence $\bar{\eta}^*(t) = (\bar{\eta}(t_n))_n$, so necessarily $\bar{\eta}^* \circ \lambda_{X_1} = \lambda_{X_2} \circ \eta$. ■

The above results may be also expressed by saying that the functor $B \to \mathcal{C}(B^*)$ is naturally equivalent to identity functor of $\mathfrak{B}$, and $X \to \mathcal{C}(X)^*$ is naturally equivalent to the identity functor of $\mathfrak{G}$.

In the same way, we see that the functors $B \to B^*$ and $X \to \mathcal{C}(X)$ are inverse isomorphisms between the categories $\mathfrak{B}$, $\mathfrak{G}$.

References

[1] S. Balcerzyk, *On Groups of functions defined on Boolean Algebras*, Fund. Math. 50 (1962), pp. 347-367.

[2] G. Birkhoff, *Lattice theory*. A. M. S. Coll. Publ. 1948.

[3] A. L. Foster, *Functional completeness in the small*, Math. Ann. 143 (1961), pp. 29-58.

[4] H. Hermes, *Einführung in die Verbandstheorie*, Springer 1955.
[5] J. Łoś, *Linear equations and pure subgroups*, Bull. Acad. Polon. Sci. 7 (1959), pp. 13-18.
[6] B. H. Neumann and S. Yamamuro, *Boolean powers of simple groups*, J. Australian M. S. 5 (1965), pp. 315-324.
[7] P. Ribenboim, *Sur quelques constructions de groupes réticulés et l'équivalence logique entre l'affinement de filtres et d'ordres*, Summa Brasil Math. 4 (1958), pp. 65-89.
[8] — *Fonctions modulo antifiltres*, Math. Z. 77 (1961), pp. 195-206.
[9] — *Théorie des groupes ordonnés*, Univ. Nac. del Sur, Bahía Blanca 1963.

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Generalized group cohomology*

by

Morton E. Harris (Chicago, Ill.)

Introduction

A permutation representation (G, X) of a group G will consist of a non-empty set X with G acting on the left such that $(\varrho\sigma)x = \varrho(\sigma x)$ for all $\varrho, \sigma \in G$ and all $x \in X$ and such that $ex = x$ for all $x \in X$ where e denotes the identity element of G.

When (G, X) is a finite permutation representation (i.e., when X is a finite set) a cohomology theory is defined and investigated in a series of papers by Snapper ([9], [10], [11], [12], [13]). The results of [13] are an application of this cohomology theory to the study of Frobenius groups.

When the finite permutation representation (G, X) is fixed point free (i.e., $\sigma x = x$ for $x \in X$ and $\sigma \in G$ implies $\sigma = e$) then this cohomology theory is just the ordinary cohomology theory for finite groups.

This cohomology theory of (finite) permutation representations is a generalization to not necessarily transitive permutation representations of the cohomology theory of [1].

These cohomology theories of permutation representations are defined by means of a "standard complex". The cohomology theory of [1] has been investigated in terms of relative homological algebra in [5].

Using recent developments in relative homological algebra, we investigate the cohomology theory of finite permutation representations of [9], [10], [11], [12] and [13]. This investigation generalizes that of [5] and the well known homological algebraic foundations of the ordinary cohomology theory of finite groups.

Our investigation will permit straightforward (standard categorical) derivations of all of the results of [9] and [10], some generalizations of these results, some new results, as well as generalizations to not necessarily

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