Continuing in this way, we arrive at the vertex \( v_{i+1n} = v_{i,1} \), from which we proceed to \( v_{i,0} = v_{(i-1)n+1} \), \( v_{i,0+1} \), \( v_{i,0+2} \), \( \cdots \), \( v_{i,n-1} \). The path \( P \) thus far contains all vertices of \( D \) with the exception of \( v_i \) so that \( D \) contains the arcs \( v_{i-1}v_i \) and \( v_{i}v_{i+1} \). Conversely, suppose \( v_{i,1} \) is an arc of \( D \) and \( j-i \neq 1 \) (mod \( n \)). We then construct a path \( P' \) which begins as follows: \( v_i, v_{i+1}, v_{i+2}, \ldots, v_{i-1}, v_{i+n}, v_{i+n+1}, \ldots, v_{i+1n-1}, v_{i+1n} \). We then continue as before until we reach the final vertex of the type \( v_{i+n} \) which is not thus far on \( P' \). The next vertex of \( P' \) would then be \( v_{i+2n}, v_{i+2n+1}, \ldots, v_{i+n-1}, v_{i+n} \). Since \( j = i + (i+1) \equiv 0 \) (mod \( n \)), the vertex of \( P' \) following \( v_{i+n} \) necessarily defines an outer transitive cycle of length less than \( n + 2 \), and this is a contradiction. Because \( v_{i,1} \) obviously belongs to \( D \), we have \( 1-p = 1 \) (mod \( n \)), or there exists an integer \( k \) such that \( p = nk \). If for each \( i, 1 < i < n \), we let \( V_i = \{ v_i | x = i \) (mod \( n ) \} \), \( D \) is seen to be the digraph \( D(n, k) \). This completes the proof.

Each randomly hamiltonian graph may be considered a randomly hamiltonian digraph (obtained by replacing each edge by a symmetric arc), but among the randomly hamiltonian digraphs with \( p \) vertices, only \( S_p, K_p \), and \( D(2, p/2) \) are (ordinary) graphs. Thus, we obtain as a corollary the result presented in [1].

**Corollary.** A graph is randomly hamiltonian if and only if it is a cycle, a complete graph, or a regular complete bipartite graph.

### References


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**Extended operations and relations on the class of ordinal numbers**

by

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§1. Introduction. This is intended as a sequel to the paper *An extended arithmetic of ordinal numbers* by John Doner and Alfred Tarski. Thus, our notation is the same as theirs. For the sake of convenience we shall repeat several of their definitions. When referring to a theorem, lemma, etc., in the Doner–Tarski paper we shall prefix the numeral by the symbol "D–T".

Lower case greek letters \( \alpha, \beta, \gamma, \ldots \) represent ordinal numbers and the class of all ordinal numbers is denoted by \( \Omega \).

**Definition 1.** For each \( \gamma \in \Omega \), \( O_\gamma \) is a binary operation from \( \Omega \times \Omega \) to \( \Omega \) such that for all \( \alpha, \beta \in \Omega \),

(i) \( \alpha O_\gamma \beta = \alpha + \beta \) if \( \gamma = 0 \);

(ii) \( \alpha O_\gamma \beta = \bigcup_{\gamma \leq \gamma} \{ (a O_\gamma a) | \alpha \leq a \} \) if \( \gamma > 1 \).

**Definition 2.** For each \( \gamma \in \Omega \), \( R_\gamma \) and \( L_\gamma \) are relations such that

(i) \( R_\gamma \subseteq \Omega \times \Omega \)

(ii) For all \( \alpha, \beta \in \Omega \)

\[ a R_\gamma \beta \iff (\exists \theta)(\theta \neq 0 \text{ and } \alpha O_\gamma \beta = \beta) \]

\[ a L_\gamma \beta \iff (\exists \theta)(\theta \neq 0 \text{ and } \delta O_\gamma \alpha = \beta) \]

(For \( \gamma = 0, 1 \), \( R_\gamma \) and \( L_\gamma \) have been described in Rubin [3].)

Our results include the following: If \( A = \{ a : a R_\gamma \beta \} \) for some \( \beta, \gamma \in \Omega \), and \( A \neq \emptyset \), then \( \bigcup \gamma \in A \). If \( \gamma \) is a limit ordinal and \( \Omega' = \Omega \sim 0 \), then \( \Omega', R_\gamma' \) is a complete lattice. Moreover, for \( \gamma \) a limit ordinal we have obtained necessary and sufficient conditions for \( O_\gamma \) to be commutative and associative. Also, for \( a, \beta, \gamma \in \Omega \) we have obtained necessary and sufficient conditions on \( \alpha \) such that \( a O_\gamma \beta = a' O_\gamma \beta \).

We shall assume the traditional arithmetic of ordinal numbers. (Sierpiński [5] is an excellent reference.) We frequently use the following well-known result.
LEMMA 3. For each $a \in \Omega$, $a \neq 0$, there is a unique $n \in \omega$, $n \neq 0$, unique ordinal numbers $a_0, a_1, \ldots, a_n$ such that $a_0 > a_1 > \cdots > a_n$, and unique natural numbers $i_0 \neq 0$, $i_1 = 1, i_2, \ldots, i_n$ such that
\[ a = a_0^{\omega_{i_0}} + a_1^{\omega_{i_1}} + \cdots + a_n^{\omega_{i_n}}. \]

The form $(\times)$ will be called the normal form of $a$, and $a_0$ is called the degree of $a$. See for example, Sierpinski [5], pp. 319–333 for a proof of Lemma 3.

Main numbers also play an important role in what follows, so that even at the risk of being redundant, we shall restate their definition and some of their properties.

DEFINITION 4. (i) If $O$ is a binary operation from $\Omega \times \Omega$ to $\Omega$ then $\delta \geq a$ is a main number of $O$ if and only if for all $a, \beta \leq \delta$, $aO \beta \leq \beta$.

(ii) $M(O)$ denotes the class of all main numbers of $O$.

(iii) If $\gamma$ is a limit ordinal $M_\gamma = \bigcup_{\alpha < \gamma} M(O_\alpha)$.

The main numbers of $O_\alpha$ are its fixed points. That is, $\delta$ is a main number of $O_\alpha$ if and only if $\delta \geq 3$ and $aO_\alpha \delta = \delta$ for all $a, 2 < a < \delta$ (D-T 46). In the case that $\gamma \geq 2$, we have that for all $a, 2 < a < \delta$, $\delta \in M(O_\alpha)$ if and only if $aO_\alpha \delta = \delta$ (D-T 47). Thus, for example, the main numbers of $O_\alpha$ (addition) are the powers of $\omega$; the main numbers of $O_\lambda$ (multiplication) are all ordinals of the form $\omega^\alpha, \eta \in \Omega$, and the main numbers of $O_\sigma$ (essentially exponentiation) are $\omega$ and the epsilon numbers.

LEMMA 5. If $\delta \in M(O_\alpha)$ then $\delta$ is a positive power of $\omega$.

Proof: D-T 43 (ii), D-T 52 (ii), and D-T 57.

LEMMA 6. (i) If $\gamma \geq 1$ then $2O_{i_0+2}[\omega(1+\eta)]$ is the $\eta$-th successive element of $M(O_{i_0}) = M(O_{i_0+1})$.

(ii) If $\gamma > 1$ and $a \geq 2$ then a $aO_{i_0+2}[\omega(1+\eta)]$ is the $\eta$-th successive element of $M(O_{i_0}) = M(O_{i_0+1})$ exceeding $a$.

(iii) If $\gamma \geq 1$ and $a \geq 2$ then $aO_{i_0+2}[\omega(1+\eta)]$ is the $\eta$-th successive element of $M_{i_0}$.

(iv) If $\gamma > \eta \neq 0$ and $a \geq 3$ then $aO_{i_0+2}[\omega(1+\eta)]$ is the $\eta$-th successive element of $M_{i_0}$ exceeding $a$.

Proof. Part (i) follows from D-T 49 (ii); (ii) from D-T 48 (ii); (iii) from D-T 57 and D-T 55; and (iv) from D-T 54.

LEMMA 7. $aO_\alpha \beta$ is a limit ordinal if any one of the following conditions holds.

(i) $\gamma > a, a, \beta \geq 2$, and $a = a \beta = 2$ does not hold.

(ii) $2 < a < a_1, a \geq 2$, and $\beta \geq a_2$.

(iii) $3 < a < a_1, a \geq a_2$, and $\beta \geq 2$.

Proof. Part (i) is the same as D-T 34. Part (ii) follows from D-T 33 (iii) and D-T 48 (ii) if $\beta$ is a limit ordinal. If $\beta$ is not a limit ordinal, use D-T 9 to reduce it to the case where $\beta$ is a limit ordinal.

To prove (iii) note that if $\beta$ is infinite (iii) follows from (ii). If $\beta$ is finite use D-T 9 to reduce it to (ii).

§ 2. The equation $aO_\alpha \beta = a_0^\omega \beta$. If $\beta < \omega$ then it follows from D-T 4 (ii) that $aO_\alpha \beta = a_0^\omega \beta$. However, we do not have strict monotonicity in the first argument of $O_\alpha$. For example, $n \neq \omega = \omega$ for all $a \in \omega$; $n \neq \omega = \omega$ for all $a \in \omega$; $n \neq \omega$ because $\omega$ is a main number of $O_\alpha$ for all $\alpha \in \omega$ (D-T 2 (iv)). It does follow from D-T 6 that if $a < a_1$ then $aO_\alpha \beta = a_0^\omega \beta$. It is the purpose of this section to determine for which values of $a_0, aO_\alpha \beta = a_0^\omega \beta$.

First we give some negative results—values of $a, \beta, \gamma_1$ and $a' \neq \gamma$ for which equality does not hold.

THEOREM 8. If $2 < a < a_1, \cup \beta \neq \beta$, and $\cap \gamma \neq \gamma$ then $aO_\alpha \beta = a_0^\omega \beta$.

Proof. D-T 11.

The cases in which $a < a_1, a < a_2, \beta < \omega$ are trivial (see D-T 2) so we shall omit them.

Our next two results hold for all $a \geq 1$.

THEOREM 9. If $\gamma \geq 1, \beta \cap \beta \neq 0$, and $3 < a < a_1 < \omega$ then $aO_\alpha \beta = a_0^\omega \beta$.


THEOREM 10. If $\gamma \geq 1, \beta \cap \beta \neq 0$ and $a = a_0^\omega a_1 + \cdots + a_0^\omega a_n$, $a \neq 0$ is the normal form of $a_0$, then $aO_\alpha \beta = a_0^\omega a_0 \beta$.

Proof. D-T 3.

If $\gamma = 1, 2, 3$, or the theorem follows from the traditional arithmetic of ordinal numbers. (See for example Rubin [4], § 9.1.)

If $\gamma = \omega^\omega$, the theorem clearly holds. If $\gamma > \omega^\omega$ then by Lemma 5, there are no main numbers between $\omega^\omega$ and $\omega$. Consequently, if $\gamma = 2\omega^\omega + 2$ for some $\zeta \geq 1$, the theorem follows from Lemma 6 (ii); and if $\gamma = \gamma \cap \gamma \neq 0$, use Lemma 6 (iv) to get the desired result.

Suppose $\gamma = 2\omega^\omega + 1$ for some $\zeta \geq 2$. Then, by D-T 33 (iii),

$aO_\alpha \beta = aO_\alpha a_0 ^\omega \beta$.

It follows from elementary properties of ordinal numbers that since $\beta$ is a limit ordinal $a_0 ^\omega \beta = a_0 ^\omega a_0 ^\omega \beta$. If $\zeta = \omega^\omega + 1$ then by Lemma 6 (ii),

$aO_\alpha a_0 ^\omega \beta = a_0 ^\omega a_0 ^\omega a_0 ^\omega \beta$. 
If \( \zeta = \bigcup \zeta \neq \emptyset \) then by Lemma 6 (iv), equation (9) also holds. Thus in either case,

\[
a_0^\alpha \beta = \omega^\alpha a_0^\alpha \beta = \omega^\alpha a_0^\alpha \beta \quad \text{[D-T 33 (iii)]}.
\]

We now consider the case that \( \bigcup \gamma \neq \gamma \). It follows from Theorem 8, that in this case we need only consider the case that \( \beta \) is a limit ordinal.

**Theorem 11.** If \( \gamma = 2_1 + 1 \) for some \( \xi \geq 2 \), \( \beta = \bigcup \beta \neq 0 \) and \( 2 < \alpha < \omega \) then

\[
a_0^\alpha \beta = \omega_0^\alpha \beta \quad \text{iff} \quad a_0^\beta = \beta.
\]

**Proof.** Suppose \( \gamma = 2_1 + 1 \) for some \( \xi \geq 2 \). Then by D-T 33 (iii),

\[
a_0^\alpha \beta = a_{0_0} \beta = a_{0_0} \beta
\]

and

\[
\omega_0^\alpha \beta = \omega_0 a_0 \beta.
\]

Since there is just one main number which is larger that \( \alpha \) and not larger than \( \omega \) (namely \( \omega \) itself) and since \( 1 + \beta = \beta \), using Lemma 6 (ii) or Lemma 6 (iv), we obtain

\[
\omega_0 a_0 \beta = a_{0_0} a_0 \beta.
\]

Therefore,

\[
a_0^\alpha \beta = a_0^\alpha \beta \quad \text{iff} \quad a_0^\beta = \omega_0 a \beta \quad \text{[D-T 4 (iii)].}
\]

**Theorem 12.** If \( \gamma = 2_1 + 1 \) for some \( \xi \geq 2 \), \( \beta = \bigcup \beta \neq 0 \), and \( \alpha > 1 \) then

\[
a_0^\alpha \beta = \omega_0 a_0 \beta \quad \text{iff} \quad a_0^\beta = \beta.
\]

**Proof.** The proof is similar to the proof of Theorem 11, but there are a few more details to worry about. By D-T 33 (iii) we have

(1)

\[
a_0^\alpha \beta = \omega_0 a_0 \beta
\]

and

(2)

\[
a_0^\alpha \beta = \omega_0 a_0 \beta \quad \text{[D-T 4 (iii)].}
\]

It is clear from (1), (3) and the monotonicity laws (D-T 4 and D-T 6) that if \( a_0^\beta = \beta \) then equality does not hold. So suppose \( a_0^\beta = \beta \).

We note that there are at most \( \omega_0 \) main numbers of \( M(\omega_0) \) exceeding \( \omega_0 \) and not exceeding \( \omega_0^\alpha \).

**Case 1.** \( \zeta = \bigcup \zeta \neq 0 \). In this case it follows from Lemma 6 (iv) and the fact that

(3)

\[
a_0^\alpha \beta = \omega_0 a_0 \beta
\]

that

(4)

\[
a_0^\alpha \beta = \omega_0 a_0 \beta.
\]

**Case 2.** \( \zeta = \zeta + 1 \) and \( \alpha > 0 \). Then

\[
\omega_0 (1 + \omega_0^\alpha \beta) = \omega_0^\alpha \beta.
\]

Thus (4) also holds in this case because of (3) and Lemma 6 (ii).

**Case 3.** \( \zeta = \zeta + 1 \) and \( 1 < \alpha < \omega \). Let \( a = 1 + a' \) where \( 0 < a' < \omega \).

Then

\[
\omega_0 (1 + \omega_0^\alpha \beta) = \omega_0^\alpha \beta.
\]

If \( \delta > \omega \) then \( \alpha_0 + \delta = \alpha_0 + \delta \) and the proof proceeds as in Case 2. Suppose \( \delta > \omega \). Since \( \beta \) is a limit ordinal there exist ordinal numbers \( \xi \) and \( \eta \) such that \( \beta = \omega_0 (\eta + 1) \). Thus, if \( \xi > \omega \) then

\[
\omega_0^\alpha \beta = \omega_0^\alpha \beta
\]

and the proof proceeds as in Case 2. Suppose then that \( 1 < \delta < \omega \). In this case \( \omega_0^\alpha \beta = \beta \) contradicting our assumption.

Thus in all 3 cases (4) holds. Therefore, it follows from (1), (2) and (4) that if \( a_0^\beta = \beta \) then

\[
a_0^\alpha \beta = \omega_0 a_0 \beta.
\]

Before proceeding it is convenient at this point to introduce some notation. It follows from D-T 3 (ii) that for each \( \alpha \geq 1 \), \( \gamma \geq 1 \) and \( \delta \) there is exactly one \( \beta \) such that

\[
a_0^\alpha \beta = \delta < a_0^\beta + 1.
\]

We denote this unique \( \beta \) by \( \varphi_\alpha (\delta) \). Thus,

**Definition 13.** If \( \alpha > 0 \), \( \gamma > 0 \),

(i) \( \varphi_\alpha (\delta) = \beta \) \quad \text{iff} \quad a_0^\beta < \delta < a_0^\beta + 1.

(ii) \( \varphi_\alpha (\delta) = \beta \) \quad \text{iff} \quad 3_0^\beta < \delta < 3_0^\beta + 1.

**Lemma 14.** If \( \alpha > 3 \), \( \beta > 2 \), and \( \gamma = \bigcup \gamma \neq 0 \) then

\[
a_0^\alpha \beta = \delta_0 \varphi_\alpha (\alpha) \quad \text{for all} \ \delta > 3.
\]

(In the case \( \delta = 3 \) we get,

\[
a_0^\alpha \beta = 3_0 \varphi_\alpha (\alpha) \quad \text{for all} \ \delta > 3.
\]

**Proof.** By the definition of \( \varphi_\alpha (\alpha) \) we have

\[
\delta_0 \varphi_\alpha (\alpha) = \alpha < \delta_0 \varphi_\alpha (\alpha) + 1
\]

It follows from Lemma 6 (iv) that there are no main numbers of \( \alpha \) between \( \delta_0 \varphi_\alpha (\alpha) \) and \( \alpha \). Thus the lemma follows from Lemma 6 (iv).

**Lemma 15.** If \( \alpha > 2 \), \( \beta = \bigcup \beta \neq 0 \), and \( \gamma = 2_1 + 2 \) for some \( \xi > 1 \) then

\[
a_0^\alpha \beta = \delta_0 \varphi_\alpha (\alpha) \quad \text{for all} \ \delta > 3.
\]
In the case $\alpha = 3$ we get
\[ aO_{\alpha} = (3O_{\alpha} \cdot \varphi(a))O_{\beta}. \]

Proof. The proof is the same as the proof of Lemma 14, using Lemma 6 (ii) instead of Lemma 6 (iv).

Before considering the general solution of the equation with $\gamma$ a limit ordinal there is one annoying special case to consider.

Theorem 16. If $\gamma = \bigcup \gamma \neq 0$ and $2 \leq \beta < \omega$ then
\[ 2O_{\beta} < 3O_{\beta}. \]

Proof. If $\beta = 2$ then $2O_{\beta} = 4$ (D-T 2 (iii)), and $3O_{\beta} \cdot M \cdot \varphi(\alpha)$ (Lemma 6 (iii)). If $\beta > 2$ then there is a $\beta' < \omega$ such that $\beta = 3 + \beta'$. By D-T 37,
\[ 2O_{\beta} = 3O_{\beta}(2 + \beta') \]
\[ = 3O_{\beta}(3 + \beta') \quad [\text{D-T 4 (iii)}] \]
\[ = 3O_{\beta}. \]

Theorem 17. If $\gamma = \bigcup \gamma \neq 0$, $2 \leq \alpha < \alpha'$, $\alpha' \leq \beta < \omega^{\alpha+1}$ and either $\alpha \geq 3$ and $\beta \geq 2$, or $\alpha = 2$ and $\beta > \omega$, then the following three conditions are equivalent:

1. $aO_{\beta} = a'O_{\beta}$;
2. $\varphi(a) + \beta = \varphi(a') + \beta$;
3. The ordinal number of $\langle X, < \rangle$ is less than $\omega^\alpha$ where $X = \{ x \in M \cdot \alpha : \alpha < x < a' \}$.

Proof. If $\alpha = 2$ and $\beta > \omega$ it follows from D-T 37 that
\[ aO_{\beta} = 3O_{\beta}. \]

Therefore, we can assume $\alpha \geq 3$ and $\beta \geq 2$.

Let $a = \varphi(a)$ and $\alpha' = \varphi(a')$. Then by Lemma 14,
\[ aO_{\beta} = (3O_{\beta} \cdot \varphi(a)) \]
and
\[ a'O_{\beta} = (3O_{\beta} \cdot \varphi(a')). \]

Let $\beta = 1 + \beta'$ and use D-T 27 (i), thereby obtaining,
\[ aO_{\beta} = 3O_{\beta}(1 + \beta'), \]
\[ a'O_{\beta} = 3O_{\beta}(1 + \beta'). \]

Thus the equivalence of (1) and (2) follows from D-T 4 (ii).

To prove the equivalence of (3), let $\lambda$ be the ordinal number of $\langle X, < \rangle$ where $X = \{ x \in M \cdot \alpha : \alpha < x < a' \}$, and let $\beta = 2 + \beta''$. Then it follows from Lemma 6 (iv) that
\[ a'O_{\beta} = aO_{\beta}(2 + 1 + \beta''). \]

Thus, $aO_{\beta} = a'O_{\beta}$ if and only if $\lambda + \beta'' = \beta''$. The latter equation holds if and only if $\lambda < \omega^\alpha$.

Next, we consider the case where $\gamma$ is even but not a limit ordinal.

Theorem 18. If $\gamma = 2\alpha + 2$ for some $\alpha \geq 1$, $\beta = \bigcup \beta \neq 0$, $\alpha' \leq \beta < \omega^{\alpha+1}$, and $2 \leq \alpha < \alpha'$ then the following three conditions are equivalent:

1. $aO_{\beta} = a'O_{\beta}$;
2. $\varphi(a) + \beta = \varphi(a') + \beta$;
3. The ordinal number of $\langle X, < \rangle$ is less than $\omega^\alpha$ where $X = \{ x \in M \cdot \alpha : \alpha < x < a' \}$.

Proof. The proof is similar to the proof of Theorem 17 using Lemma 15, D-T 32 (i) and Lemma 6 (ii) instead of Lemma 14, D-T 37 and Lemma 6 (iv) respectively.

Thus, theorems 8–12, 16–18 give necessary and sufficient conditions for the equation
\[ aO_{\beta} = a'O_{\beta} \]
to hold for $\gamma > 3$. If $\gamma = 0$, $O_{\alpha}$ is addition and necessary and sufficient conditions for equality are easy to obtain when $a$, $a'$ and $\beta$ are all written in normal form. If $1 \leq \gamma < 3$ then it follows from Theorem 8 that if $\beta = \bigcup \beta$ then equality does not hold and it follows from Theorem 10 that it is sufficient to consider values of $a$ and $a'$ which are either finite or powers of $\omega$. Using these results and traditional properties of ordinal numbers it is an easy matter to determine whether or not $aO_{\beta} = a'O_{\beta}$ for $\gamma = 1, 2, 3$. We leave the details to the interested reader.

§ 3. The commutative and associative laws for $O_{\alpha}$, where $\gamma = \bigcup \gamma \neq 0$. In this section we shall give necessary and sufficient conditions for the commutative and associative laws to hold for $O_{\alpha}$ when $\gamma$ is a limit ordinal.

Theorem 19. If $\gamma = \bigcup \gamma \neq 0$ and $\beta > 2$ then $2O_{\beta} = \beta O_{2}$ if and only if $\beta = 3$ or $\beta = \omega + 1$ for some $\alpha \in M \cdot O_{\beta}$.

Proof. If $\beta = 3$ then D-T 37 implies that $2O_{\beta} = \beta O_{2}$. Suppose $\beta = \omega + 1$ for some $\alpha \in M \cdot O_{\beta}$. Then
\[ 2O_{\beta} = 2O_{\beta}(\omega + 1) \]
\[ = (2O_{\beta} \cdot \omega)O_{2} \quad [\text{D-T 27}] \]
\[ = \omega O_{2} \quad [\text{D-T 46}] \]
\[ = \beta O_{2} \quad [\text{D-T 26}]. \]

Conversely, suppose $2O_{\beta} = \beta O_{2}$. 

Case 1. $2 < \beta < \omega$. There is a $\beta' < \omega$ such that $\beta = 3 + \beta'$. Thus,

$$2 \cdot \beta = 2 \cdot 2 = (2 + 2) = (2 + 2),$$

[D-T 27]

Also, it follows from D-T 26 that

$$\beta \cdot 2 = 3 \cdot 2.$$ 

Therefore, if $2 \cdot \beta = \beta \cdot 2$ then $\beta' = 0$ so $\beta = 3$.

Case 2. $\beta \geq \omega$. In this case it follows from D-T 37 that

$$2 \cdot \beta = 3 \cdot \beta.$$ 

Moreover, by Lemma 14,

$$\beta \cdot 2 = 3 \cdot \beta = (3 \cdot \beta + 2) + 1.$$ 

[D-T 27]

Thus, if $2 \cdot \beta = \beta \cdot 2$ then $\beta = \beta + 1$. It remains to be shown that $\beta \in \mathcal{M}(\omega)$.

It follows from the monotonicity law D-T 7 and Definition 13 that

$$\beta \cdot \beta = \beta \cdot (\beta + 1) = \beta \cdot \beta + \beta.$$ 

By D-T 34, $3 \cdot \beta$ is a limit ordinal, so we must have

$$\beta \cdot \beta = 3 \cdot \beta + 1.$$ 

Consequently, D-T 47 implies $\beta \in \mathcal{M}(\omega)$.

To extend the preceding result to $\alpha > 3$, it is convenient first to prove a lemma.

**Lemma 20.** If $\gamma = \bigcup \gamma \not= 0$, $\alpha > 3$, and $\beta > 2$ then $\alpha < \beta$ if and only if $\beta = \beta + 1$ for some $\lambda < \alpha \cdot \beta$.

**Proof.** Let $\delta = \beta \cdot \beta$ and suppose $\beta = \delta + \lambda$ for some $\lambda < \alpha \cdot \beta$.

Then by Definition 13,

$$\alpha \cdot \delta < \delta + \lambda < \alpha \cdot (\delta + 1).$$ 

By D-T 4, $\delta < \alpha \cdot \delta$. If $\delta < \alpha \cdot \delta$ then since $\alpha \cdot \delta \in \mathcal{M}(\omega)$,

$$\delta + (\alpha \cdot \delta) = \alpha \cdot \delta.$$ 

Since $\lambda < \alpha \cdot \delta$, we obtain

$$\delta + \lambda < \alpha \cdot \delta$$

which is a contradiction. Hence, $\delta = \alpha \cdot \delta$. Therefore, it follows from D-T 5 (ii) and D-T 47 that

$$\alpha < \delta \in \mathcal{M}(\omega).$$ 

Conversely, suppose $\alpha < \delta \in \mathcal{M}(\omega)$. Then

$$\alpha \cdot \delta = \alpha \cdot (\delta - 1) + \alpha$$

[D-T 27]

$$= \delta \cdot \alpha$$

[D-T 47]

$$\leq \delta + \alpha$$

[D-T 8]

Therefore, by D-T 47 and Definition 13,

$$\alpha \cdot \delta = \delta \leq \delta + \alpha.$$ 

This implies that there is a $\lambda < \delta = \alpha \cdot \delta$ such that $\beta = \delta + \lambda$, which completes the proof of the lemma.

**Theorem 21.** If $\gamma = \bigcup \gamma \not= 0$, $\alpha < \beta$, and $\alpha = 1 + \alpha'$ then the following conditions are equivalent

1. $\alpha \cdot \beta = \beta \cdot \alpha$;
2. $\beta = \beta + \alpha$;
3. $\beta = \gamma + \alpha'$ for some $\gamma < \alpha < \alpha'$ such that $\alpha < \alpha < \mathcal{M}(\omega)$.

**Proof.** By Lemma 14,

$$\beta \cdot \alpha = (\alpha \cdot \beta) \cdot \alpha$$

[D-T 27]

Therefore, by D-T 4 (ii), $\alpha \cdot \beta = \beta \cdot \alpha$ if and only if $\beta = \beta + \alpha'$, which proves the equivalence of (1) and (2).

By hypothesis, $3 < \alpha < \beta$. Therefore, if (2) holds it follows from Definition 13 that $\beta = \beta + \alpha'$. Moreover, by D-T 5 (ii), $\alpha' < \alpha \cdot \beta$.

Thus, Lemma 20 applies and we obtain (2) implies (3).

Suppose (3) holds. Then

$$\alpha \cdot \beta = \alpha \cdot (\gamma + \alpha')$$

[D-T 27]

$$= (\alpha \cdot \gamma) \cdot \alpha$$

[D-T 47]

On the other hand since $\alpha' < \alpha < \alpha$ and there are no main numbers between $\alpha$ and $\gamma + \alpha'$, it follows from Lemma 6 (iv) that

$$\beta \cdot \alpha = (\gamma + \alpha') \cdot \alpha = \gamma \cdot \alpha.$$ 

Therefore, (3) implies (1) and the proof is complete.

Thus, we have shown that $\gamma = \bigcup \gamma \not= 0$, $\alpha < \beta$, and $\alpha = 1 + \alpha'$ then $\alpha \cdot \beta = \beta \cdot \alpha$ if and only if either

$$\alpha = 2$$

and

$$\beta = 3.$$
or
\[
\beta = \alpha + \alpha' \quad \text{for some } \alpha \text{ such that } \alpha < \alpha \in M(O_\alpha).
\]

The next theorem describes necessary and sufficient conditions for \(O_\alpha\) to be associative, if \(\gamma\) is a limit ordinal.

**Theorem 22.** If \(\gamma = \bigcup \gamma \neq 0\) and \(\alpha, \beta, \delta \geq 2\) then
\[
(a O_\alpha \beta) O_\beta \delta = a O_\alpha (\beta O_\beta \delta)
\]
if and only if \(\beta < \delta \in M(O_\alpha)\).

**Proof.** Let \(\delta = 1 + \delta'\). Then by D-T 27,
\[
(a O_\alpha \beta) O_\beta \delta = a O_\alpha (\beta + \delta')
\]
Thus, by D-T 4 (ii),
\[
(a O_\alpha \beta) O_\beta \delta = a O_\alpha (\beta O_\beta \delta)
\]
if and only if
\[
(\delta') \beta + \beta' = \beta O_\beta \delta.
\]
If \(\beta < \delta \in M(O_\alpha)\) then it follows from D-T 47 and elementary properties of ordinal arithmetic that
\[
\beta + \delta' = \beta O_\beta \delta
\]
So (4) holds.

Conversely, suppose (4) holds. Then one of \(\beta\) or \(\delta\) must be larger than 2. (For if \(\beta = \delta = 2\) then \(\beta + \delta' = 3\) and by D-T 2 (iii), \(\beta O_\beta \delta = 4\).) If \(\beta > 2\) then it follows from D-T 37 that \(\beta O_\beta \delta \in M_\beta\). If \(\delta > 2\) and \(\beta = 2\) then
\[
\beta O_\beta \delta = 3 O_\beta \delta' \quad \text{[D-T 37]}
\]
\[
\in M_\beta. \quad \text{[D-T 37]}
\]
Thus, \(\beta + \delta' \in M_\beta \subseteq M(O_\alpha)\). So \(\beta + \delta'\) is a power of \(\omega\). This implies
\[
\beta < \delta' = \delta \in M(O_\alpha)
\]
and
\[
\beta + \delta' = \delta = \beta O_\beta \delta.
\]
Then using D-T 47, we obtain that \(\delta \in M(O_\alpha)\) thus completing the proof of the theorem.

**§ 4. Properties of \(R_\alpha\) and \(I_\alpha\).** The first few theorems describe how \(\Omega\) is ordered by \(R_\alpha\) when \(\gamma\) is a limit ordinal.

**Theorem 23.** If \(\gamma = \bigcup \gamma \neq 0\) then:
(i) If \(\alpha \geq 3\) then \(\beta: a R_\alpha \beta = (a) \cup (\beta \in M; \beta > a)\).
(ii) If \(a = 2\) then \(\beta: a R_\alpha \beta = (2, 4) \cup M_\alpha\).
(iii) If \(\beta 
eq 0\), 4 and \(\beta \in M_\alpha\) then \((a: a R_\alpha \beta = (1, \beta)\).
(iv) If \(\beta = 4\) then \((a: a R_\alpha \beta = (1, 2, 4)\).
(v) If \(\beta \in M_\alpha\) then \((a: a R_\alpha \beta = (a: 1 \leq \alpha < \beta)\).

**Proof.** Part (i) follows from D-T 2 and D-T 54; (ii) follows from D-T 37 and D-T 54; and (iii)-(v) follow from (i) and elementary properties of \(O_\alpha\).

**Theorem 24.** If \(\gamma = \bigcup \gamma \neq 0\) and \(\gamma = \Omega \cap (\alpha)\) then \(\gamma \cap R_\alpha\) is a complete lattice. (That is, a lattice in which each subset of \(\gamma\) has an \(R_\alpha\)-least upper bound and an \(R_\alpha\)-greatest lower bound.)

**Proof.** \(R_\alpha\) is reflexive because of D-T 2 (ii); anti-symmetric D-T 5 (ii) and D-T 7; and transitive, D-T 27.

Suppose \(\emptyset \neq x \subseteq \Omega\) and \(x\) is a set. Then \(\bigcup x \in \Omega\) and \(\bigcup x\) is the \(\leq\)-least upper bound of \(x\). Let \(\beta\) be an element of \(M_\alpha\), larger than \(\bigcup x\). (It follows from D-T 39 that there is an element of \(M_\alpha\), with the required property.) Then it follows from Theorem 33 (v) that \(\beta\) is an \(R_\alpha\)-upper bound of \(x\) and that the smallest \(R_\alpha\)-upper bound is the \(R_\alpha\)-least upper bound.

Suppose again that \(\emptyset \neq x \subseteq \Omega\), 1 is an \(R_\alpha\)-lower bound of \(x\). Moreover, it follows from Theorem 23 (iii)-(v) that the set of \(R_\alpha\)-lower bounds of \(x\) is an intersection of closed sets and is therefore closed. Therefore, \(x\) has an \(R_\alpha\)-greatest lower bound. This completes the proof.

In the case that \(\gamma \neq \bigcup \gamma\) the explicit description of \((a: a R_\beta \beta)\) and \((\beta: a R_\alpha \beta)\) is rather complicated and not very instructive. However, we did obtain some results for the case that \(\gamma\) is not a limit ordinal, the most important of which is that \((a: a R_\alpha \beta)\) is a closed set for all \(\beta\) and \(\gamma\).

**Theorem 25.** If \(A = (a: a R_\beta \beta)\) and \(\emptyset \neq x \subseteq A\), then \(\bigcup x \in A\).

**Proof.** Suppose \(A = (a: a R_\beta \beta)\) and \(\emptyset \neq x \subseteq A\). If \(x\) is finite the theorem is trivial, thus, let us suppose \(x\) is infinite. Let
\[
x = \{a: \kappa \in B\}
\]
where \(B \subseteq \Omega\), \(B\) is infinite, and \(\alpha_\kappa < \alpha_\tau\) if \(\kappa < \tau\). Therefore, for each \(\kappa \in B\) there is a \(\delta \in \Omega\), \(\delta_\kappa \neq \delta\), such that
\[
a_\alpha \delta_\delta = \beta.
\]
Consequently, it follows from the monotonity laws, that if \(\kappa < \tau\), \(\delta_\kappa > \delta_\tau\). Thus the \(\delta_\kappa\)'s form a decreasing sequence, \(\delta_\kappa > \delta_\tau > \ldots > \delta_n = \delta\), with \(\alpha \in \alpha\).

Let
\[
x = \{a \in x: a R_\alpha \beta = \beta\}.
\]
Then \(\bigcup x = \bigcup x\). We need only consider the case where \(x\) is infinite and show that \(\bigcup x' R_\beta \in x\). Let
\[
\theta = \bigcup x'.
\]
Case 1. \( \gamma = \bigcup \gamma \neq 0 \). By Theorem 23 (iii)--(v), \( A \) is closed for each \( \beta \in \Omega \), so \( \theta \in A \).

It follows from Theorem 8, that if \( \gamma \neq \bigcup \gamma \) then \( \delta \) is a limit ordinal, otherwise \( \chi' \) would not be infinite.

Case 2. \( \gamma = 2^\zeta + 2 \) for some \( \zeta \geq 1 \). Suppose that \( \omega^{\sigma} \alpha < \delta < \omega^{\sigma+1} \).

By Theorem 15, if \( \alpha, \beta \in \chi' \), \( \alpha < \beta \), then there are less than \( \omega^{\sigma} \) elements in the set

\[ Y_{\omega^\sigma} = \{ \alpha \in \mathcal{M}(\Omega_{\omega^\sigma}) : \alpha < \omega^{\sigma} \} \]

Therefore, if \( \theta \notin \chi' \) then each of the sets \( Y_{\omega^\sigma} \), \( \alpha \in \chi' \) has at least \( \omega^{\sigma} \) elements. Since \( \theta = \bigcup \chi' \), this implies \( \theta \in \mathcal{M}(\Omega_{\omega^\sigma}) \). Also, it follows from Lemma 6 (ii) that \( \beta \in \mathcal{M}(\Omega_{\omega^\sigma}) \). Clearly \( \theta < \beta \). If \( \theta = \beta \) then \( \theta \Omega_{\omega^\sigma} = \beta \) so \( \theta \in A \). If \( \theta < \beta \), then by Lemma 6 (ii) there exist ordinal numbers \( \eta \) and \( \eta' \), \( \eta < \eta' \) such that

\[ \theta = 2\Omega_{\omega}(1 + \eta) \]

and

\[ \beta = 2\Omega_{\omega}(1 + \eta') \]

Since \( \eta < \eta' \), there is a \( \xi > 0 \) such that

\[ \eta + \xi = \eta' \]

Thus,

\[ \beta = 2\Omega_{\omega}(1 + \eta + \xi) = 2\Omega_{\omega}(1 + \eta + \omega^\xi) \]

\[ = 2\Omega_{\omega}(1 + \eta + \omega^\xi) \]

\[ = \theta \Omega_{\omega} \omega^\xi \]

Therefore, \( \theta \in A \).

Case 3. \( \gamma = 2^\zeta + 1 \) for some \( \zeta \) such that \( \bigcup \zeta = \zeta \neq 0 \). Since \( \delta \) is a limit ordinal there is an \( \eta \) such that

\[ \delta = \omega^{1 + \eta} \]

Suppose \( \delta \notin \chi' \). If all the elements of \( \chi' \) are finite ordinals then \( \omega = \omega \) and for each \( \alpha \in \chi' \), \( \alpha > 2^\zeta \).

\[ \beta = \alpha \Omega_{\omega} \delta \]

\[ = \alpha \Omega_{\omega} \omega^\delta \]

\[ = \alpha \Omega_{\omega} (1 + \eta) \]

\[ = \omega \Omega_{\omega} \omega(1 + \eta) \]

\[ = \omega \Omega_{\omega} (1 + \eta) \]

\[ = \omega \Omega_{\omega} (1 + \eta) \]

which implies

\[ (1) \]

\[ \beta = \alpha \Omega_{\omega} (1 + \eta) \]

But, \( x + x' \in \Omega_1 \), so

\[ \beta = \omega \Omega_{\omega} (1 + \eta) \]

Therefore, by Theorem 12, \( \omega \chi' \delta = \delta \), or equivalently, \( x' + \mu = \mu \). This latter equation implies \( x' < \mu \) which contradicts (1). Thus, we must have,

\[ \sigma < x' + \mu \]
This implies there is a $\sigma'$ such that
\[ \sigma + \sigma' = n + \mu. \]
Now we have
\[ \beta = \omega \sigma \omega \rho \omega \tau (n' + 1). \]
If
\[ \beta = \sigma' \omega \rho \omega \tau (n' + 1) = \omega \sigma [1 + \omega \tau (n' + 1)] \quad \text{[D-T 33 (iii)]} \]
then $\theta E_{\omega} \beta$. Otherwise, it follows from Theorem 17 that $\sigma' \in \mathcal{M}_z$ and the proof follows along the same lines as the end of the proof of case 2, using Lemma 6 (iv) instead of Lemma 6 (ii).

**Case 4.** $\gamma = 2^{\zeta} + 3$ for some $\zeta > 1$. If $\zeta > \omega$ or $X'$ contains an infinite ordinal number the proof is similar to the proof of case 3, using Lemma 6 (ii) instead of Lemma 6 (iv) and Theorem 18 instead of Theorem 17. If $\zeta < \omega$ and $X' \subset \omega$ the proof is modified as follows.

First, we have as before that $\delta$ is a limit ordinal so there is an $\eta$ such that $\delta = \omega (1 + \eta)$. Also, since $X'$ is an infinite subset of $\omega$, $\theta = \omega$ and for each $\alpha \in X'$, $\alpha > 2$,
\[ \beta = \omega \alpha \beta = \omega \sigma [1 + \omega (1 + \eta)]. \]

If $\eta > 0$ then since $\alpha, \zeta < \omega$, it follows from Definition 1 that $\beta = \omega = \theta$.

If $\eta > 0$ then since $\omega \in \mathcal{M}(\mathcal{O})$ for all finite $\zeta$, it follows from Lemma 6 (ii), that
\[ \beta = \omega \sigma [1 + \omega (1 + \eta)]. \]

Thus, in either case $\beta \in A$.

The proof of the theorem for the remaining cases, $\gamma = 0, 1, 2, 3$, is an exercise in the traditional arithmetic of ordinal numbers. We leave the details for the interested reader. (For $\gamma = 1, 2$, see Carruth [2].)

Before stating the results for $L_\gamma$ it is convenient to introduce some notation:

**Definition 26.** If $\omega \neq \delta = \omega \alpha \rho \omega \delta + \ldots + \omega \alpha \rho \omega \delta$ is the normal form of $\delta$, then for each $m, n \in \omega$ and $0 < n < \omega$ such that $0 < m < n$ and $0 < \alpha _m < \omega$,
\[ \tau (m, \alpha _m) = \omega \rho \omega \tau (m) + \omega \rho \omega \tau (m+1) + \ldots + \omega \rho \omega \tau (n). \]

**Lemma 27.** If $\gamma = \omega \sigma$, $\delta = \omega \rho \omega \sigma + \ldots + \omega \rho \omega \sigma$ is the normal form of $\delta$ then $\alpha L_\delta \gamma$ if and only if there exist $m, n \in \omega$ such that $0 < m < n$, $0 < \alpha _m < \omega$, and
\[ a = \tau (m, \alpha _m). \]

**Proof.** The lemma follows from elementary properties of ordinal arithmetic.

**Theorem 28.** If $\gamma = \omega \sigma$ then
\[
(i) \quad \{ \beta : 2 \alpha \beta \gamma = \omega \rho \omega \sigma \} \cup \{ \beta : (\mathcal{O} \beta) [\gamma > 2, \beta = 3 \alpha \xi, \text{and } 1 \alpha \xi] \}
\]
\[ = \{ \omega \rho \omega \sigma \} \cup \{ \beta : (\mathcal{O} \beta) [\beta \text{ is the smallest element of } M, \text{ exceeding } \delta] \}. \]

(ii) If $\alpha > 0$ then
\[
\{ \beta : 2 + a \alpha \beta \gamma = \omega \rho \omega \sigma \} \cup \{ \beta : (\mathcal{O} \beta) [\beta = 3 \alpha \xi, \text{ and } (1 + a) \alpha \xi] \}
\]
\[ = \{ \omega + a \} \cup \{ \beta : (\mathcal{O} \beta) [\beta \text{ is the } \alpha \text{th successor element of } M, \text{ exceeding } \delta] \}. \]

(iii) If $\beta \neq 0, 4$ and $\beta \in M$, then $\alpha : a \alpha L_\beta = (1, \beta)$.

(iv) If $\beta = 4$ then $a \alpha L_\beta = (1, 2, 4)$.

(v) If $\beta \in M$, then $\beta = 3 \alpha \beta$ for some $\delta > 2$ and $\alpha : (1 + a) \beta L_\beta = (\delta \omega \upsilon (\delta) \omega \upsilon (\delta) + 1) \quad \text{[D-T 27]}$

Thus, part (i) is true.

The proof of (ii) is similar. The proofs of (iii) and (iv) follow from (i) and (ii). To prove (v) use Lemma 14 and D-T 27.

Theorem 24 does not hold if “$E$” is replaced by “$L$”, but we do have the following result for $L_\gamma$.

**Theorem 29.** If $\gamma = \omega \sigma$ and $a < a'$ then $a + 2$ and $a + \alpha'$ have an $L_\delta$-upper bound if and only if $a \alpha L_\sigma$ or $(1 + a) \alpha L_\sigma$.

**Proof.** The proof follows from Theorem 28 (v).

It is clear that 1 is an $L_\delta$-lower bound for every subset of $\Omega = \mathcal{O} \sim (0)$, if $\gamma > 0$ (D-T 2 (ii)). Since the set of all $L_\delta$-lower bounds of an
ordinal number is finite (D,T 14), it follows that every non-empty subclass of \( \mathcal{O} \) has an \( L \)-greatest lower bound.

It is also clear that \( L \) is reflexive and anti-symmetric for all \( \gamma \) and that \( L \) is transitive for \( \gamma = 0, 1, 2, 3 \). But we know that \( L \), for example, is not transitive. We do not know whether or not \( L \) is transitive even when \( \gamma \) is a limit ordinal. Our results for \( K \) and \( L \) are incomplete and it is probably that much more could be learned about these relations by additional study.

Our bibliography just includes those books and articles explicitly referred to in the paper. Additional references may be found in the bibliography of the Dones-Tarski paper [1].

Bibliography


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